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一个演化方程族约束系统的r-矩阵

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摘要 本文给出一个演化方程族约束系统的Lax表示、r-矩阵、经典Poisson结构与可积性的证明.
关键词 约束系统，Lax表示，r-矩阵，经典Poisson结构.
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r-MATRIX FOR THE CONSTRAINED SYSTEM OF A EVOLUTION EQUATION HIERARCHY

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Abstract For the constrained systems of a evolution equation hierarchy, the Lax representations, r-matrices and the classical Poisson structures are given in this paper. The proof of integrability of these constrained systems are also presented.
Keywords Constrained System, Lax Representation, r-Matrix, Classical Poisson Structure.
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1 引言

r-矩阵与Poisson结构在可积系统的研究中有着重大的作用，因为它们蕴含了可积系统的许多内在性质[1].人们所关注的最新研究是由动力变量所确定的动力r-矩阵及其相应的经典、动力Yang-Baxter方程[2，3]和约束系统的r-矩阵[4，5]的构造问题.

本文用文［5］的方法研究一个演化方程族的约束系统，给出它的Lax表示、r-矩阵和经典Poisson结构，并用r-矩阵的方法证明了约束系统的可积性.

2 约束系统

考虑如下的谱问题[6]

$$\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}_x = U(u,\lambda)\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix},\quad U(u,\lambda)=\begin{pmatrix}\lambda & q\\ \lambda r & -\lambda\end{pmatrix},\quad u=\begin{pmatrix}q\\ r\end{pmatrix}. \tag{2.1}$$

解(2.1)的伴随方程

$$V_x=[U,V],\quad V(u,\lambda)=\sum_{m=0}^{\infty}\begin{pmatrix}a_m & b_m\\ c_m & -a_m\end{pmatrix}\lambda^{-m}, \tag{2.2}$$

则得到

$$a_0=1,\quad b_0=0,\quad c_0=r,\quad a_1=-\frac{1}{2}qr,\quad b_1=q,\quad c_1=-\frac{1}{2}(r_x+qr^2),$$

$$a_2=-\frac{1}{4}(q_xr-qr_x-\frac{3}{2}q^2r^2),$$

$$b_2=\frac{1}{2}(q_x-q^2r),\quad c_2=\frac{1}{4}(r_{xx}+\frac{3}{2}q^2r^3+3qrr_x),$$

$$\begin{pmatrix}c_{m+1}\\ b_{m+2}\end{pmatrix}=L\begin{pmatrix}c_m\\ b_{m+1}\end{pmatrix},\quad L=\frac{1}{2}\begin{pmatrix}-\partial-r\partial^{-1}q\partial & -r\partial^{-1}r\partial\\ -q\partial^{-1}q\partial & \partial-q\partial^{-1}r\partial\end{pmatrix},$$

$$a_{m+1}=\frac{1}{2r}(c_{m,x}+2c_{m+1}),\quad \partial=\frac{\partial}{\partial x},\partial\partial^{-1}=\partial^{-1}\partial=1. \tag{2.3}$$

令

$$V^{(n)}(u,\lambda)=(\lambda^{n+1}V)_++\Delta_n=\sum_{m=0}^{n+1}\begin{pmatrix}a_m & b_m\\ c_m & -a_m\end{pmatrix}\lambda^{n+1-m}+\begin{pmatrix}-a_{n+1} & 0\\ -c_{n+1} & a_{n+1}\end{pmatrix}, \tag{2.4}$$

则(2.1)与(2.4)的相容性条件，即零曲率表示

$$U_{t_n}-V_x^{(n)}+[U,V^{(n)}]=\begin{pmatrix}0 & q_{t_n}-b_{n+1,x}\\ \lambda(r_{t_n}-c_{n,x}) & 0\end{pmatrix}, \tag{2.5}$$

给出演化方程族

$$u_{t_n}=\begin{pmatrix}q\\ r\end{pmatrix}_{t_n}=J\begin{pmatrix}c_n\\ b_{n+1}\end{pmatrix}=J\frac{\delta H_n}{\delta u}, \tag{2.6}$$

其中

$$J=\begin{pmatrix}0 & \partial\\ \partial & 0\end{pmatrix},\quad H_n=-\frac{2a_{n+1}+rb_{n+1}}{n}\quad (n>0),\quad H_0=qr.$$

取N个不同的λ\-j,考虑由(2.1)产生的系统

$$\begin{pmatrix}\psi_{1j}\\ \psi_{2j}\end{pmatrix}_x=U(u,\lambda_j)\begin{pmatrix}\psi_{1j}\\ \psi_{2j}\end{pmatrix},\quad j=1,2,\cdots,N. \tag{2.7}$$

对于系统(2.7)不难证明下面的等式成立

$$L\frac{\delta\lambda_j}{\delta u}=\lambda_j\frac{\delta\lambda_j}{\delta u},\quad \frac{\delta\lambda_j}{\delta u}=\begin{pmatrix}\delta\lambda_j/\delta q\\ \delta\lambda_j/\delta r\end{pmatrix}=\begin{pmatrix}\psi_{2j}^2\\ -\lambda_j\psi_{1j}^2\end{pmatrix}. \tag{2.8}$$

对任意非负整数k_0,考虑如下约束

$$\begin{pmatrix} c_{k_0} \\ b_{k_0+1} \end{pmatrix} - \frac{1}{2}\beta\begin{pmatrix} \langle\Psi_2,\Psi_2\rangle \\ -\langle\Lambda\Psi_1,\Psi_1\rangle \end{pmatrix} = \frac{\delta H_{k_0}}{\delta u} - \beta\sum_{j=1}^{N}\frac{\delta\lambda_j}{\delta u} = 0, \tag{2.9}$$

其中β为适当选取的常数，〈·　，·〉为R^N中的内积，$\Lambda=\mathrm{diag}(\lambda_1,\lambda_2,\cdots,\lambda_N)$，$\Psi_i=(\psi_{i1},\psi_{i2},\cdots,\psi_{iN})^T$, i=1,2.

(1) 当$k_0=0$时得到第一个约束

$$\begin{pmatrix} r \\ q \end{pmatrix} = \begin{pmatrix} c_0 \\ b_1 \end{pmatrix} = \frac{\delta H_0}{\delta u} = \frac{1}{2}\beta\begin{pmatrix} \langle\Psi_2,\Psi_2\rangle \\ -\langle\Lambda\Psi_1,\Psi_1\rangle \end{pmatrix}, \tag{2.10}$$

在(2.10)下(2.7)变换到如下正则Hamiltonian系统

$$\Psi_{1,x} = \frac{\partial\widetilde{H}_0}{\partial\Psi_2},\quad \Psi_{2,x} = -\frac{\partial\widetilde{H}_0}{\partial\Psi_1}, \tag{2.11}$$

$$\widetilde{H}_0=\langle\Lambda\Psi_1,\Psi_2\rangle-\frac{1}{4}\beta\langle\Lambda\Psi_1,\Psi_1\rangle\langle\Psi_2,\Psi_2\rangle,\beta\text{ 为任意}.$$

(2) 当$k_0=1$且$\beta=\frac{1}{2}$时，由(2.9)得第二个约束

$$\begin{pmatrix} -\frac{1}{2}(r_x+qr^2) \\ \frac{1}{2}(q_x-q^2r) \end{pmatrix} = \begin{pmatrix} c_1 \\ b_2 \end{pmatrix} = \frac{\delta H_1}{\delta u} = \frac{1}{4}\begin{pmatrix} \langle\Psi_2,\Psi_2\rangle \\ -\langle\Lambda\Psi_1,\Psi_1\rangle \end{pmatrix}, \tag{2.12}$$

其中$H_1=-\frac{1}{4}\int_{-\infty}^{+\infty}(-2q_xr+q^2r^2)\mathrm{d}x.$ 引入Jacobi-Ostrogradsky坐标[7]

$$q_{N+1}=q,\quad p_{N+1}=r \tag{2.13}$$

则(2.7)可变换到如下正则Hamiltonian系统

$$\begin{aligned} &Q_x=\frac{\partial\widetilde{H}_1}{\partial P},P_x=-\frac{\partial\widetilde{H}_1}{\partial Q},\\ &\widetilde{H}_1=\langle\Lambda\Psi_1,\Psi_2\rangle+\frac{1}{2}q_{N+1}\langle\Psi_2,\Psi_2\rangle-\frac{1}{2}p_{N+1}\langle\Lambda\Psi_1,\Psi_1\rangle+\frac{1}{2}q_{N+1}^2p_{N+1}^2,\\ &Q=(\psi_{11},\psi_{12},\cdots,\psi_{1N},q_{N+1})^T,P=(\psi_{21},\psi_{22},\cdots,\psi_{2N},p_{N+1})^T. \end{aligned} \tag{2.14}$$

(3) 当$k_0=2$且$\beta=\frac{1}{2}$时，由(2.9)得第三个约束

$$\frac{1}{4}\begin{pmatrix} r_{xx}+3qrr_x+\frac{3}{2}q^2r^3 \\ q_{xx}-3rqq_x+\frac{3}{2}q^3r^2 \end{pmatrix}=\begin{pmatrix} c_2 \\ b_3 \end{pmatrix}=\frac{\delta H_2}{\delta u}=\frac{1}{4}\begin{pmatrix} \langle \Psi_2,\Psi_2\rangle \\ -\langle \Lambda\Psi_1,\Psi_1\rangle \end{pmatrix}, \tag{2.15}$$

其中 $H_2=\frac{1}{8}\int_{-\infty}^{+\infty}(-4q_xr_x+3q^2rr_x-6r^2qq_x+4q^3r^3)\mathrm{d}x.$ 引入修正的Jacobi-Ostrogradsky坐标

$$q_{N+1}=q,\quad q_{N+2}=\frac{1}{2}(q_x-\frac{3}{4}q^2r),\quad p_{N+1}=-\frac{1}{2}(r_x+\frac{3}{4}qr^2),\quad p_{N+2}=r, \tag{2.16}$$

则(2.7)可变换到如下的正则Hamiltonian系统

$$\begin{aligned} &Q_x=\frac{\partial\widetilde{H}_2}{\partial P},P_x=-\frac{\partial\widetilde{H}_2}{\partial Q}, \\ &\widetilde{H}_2=\langle \Lambda\Psi_1,\Psi_2\rangle+\frac{1}{2}q_{N+1}\langle\Psi_2,\Psi_2\rangle-\frac{1}{2}p_{N+2}\langle\Lambda\Psi_1,\Psi_1\rangle+2q_{N+2}p_{N+1}+ \\ &\qquad \frac{3}{4}q_{N+1}^2p_{N+1}p_{N+2}+\frac{3}{4}q_{N+1}q_{N+2}p_{N+2}^2+\frac{1}{32}q_{N+1}^3p_{N+2}^3, \\ &Q=(\psi_{11},\cdots,\psi_{1N},q_{N+1},q_{N+2})^T,\quad P=(\psi_{21},\cdots,\psi_{2N},p_{N+1},p_{N+2})^T. \end{aligned} \tag{2.17}$$

3　Lax表示与r-矩阵

由(2.7)和(2.9)所确定的约束系统的Lax表示为 [4,5,8]

$$M_x^{(k_0)}(\lambda)=[U,M^{(k_0)}(\lambda)],\quad M^{(k_0)}(\lambda)=\begin{pmatrix} A^{(k_0)}(\lambda) & B^{(k_0)}(\lambda) \\ C^{(k_0)}(\lambda) & -A^{(k_0)}(\lambda) \end{pmatrix}. \tag{3.1}$$

用文［8］的方法求得约束系统(2.11)，(2.14)和(2.17)的Lax表示(3.1)中的M(λ)分别如下：

$$M^{(0)}(\lambda)=\begin{pmatrix} A^{(0)}(\lambda) & B^{(0)}(\lambda) \\ C^{(0)}(\lambda) & -A^{(0)}(\lambda) \end{pmatrix}=\begin{pmatrix} 1 & 0 \\ \frac{1}{2}\beta\langle\Psi_2,\Psi_2\rangle & -1 \end{pmatrix}+\frac{\beta}{2}\sum_{j=1}^{N}\frac{\lambda_j}{\lambda-\lambda_j}\begin{pmatrix} \psi_{1j}\psi_{2j} & -\psi_{1j}^2 \\ \psi_{2j}^2 & -\psi_{1j}\psi_{2j} \end{pmatrix}, \tag{3.2}$$

$$\begin{aligned} M^{(1)}(\lambda)&=\begin{pmatrix} A^{(1)}(\lambda) & B^{(1)}(\lambda) \\ C^{(1)}(\lambda) & -A^{(1)}(\lambda) \end{pmatrix}=\begin{pmatrix} \lambda-\frac{1}{2}q_{N+1}p_{N+1} & q_{N+1} \\ p_{N+1}\lambda+\frac{1}{2}\beta\langle\Psi_2,\Psi_2\rangle & -\lambda+\frac{1}{2}q_{N+1}p_{N+1} \end{pmatrix}+ \\ &\qquad \frac{\beta}{2}\sum_{j=1}^{N}\frac{\lambda_j}{\lambda-\lambda_j}\begin{pmatrix} \psi_{1j}\psi_{2j} & -\psi_{1j}^2 \\ \psi_{2j}^2 & -\psi_{1j}\psi_{2j} \end{pmatrix},\quad \beta=\frac{1}{2}, \end{aligned} \tag{3.3}$$

$$M^{(2)}(\lambda)=\begin{pmatrix} A^{(2)}(\lambda) & B^{(2)}(\lambda) \\ C^{(2)}(\lambda) & -A^{(2)}(\lambda) \end{pmatrix}=$$

$$\begin{bmatrix} \lambda^2-\frac{1}{2}q_{N+1}p_{N+2}\lambda-\frac{1}{2}(l_1+l_2) & q_{N+1}\lambda+(q_{N+2}-\frac{1}{8}q_{N+1}^2p_{N+2}) \\ p_{N+2}\lambda^2+(p_{N+1}-\frac{1}{8}q_{N+1}p_{N+2}^2)\lambda+\frac{1}{2}\beta\langle\Psi_2,\Psi_2\rangle & -\lambda^2+\frac{1}{2}q_{N+1}p_{N+2}\lambda+\frac{1}{2}(l_1+l_2) \end{bmatrix}+$$

$$\frac{\beta}{2}\sum_{j=1}^{N}\frac{\lambda_j}{\lambda-\lambda_j}\begin{bmatrix} \psi_{1j}\psi_{2j} & -\psi_{1j}^2 \\ \psi_{2j}^2 & -\psi_{1j}\psi_{2j} \end{bmatrix},\quad \beta=\frac{1}{2},\quad l_1\triangleq q_{N+1}p_{N+1},\quad l_2\triangleq q_{N+2}p_{N+2}$$

(3.4)

利用通常的Poisson括号的定义直接计算知(3.2)，(3.3)和(3.4)中的$A^{(i)}(\lambda)$, $B^{(i)}(\lambda)$, $C^{(i)}(\lambda)$ (i=0,1,2)都满足如下关系式

$$\{A(\lambda),A(\mu)\}=\{B(\lambda),B(\mu)\}=\{C(\lambda),C(\mu)\}=0,$$
$$\{A(\lambda),B(\mu)\}=\frac{\beta}{\mu-\lambda}(\mu B(\mu)-\lambda B(\lambda)),$$
$$\{A(\lambda),C(\mu)\}=\frac{\beta}{\mu-\lambda}(\lambda C(\lambda)-\mu C(\mu))+\beta C(\lambda),$$
$$\{B(\lambda),C(\mu)\}=\frac{2\beta}{\mu-\lambda}(\mu A(\mu)-\lambda A(\lambda))-2\beta A(\lambda). \quad (3.5)$$

设I为2×2单位矩阵，$M^{(1)}(\alpha_1)=I\otimes M(\alpha_1), M^{(2)}(\alpha_2)=M(\alpha_2)\otimes I$，则由(3.5)式容易证明约束系统(2.11)，(2.14)和(2.17)具有经典Poisson结构 [5]

$$\{M^{(1)}(\alpha_1),M^{(2)}(\alpha_2)\}=[r^{(12)}(\alpha_1,\alpha_2),M^{(1)}(\alpha_1)]-[r^{(21)}(\alpha_2,\alpha_1),M^{(2)}(\alpha_2)],$$

(3.6)

其中$\{M^{(1)}(\alpha_1),M^{(2)}(\alpha_2)\}$表示4×4矩阵$M^{(1)}(\alpha_1),M^{(2)}(\alpha_2)$相乘，但元素之间乘积换为Poisson括号，而$r^{(21)}(\alpha_1,\alpha_2)=P^{(12)}r^{(12)}(\alpha_1,\alpha_2)P^{(12)}$，r-矩阵为

$$r^{(ij)}(\alpha_i,\alpha_j)=\frac{\beta\alpha_i}{\alpha_j-\alpha_i}P^{(ij)}+\beta S^{(ij)},\quad S^{(ij)}=\sigma_+^{(i)}\otimes\sigma_-^{(j)}, \quad (3.7)$$

这里 $P^{(ij)}=\frac{1}{2}\sum_{n=0}^{3}\sigma_n^{(i)}\otimes\sigma_n^{(j)}$ 为置换矩阵，σ表示标准的Pauli矩阵 [1].

4　可积性的证明

由文献［2，3，5］知，当r-矩阵满足经典Poisson结构(3.6)时，守恒积分的生成函数$TrM^2(\lambda)$的Poisson括号为零，这个性质可用来证明守恒积分的对合性.

对系统(2.11)，由(3.2)得

$$\mathrm{Tr}(M^{(0)}(\lambda))^2 = (A^{(0)}(\lambda))^2 + B^{(0)}(\lambda)C^{(0)}(\lambda) = 1 + \beta\sum_{j=1}^{N}\frac{F_0^{(j)}}{\lambda-\lambda_j}, \tag{4.1}$$

其中

$$F_0^{(j)} = \lambda_j\psi_{1j}\psi_{2j} - \frac{\beta}{4}\langle\Psi_2,\Psi_2\rangle\lambda_j\psi_{1j}^2 + \frac{\beta}{4}\sum_{\substack{k=1\\(k\neq j)}}^{N}\frac{(\lambda_j\psi_{1j}\psi_{2k}-\lambda_k\psi_{1k}\psi_{2j})^2}{\lambda_k-\lambda_j},$$
$$j=1,2,\cdots,N, \tag{4.2}$$

为守恒积分并且有$\tilde{H}_0 = \sum_{j=1}^{N}F_0^{(j)}$.

对系统(2.14)，由(3.3)得

$$\mathrm{Tr}(M^{(1)}(\lambda))^2 = (A^{(1)}(\lambda))^2 + B^{(1)}(\lambda)C^{(1)}(\lambda) = \lambda^2 + \tilde{H}_1 + \frac{1}{4}\sum_{j=1}^{N}\frac{F_1^{(j)}}{\lambda-\lambda_j}, \tag{4.3}$$

其中 $\tilde{H}_1$和

$$F_1^{(j)} = 2\lambda_j^2\psi_{1j}\psi_{2j} - q_{N+1}p_{N+1}\lambda_j\psi_{1j}\psi_{2j} - p_{N+1}\lambda_j^2\psi_{1j}^2 - \frac{1}{4}\langle\Psi_2,\Psi_2\rangle\lambda_j\psi_{1j}^2 +$$
$$q_{N+1}\lambda_j\psi_{2j}^2 + \frac{1}{4}\sum_{\substack{k=1\\(k\neq j)}}^{N}\frac{(\lambda_j\psi_{1j}\psi_{2k}-\lambda_k\psi_{1k}\psi_{2j})^2}{\lambda_k-\lambda_j}, j=1,2,\cdots,N. \tag{4.4}$$

构成(2.14)的守恒积分.

对系统(2.17)，由(3.4)得

$$\mathrm{Tr}(M^{(2)}(\lambda))^2 = (A^{(2)}(\lambda))^2 + B^{(2)}(\lambda)C^{(2)}(\lambda) = \lambda^4 + \frac{1}{2}\tilde{H}_2\lambda + \frac{1}{4}F_2^{(0)} + \frac{1}{4}\sum_{j=1}^{N}\frac{F_2^{(j)}}{\lambda-\lambda_j}, \tag{4.5}$$

其中 $\tilde{H}_2, F_2^{(0)}$和 $F_2^{(j)}$构成(2.17)的守恒积分，这里

$$F_2^{(0)} = q_{N+1}\langle\Lambda\Psi_2,\Psi_2\rangle - q_{N+1}p_{N+2}\langle\Lambda\Psi_1,\Psi_2\rangle - p_{N+2}\langle\Lambda^2\Psi_1,\Psi_1\rangle +$$
$$2\langle\Lambda^2\Psi_1,\Psi_2\rangle - (p_{N+2} - \frac{1}{8}q_{N+1}p_{N+2}^2)\langle\Lambda\Psi_1,\Psi_1\rangle +$$
$$(q_{N+2} - \frac{1}{8}q_{N+1}^2p_{N+2})\langle\Psi_2,\Psi_2\rangle + (q_{N+1}p_{N+1} + q_{N+2}p_{N+2})^2, \tag{4.6}$$

$$F_2^{(j)} = 2\lambda_j^3\psi_{1j}\psi_{2j} - p_{N+2}\lambda_j^3\psi_{1j}^2 + q_{N+1}\lambda_j^2\psi_{2j}^2 - q_{N+1}p_{N+2}\lambda_j^2\psi_{1j}\psi_{2j} - \frac{1}{4}\langle\Psi_2,\Psi_2\rangle\lambda_j\psi_{1j}^2 - (p_{N+1} - \frac{1}{8}q_{N+1}p_{N+2}^2)\lambda_j^2\psi_{1j}^2 + (q_{N+2} - \frac{1}{8}q_{N+1}^2p_{N+2})\lambda_j\psi_{2j}^2 - (q_{N+1}p_{N+1} + q_{N+2}p_{N+2})\lambda_j\psi_{1j}\psi_{2j}, \quad j = 1,2,\cdots,N. \tag{4.7}$$

由$\mathrm{Tr}M^2(\lambda)$的对合性，(2.11)，(2.14)和(2.17)的守恒积分$\{F_0^{(j)}\}_{j=1}^N$，$\{\widetilde{H}_1, F_1^{(j)}\}_{j=1}^N$和$\{\widetilde{H}_2, F_2^{(j)}\}_{j=0}^N$是对合的，而其独立性则易验证.从而知约束系统(2.11)，(2.14)和(2.17)在Liouville意义下完全可积.

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