Painlevé Properties and Solution of Revised Camassa-Holm Equation

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Abstract: Expansion of Painlevé is one of most effictive methods for solving non-linear partial differential equations. In this paper, using the Painlevé standard and non-standard cut-expansion as well as Maple softwar, the revised pricise solution of Camassa-Holm (mCH) is obtained.

Key words: revised Camassa-Holm equation; standard cut-expansion; non-standard cut-expansion; precise solution

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The investigation of integrability and the exact solutions of nonlinear equations is an important aspect in the study of nonlinear physical phenomena. One of the most powerful method to prove the integrability of a model is the so-called Painlevé analysis developed by WTC (Weiss-Tabor-Canvela)[1]. More methods have been proposed in succession through the efforts of researchers, such as the Conte's invariant method[2], the Pickering method[3], the extended Painlevé analysis method[4-5], and the W-K algorithm[6-7]. With the help of these methods, some interesting results are obtained[8]. Though the precise equivalence of Painlevé property and Painlevé integrability remains to be determined, a connection between integrability and the Painlevé property has been noted since the work of Kowalevskaya[9]. Therefore, the integrability, the Lax pairs and Bäcklund transformations of equations can be studied. The rest of this paper is organized as followed, In section 2, we review the Painlevé integrability of the modified Camassa-Holm equation simply. In section 3, we use the standard truncated Painlevé expansion to obtain some exact solutions of the model. The section 4 , we use the nostandard truncated Painlevé expansion to obtain some exact solutions of the model. The section 5 is a short conclusion.

1 Painlevé integrability of the modified Camassa-Holm equation

In this paper, we consider the modified Camassa-Holm equation shallow water equation[10]. Many powerful methods such as the inverse scattering transform, GRE method[11], discrete variational derivative method[12], the homotopy analysis method[13], tanh-sech method, sine-cosine method[14] and exp-function method[15] were used to investigate this equation. The study of its integrability and exact solutions plays an important role in the research of nonlinear physical phenomena[16-18]. That is the equation:

$$u_t - u_{xxt} + 3u^2 u_x - 2u_x u_{xx} - u u_{xxx} = 0, \tag{1}$$

where u can be written as

$$u = \sum_{j=0}^{+\infty} u_j \xi^{j-\alpha}, \xi = \xi(x,t), u_i = u_i(x,t),$$

α is a negative integer. Substituting $u = u_0 \xi^{\alpha}$ into equation (1), one can easily find that $\alpha = -2$, by using

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the leading order analysis, the recursion relation of the expansion coefficient u_j are given by,

$$(j+1)(j-6)(j-8)u_j = F_j(\xi_x,\xi_t,\cdots,u_0,u_1,\cdots,u_{j-1}), \tag{2}$$

where F_j is a function of $u_0,u_1,\cdots,u_{j-1}$ and the derivatives of ξ, from equation (2), we know that the resonances are located at $j=-1,6,8$, the resonance at $j=-1$ represents the arbitrariness of the expansion function. So, we only have to prove that $j=6,8$ should be satisfied identically. From[19], we know that by using the Kruskal's method[20], the resonance at $j=6,8$ is satisfied identically.

So, from the above simple analysis, we can review the conclusion that the modified Camassa-Holm equation possesses the Painleve property. It is Painlevé integrable.

2 Standard truncated expansion

In oeder to obtain the exact soliton solutions of equation (1), we truncate the Painlevé expansion at the constant level term[21]

$$u(x,t)=\xi^{-j}(x,t)\sum_{l=0}^{j}u_j(x,t)\xi^l(x,t). \tag{3}$$

On balancing the highest-order contribution from the linear term with the highest-order contributions from the nonlinear term, we have $j=2$. namely, equation (3) can expressed as

$$u(x,t)=\frac{u_0}{\xi^2}+\frac{u_1}{\xi}+u_2. \tag{4}$$

Substituting equation (4) into equation (1), and letting the coefficient of the power of ξ vanish, through a complicated calculation, the following results can be obtained directly.

$$\begin{aligned}
&u_0=8\xi_x^2, u_1=-8\xi_{xx},\\
&u_2=\frac{-6\xi_{xx}^2+8\xi_x\xi_{xxx}+\xi_t\xi_x}{3\xi_x^2},\\
&\xi_{xt}(\xi_x)^2=\xi_x\xi_t\xi_{xx}-6(\xi_{xx})^3+8\xi_x\xi_{xx}\xi_{xxx}-\\
&\quad 2(\xi_{xx})^2\xi_{xxxx},\\
&12(\xi_x)^3\xi_{xxxxx}=60(\xi_x)^2\xi_{xx}\xi_{xxxx}+90(\xi_{xx})^4-\\
&\quad 180\xi_x(\xi_{xx})^2\xi_{xxx}+40(\xi_x)^2\xi_{xxx}-\\
&\quad 3\xi_t(\xi_x)^3-(\xi_t)^2(\xi_x)^2,\\
&144(\xi_{xx})^5=24\xi_x(2\xi_t+12\xi_{xxx}+3\xi_x)\cdot\\
&\quad (\xi_{xx})^3-48(\xi_{xx})^2(\xi_x)^2\xi_{xxxx}-32\xi_{xxx}(\xi_x)^2\cdot\\
&\quad (2\xi_t+3\xi_x+4\xi_{xxx})\xi_{xx}+8(\xi_x)^3\xi_{xxxx}\cdot\\
&\quad (2\xi_t+3\xi_x+4\xi_{xxx}),\\
&216\xi_{xxxxxx}=36\xi_x(2\xi_t+12\xi_{xxx}+3\xi_x)(\xi_{xx})^4-\\
&\quad 144\xi_{xxxx}(\xi_x)^2(\xi_{xx})^3+12(\xi_x)^2+\\
&\quad (3\xi_t\xi_x-8\xi_t\xi_{xxx}-12\xi_x\xi_{xxx}+\xi_t-8(\xi_{xxx})^2)\cdot\\
&\quad (\xi_{xx})^2+24(\xi_x)^3\xi_{xxxx}(2\xi_t+3\xi_x+4\xi_{xxx})\xi_{xx}-\\
&\quad 2(\xi_x)^3(3\xi_t\xi_x+8(\xi_{xxx})^2+(\xi_t)^2)\cdot\\
&\quad (2\xi_t+3\xi_x+4\xi_{xxx}).
\end{aligned} \tag{5}$$

In order to simplify the solutions, we may let $u_2=0$. Firstly, we take $\xi(x,t)=1+e^{kx+\omega t+\xi_0}$ into consideration. Substituting it into equation (5), we get $\omega=-2k^3$, $k=\pm1$, $\xi=1+e^{\pm(x-2t)+\xi_0}$, so, the modified Camassa-Holm equation have the following exact solution:

$$u_1(x,t)=\frac{-4}{1+\cosh(x-2t+\xi_0)}, \tag{6}$$

$$u_2(x,t)=\frac{-4}{1+\cosh(-x+2t+\xi_0)}, \tag{7}$$

where ξ_0 is an arbitrary constant (Fig. 1). It is noted that solutions (6) and (7) are the same by using the extended tanh method in [22].

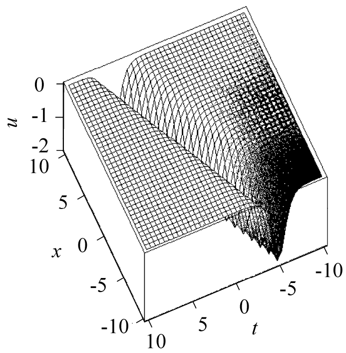

Fig. 1 Plots of $u_1(x,t)$ at $\xi_0=0$

Meanwhile, we introduce the the mobious transformation:

$$\xi\to\frac{a+b\xi}{c+d\xi},\quad ad\neq bc, \tag{7}$$

where a,b,c are arbitrary constant and c,d can not to be zero at the same time, with the mobious transformation, we can get another ξ_1, and

$$\xi_1 = \frac{a+b(1+e^{\pm(x-2t)+\xi_0})}{c+d(1+e^{\pm(x-2t)+\xi_0})}. \tag{8}$$

Substituting equation (9) into equation (5), we get $d=-c$ or $d=0$, when $d=0$, we just get $u_1(x,t)$, when $d=-c$, exact solutions of the mCH equation obtained:

$$u(x,t) = \frac{8b(a+b)e^{\pm(x-2t)+\xi_0}}{(a+b+be^{\pm(x-2t)+\xi_0})^2}, \tag{9}$$

where a,b and ξ_0 are arbitrary constants and can not to be zero at the same time. When $a=0$, solution (10) reduced to be solution (6).

3 Nostandard truncated expansion

Applying the extended truncated expansion and obtaining the partial derivatives of variable ξ with respect to x and t respectively as

$$\xi_x = 1+\frac{S\xi^2}{2},$$

$$\xi_t = -C + C_x\xi - \frac{1}{2}(C_{xx}+CS)\xi^2, \tag{10}$$

where

$$S = \frac{\phi_{xxx}}{\phi_x} - \frac{3}{2}\frac{(\phi_{xx})^2}{(\phi_x)^2}, C = -\frac{\phi_t}{\phi_x}, \tag{11}$$

the solutions of the system can be expressed as

$$u(x,t) = \sum_{j=0}^{M} u_j(x,t)\xi^{j-2}, \tag{12}$$

when $M=2$, the results are equivalent to the soliton solutions of the standard truncated expansion,and can be verified easily.

Now, let $M=4$, then $u(x,t)=\sum_{j=0}^{4}u_j(x,t)\xi^{j-2}$, substituting it into equation (1), and take the negative power of ξ as zero, we obtain the following equations:

$$u_0 = 8, u_1 = 0, u_2 = \frac{8}{3}S - \frac{1}{3}C,$$

$$u_3 = \frac{1}{3}C_x - \frac{2}{3}S_x,$$

$$u_4 = -\frac{1}{60}C^2 + \frac{1}{20}C - \frac{1}{6}C_{xx} - \frac{2}{15}S^2 + \frac{2}{15}S_{xx},$$

$$9C_x + 90C_xS + 40CS_x + 40S_t + 30C_{xxx} + 4S_{xxx} - 6CC_x - 28SS_x = 0,$$

$$10S_{xt} + 9C_{xx} + 30SC_{xx} + 10CS_{xx} + 40C_xS_x + 4S_{xxxx} - 6CC_{xx} - 28SS_{xx} - 6(C_x)^2 - 28(S_x)^2 = 0. \tag{13}$$

In order to simplify the soliton solution, we may set $u_3=0, u_4=0$, and we have following results:

Case 1 Firstly, we assume that ϕ have the following form:

$$\phi = \tan(kx+\omega t+c). \tag{14}$$

Substituting equation (15) into equation (12), then to equation (14), with the help of maple software, we can get that

$$\omega = (-\frac{3}{2} \pm \frac{\sqrt{9-128k^4}}{2})k, \tag{15}$$

firstly, when we take

$$\omega = (-\frac{3}{2} + \frac{\sqrt{9-128k^4}}{2})k,$$

$$\phi = \tan(kx + (-\frac{3}{2} + \frac{\sqrt{9-128k^4}}{2})kt + c), \tag{16}$$

here, we introduce the Mobious transformation:

$$\phi \to \frac{a+b\phi}{c+d\phi}, ad \neq bc, \tag{17}$$

where a,b,c,d are arbitrary constant and c,d can not to be zero at the same time, so we can get another ϕ that satisfied equation (12) and equation (14) through the mobious transformation, namely,

$$\phi = \frac{a+b\tan(kx+(-\frac{3}{2}+\frac{\sqrt{9-128k^4}}{2})kt+c)}{c+d\tan(kx+(-\frac{3}{2}+\frac{\sqrt{9-128k^4}}{2})kt+c)}. \tag{18}$$

Substitute equation (19) into equation (12), the value of C and S obtained

$$C = \frac{3}{2} - \frac{\sqrt{9-128k^4}}{2}, S = 2k^2, \tag{19}$$

then substituting equation (20) into equation (14), after a complex calculation, we get

$$\xi=\frac{\tan(kx+(-\frac{3}{2}t+\frac{\sqrt{9-128k^4}}{2}t+c)k)}{k}, \tag{20}$$

$$u_2=\frac{16k^2}{3}-\frac{1}{2}+\frac{\sqrt{9-128k^4}}{6}, \tag{21}$$

another solution of the mCH equation is also obtained:

$$u_3(x,t)=\frac{16k^2}{3}-\frac{1}{2}+\frac{\sqrt{9-128k^4}}{6}+8k^2\cot(kx+(-\frac{3}{2}t+\frac{\sqrt{9-128k^4}}{2}t+c)k)^2, \tag{22}$$

where k,c are arbitrary constants (Fig. 2).

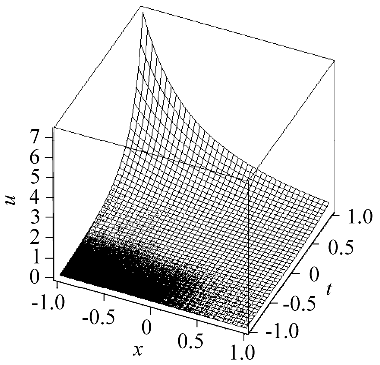

Fig. 2 Plots of $u_3(x,t)$ at $k=1/2, c=1$

When take $\omega=(-3/2-\sqrt{9-128k^4}/2)k$, another solution is obtained, namely,

$$u_4(x,t)=\frac{16k^2}{3}-\frac{1}{2}-\frac{\sqrt{9-128k^4}}{6}+8k^2\cot(kx+(-\frac{3}{2}t-\frac{\sqrt{9-128k^4}}{2}t+c)k)^2, \tag{23}$$

where k,c are arbitrary constant.

Case 2 In this case, take $\phi=\tanh(kx+\omega t+c)$, similar to case 1, we get

$$\xi=-\frac{e^{\eta}+1}{k(e^{\eta}-1)}, \tag{24}$$

where $\eta=-2kx+(3-\sqrt{9-128k^4})kt-ck$.

Some exact solutions of the mCH equation are obtained:

$$u_5(x,t)=-\frac{16k^2}{3}-\frac{1}{2}+\frac{\sqrt{9-128k^4}}{6}+8k^2\tanh(kx+(-\frac{3}{2}t-\frac{\sqrt{9-128k^4}}{2}t+c)k)^2, \tag{25}$$

$$u_6(x,t)=-16k^2/3-1/2-\sqrt{9-128k^4}/6+8k^2\cdot\tanh(kx+(-\frac{3}{2}t-\frac{\sqrt{9-128k^4}}{2}t+c)k)^2, \tag{26}$$

where k,c are arbitrary constants.

Case 3 We also can take $\phi=\coth(kx+\omega t+c)$, in this case, the value of ξ is equal to the ξ in case 2. The solutions of the mCH equation are getted:

$$u_7(x,t)=-\frac{16k^2}{3}-\frac{1}{2}+\frac{\sqrt{9-128k^4}}{6}+8k^2\coth(kx+(-\frac{3}{2}t+\frac{\sqrt{9-128k^4}}{2}t+c)k)^2, \tag{27}$$

$$u_8(x,t)=-\frac{16k^2}{3}-\frac{1}{2}-\frac{\sqrt{9-128k^4}}{6}+8k^2\coth(kx+(-\frac{3}{2}t-\frac{\sqrt{9-128k^4}}{2}t+c)k)^2, \tag{28}$$

where k, c are arbitrary constants.

4 Conclusions

In this paper, we have reintroduced the Painlevé property of the modified CH equation simply, then drawn the conclusion that it is Painlevé integrable. Applying the standard truncated expansion and nonstandard truncated expansion methods, we obtained two types of exact soliton solutions, respectively. We can use these methods to derive more solutions of other nonlinear equations.

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修正的 Camassa-Holm 方程的 Painlevé 性质及其精确解

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摘要：Painlevé 展开法是求解非线性偏微分方程的最有效的方法之一，主要利用 Painlevé 标准截断展开和非标准截断展开法及 Maple 软件来求得修正的 Camassa-Holm (mCH)方程的精确解.

关键词：修正的 Camassa-Holm 方程；标准截断展开；非标准截断展开；精确解

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