L^p estimates for quantities advected by a compressible flow

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Abstract: We consider the evolution of a quantity advected by a compressible flow and subject to diffusion. When this quantity is scalar it can be, for instance, the temperature of the flow or the concentration of some pollutants. Because of the diffusion term, one expects the equations to have a regularizing effect. However, in their Euler form, the equations describe the evolution of the quantity multiplied by the density of the flow. The parabolic structure is thus degenerate near vacuum (when the density vanishes). In this paper we show that we can nevertheless derive uniform L^p bounds that do not depend on the density (in particular the bounds do not degenerate near vacuum). Furthermore the result holds even when the density is only a measure.

We investigate both the scalar and the system case. In the former case, we obtain L^∞ bounds. In the latter case the quantity being investigated could be the velocity field in compressible Navier-Stokes type of equations, and we derive uniform L^p bounds for some p depending on the ratio between the two viscosity coefficients (the main additional difficulty in that case being to deal with the second viscosity term involving the divergence of the velocity). Such estimates are, to our knowledge, new and interesting since they are uniform with respect to the density. The proof relies mostly on a method introduced by De Giorgi to obtain regularity results for elliptic equations with discontinuous diffusion coefficients.

1 Introduction

Let $\theta(t,x)$ be a function defined on $[0,T]\times\mathbb{R}^3$, solution to the following equation:

$$\begin{aligned} &\partial_t(\rho\,\theta)+\operatorname{div}(\rho\,v\,\theta)-\operatorname{div}(\mu\nabla\theta)=\rho F+\operatorname{div}(\rho\,G),\\ &\theta(0,x)=\theta_0(x), \end{aligned} \tag{1}$$

where F, G_1, G_2, G_3, ρ, and v are given functions such that (ρ,v) satisfies the following continuity equation:

$$\begin{aligned} &\partial_t\rho+\operatorname{div}(\rho\,v)=0,\\ &\rho(0,x)=\rho_0(x). \end{aligned} \tag{2}$$

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At this point, we wish to stress out the fact that we will be considering very general diffusion coefficients $\mu(t,x)$ throughout the paper: We will only assume that $\mu(t,x)$ is measurable and verifies

$$\mu(t,x) \geq 1 \qquad \text{for} \ \ t\in[0,T], \ x\in\mathbb{R}^3. \tag{3}$$

Such a system of equations arises in a lot of contexts. The function θ can, for instance, model the temperature of a fluid with density ρ and velocity v, or the density of pollutant spreading in this fluid. When μ satisfies (3), it is well known that the usual energy inequality gives some bounds on θ (typically in $L^2(0,T,L^6(\mathbb{R}^3))$). However no further L^p estimates can be obtained on θ directly. This is because the conserved quantities always involve the density ρ (they are typically of the form $\rho\theta^\alpha$), so that any bounds obtained by this mean will degenerate on cavities (i.e. when $\rho = 0$). The goal of this paper is to establish L^∞ estimates on θ that do not degenerate on cavities and that do not depend too much on the density ρ.

Note that this kind of estimates is quite natural if one looks at the evolution of θ in the Lagrangian description of the flow. However, working in the Lagrangian framework would be of little help here, since all the bounds on F and G would be tampered (and we would lose the divergence structure of the term $\mathrm{div}(\rho G)$).

Let us now state our results more precisely. First, we will be considering only "suitable" solutions of (1). By this, we mean functions that verify the following inequality:

$$\frac{d}{dt}\int_{\mathbb{R}^3}\rho\phi(\theta)\,dx + \int_{\mathbb{R}^3}\mu\phi''(\theta)|\nabla\theta|^2\,dx \leq \int_{\mathbb{R}^3}\rho F\phi'(\theta)\,dx + \int_{\mathbb{R}^3}\phi'(\theta)\mathrm{div}(\rho G)\,dx, \tag{4}$$

for every convex function $\phi\in W^{2,\infty}_{\mathrm{loc}}(\mathbb{R})$ verifying:

$$\lim_{y\to\infty}\frac{\phi(y)}{y^2} \leq 1. \tag{5}$$

As we will see in Section 2, any regular solution of (1), (2) verifies (4) (with an equality). It is thus very natural to consider weak solutions verifying (4), in the same spirit as that of Leray's weak solutions for incompressible Navier-Stokes equations.

We denote by $\mathcal{M}^+(\mathbb{R}^3)$ the set of positive measures in $\mathbb{R}^3$, and for any $\rho\in\mathcal{M}^+(\mathbb{R}^3)$, we denote by $L^2(\rho)$ the set of ρ-measurable function h satisfying $\int h^2 d\rho < \infty$. The first result of this paper is the following:

Theorem 1 *Take T finite or $T=+\infty$. Assume that the viscosity coefficient μ verifies (3), that ρ lies in $L^\infty(0,T;\mathcal{M}^+(\mathbb{R}^3))$ and that F and G are such that there exists $0<\alpha<1$ such that*

$$\rho^\alpha|F| + \sum_{i=1}^3\rho^{1+\alpha}|G_i|^2/\mu \in L^p(0,T;L^q(\mathbb{R}^3)), \tag{6}$$

for some p and q satisfying:

$$p > \frac{1}{1-\alpha}, \qquad \frac{2}{p} + \frac{3}{q} < 2.$$

Let $\theta \in L^\infty(0,T;L^2(\rho(t)))$ *with* $\nabla\theta \in L^2((0,T)\times\mathbb{R}^3))$ *be a solution of (4) for every* $\phi \in W^{2,\infty}_{\mathrm{loc}}(\mathbb{R})$ *verifying (5).*

Then the following results hold:

- *If* $\theta_0 \in L^\infty(\mathbb{R}^3)$*, then* $\theta \in L^\infty([0,T]\times\mathbb{R}^3)$*.*
- *If* θ_0 *is only bounded in* $L^2(\rho_0)$ *and if* $\rho \in L^\infty(0,T;L^r(\mathbb{R}^3))$ *for some* $r > 3/2$*, then* $\theta \in L^\infty(t_0,T;L^\infty(\mathbb{R}^3))$ *for every* $t_0 > 0$*.*

One of the main motivation of this article is to derive some bounds for the velocity field of compressible flows (see the motivations subsection below). In the compressible Navier-Stokes system of equations, the velocity is advected by itself and subject to viscosity effects, as previously. However, there are now two viscosity terms, one of which involves $\operatorname{div} u$ which induces a strong system structure. Our first result does not apply to that case, so we will also study the system case: Consider a vector-valued function $u \in \mathbb{R}^3$ solution in $[0,T]\times\mathbb{R}^3$ to the following system of equations:

$$\partial_t\rho u + \operatorname{div}(\rho\, v\otimes u) - \operatorname{div}(2\mu\nabla u) - \nabla(\lambda \operatorname{div} u) = \rho F + \operatorname{div}(\rho\, G), \tag{7}$$

$$u(0,x) = u_0(x), \tag{8}$$

where (ρ, v) still verifies (2). In addition to (3), we will assume that the second viscosity coefficient λ (which can also depends on (t,x)) verifies for every $(t,x) \in [0,T]\times\mathbb{R}^3$:

$$\nu(t,x) = 2\mu(t,x) + 3\lambda(t,x) \geq 1, \tag{9}$$

$$3|\lambda(t,x)| \leq \kappa\nu(t,x), \tag{10}$$

for some $0 < \kappa < 1/2$.

As in the scalar case, we will consider "suitable" solutions of (7), which verify:

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^3} \rho\phi(|u|)\,dx \\
&+ \int_{\mathbb{R}^3} \nu\frac{\phi'(|u|)}{|u|}|\nabla u|^2\,dx + \int_{\mathbb{R}^3} \nu\left[\phi''(|u|) - \frac{\phi'(|u|)}{|u|}\right]|\nabla|u||^2\,dx \\
&\leq -\int_{\mathbb{R}^3} \lambda\left[\phi''(|u|) - \frac{\phi'(|u|)}{|u|}\right]\sum_{ij}\partial_i u_j\frac{u_i u_j}{|u|^2}(\operatorname{div} u)\,dx + \int_{\mathbb{R}^3}\rho\frac{u}{|u|}\cdot F\phi'(|u|)\,dx \\
&+ \int_{\mathbb{R}^3}\phi'(|u|)\sum_{i=1}^3\sum_{j=1}^3\frac{u_j}{|u|}\partial_i(\rho G_{ij})\,dx,
\end{aligned} \tag{11}$$

for every convex function $\phi \in W^{2,\infty}_{\text{loc}}(0,\infty)$ verifying:

$$\phi''(y) - \phi'(y)/y \geq 0 \qquad \lim_{y\to\infty} \frac{\phi(y)}{y^2} \leq 1. \tag{12}$$

Again, we will see in section 2 that any regular solution of (7), (2) verifies (11).

The system structure weakens the bounds, and our second result is the following:

Theorem 2 *Take T finite or $T = +\infty$. Assume that $\rho \in L^\infty(0,T;\mathcal{M}^+(\mathbb{R}^3))$, that μ and λ verify (3), (9) and (10), and that F and G are such that*

$$\sum_i \rho^\alpha |F_i| + \sum_{i,j} \rho^{1+\alpha}|G_{ij}|^2/\nu \in L^p(0,T;L^q(\mathbb{R}^3)), \tag{13}$$

for some p and q satisfying

$$p > \frac{1}{1-\alpha}, \qquad \frac{2}{p} + \frac{3}{q} < 2. \tag{14}$$

Consider $u \in L^\infty(0,T;L^2(\rho(t)))$, with $\nabla u \in L^2((0,T)\times\mathbb{R}^3)$, solution to (11) for every $\phi \in W^{2,\infty}_{\text{loc}}(0,\infty)$ verifying (12). Then the two following results hold true:

- *If $u_0 \in L^\infty(\mathbb{R}^3)$, then for any ball $B \subset \mathbb{R}^3$, $u \in L^{2+\log_2(1/\kappa)}([0,T]\times B)$ where κ is defined in (10).*
- *If u_0 is only bounded in $L^2(\rho_0)$ and $\rho \in L^\infty(0,T;L^r(\mathbb{R}^3))$ for some $r > 3/2$, then $u \in L^{2+\log_2(1/\kappa)}_{\text{loc}}((0,T]\times\mathbb{R}^3)$.*

Note that the proofs of both theorems will only make use of the relations (4) or (11) which do not depend on the advection velocity v. In particular the results hold true even if we cannot give a meaning to (1), (2) or (7). This explains why no assumption is required on v. Let us aslo emphasize that no lower bound is required on the density ρ, and that we can also deal with a density which is merely a measure. Note also that when $\rho \equiv 1$, $v = 0$, $\mu \equiv 1$ and $\lambda = 0$, the equation (1) is nothing but the heat equation with a given right hand side. In that case, the conditions on p and q in (6) and (13) are the usual one to obtain L^∞ regularity of the solutions of parabolic equations. Our conditions are thus optimal in that sense. Finally, we stress out the fact that these conditions also allow us to consider forces that blow up near vacuum like $1/\rho^\alpha$ for some $0 \leq \alpha < 1$.

In the next subsections, we show how those results apply to compressible Navier-Stokes equations and we give the main idea of the proof. It relies on a method introduced by De Giorgi in [5] to show C^α regularity of solutions to elliptic equations with rough diffusion coefficients (measurable). This method was used for the first time on Navier-Stokes equation in [18], where an alternative proof of partial regularity for solutions to incompressible Navier-Stokes equation, first proven by Caffarelli, Kohn and Nirenberg in [4], was provided. Note that the method makes use of inequalities (4) and (11) which describe the evolution of quantities of the form:

$$\int_{\mathbb{R}^3} \rho\phi(\theta)\,dx, \qquad \int_{\mathbb{R}^3} \rho\phi(|u|)\,dx.$$

This idea was already the key stone of the paper [15].

1.1 Motivation: Compressible Navier-Stokes equation

In this subsection we describe the consequences of our results on compressible barotropic Navier-Stokes equations. Our aim is to show the pertinence of the result but also its limits. We consider the following system of equations:

$$\begin{aligned} &\partial_t\rho + \mathrm{div}(\rho u) = 0 \\ &\partial_t(\rho u) + \mathrm{div}(\rho u \otimes u) + \nabla_x \rho^\gamma - \mathrm{div}(2\mu\,\nabla u) - \nabla(\lambda\,\mathrm{div} u) = 0, \end{aligned} \tag{15}$$

where the viscosity coefficients verify:

$$\mu \geq 1 \qquad 2\mu + 3\lambda \geq 1. \tag{16}$$

Note that this system can be written in the form of (7), (2) with $v = u$, $F = 0$ and $G = -\rho^{\gamma-1}$.

The problem of the existence of solutions defined globally in time for this type of system was addressed in one dimension for smooth enough data by Kazhikov and Shelukhin [12], and for discontinuous one, but still with densities away from zero, by Serre [16] and Hoff [8]. Those results have been generalized to higher dimensions by Matsumura and Nishida [14] for smooth data close to equilibrium and by Hoff [10], [9] in the case of discontinuous data.

Concerning large initial data, Lions showed in [13] the global existence of weak solutions for $\gamma \geq 3/2$ for $N = 2$ and $\gamma \geq 9/5$ for $N = 3$. This result has been extended later by Feireisl, Novotny, and Petzeltova to the range $\gamma > 3/2$ in [7]. Other results provide the full range $\gamma > 1$ under symmetries assumptions on the initial datum (see for instance Jiang and Zhang [11]). Notice that all those results hold with constant viscosity coefficients μ and λ. Unfortunately, those theories do not provide enough L^p bounds on the pressure ρ^γ to apply Theorem 2. Indeed, Vaigant in [17] showed that, even with rather smooth data, we cannot expect such bound for p large. Hence Theorem 2 provides only the following partial result:

Corollary 3 *Let μ and λ be two constants verifying (9) and (10), and let $\gamma > 3/2$. Let (ρ, u) be the solution of (15) constructed in [7]. If T is such that $\rho \in L^\infty(0,T; L^p(\mathbb{R}^3))$ with $p > 3\gamma$, then*

$$u \in L^{2+\log_2(1/\kappa)}_{\mathrm{loc}}((0,T] \times \mathbb{R}^3).$$

Note that the condition $\gamma > 3/2$ is necessary to make use of [7]. Naturally, the initial data is assumed to have finite energy, which, in particular guarantees that u_0 is bounded in $L^2(\rho_0)$.

All the previously mentioned results only hold for constant viscosity coefficients. From a physical point of view, however, the viscosity coefficients λ and μ are known to depend on the temperature and thus, for the barotropic model, on the density. Unfortunately, very little is known on the existence of solutions in this case (the main difficulty being to get some compactness on the density ρ). Bresch and Desjardins have found a new mathematical entropy for a class of such coefficients which gives some norms on the gradient of ρ (see [3],[2]). This allowed them to construct solutions of (15) when additional physical terms are added (drag force, Korteweg type term or cold pressure). It was shown in [15] that these additional terms can be droped. However Theorem 2 cannot be applied in this framework since the new BD entropy requires to violate condition (3). Thus, in this context, our results only give a priori bound for a system whose solutions are not yet known to exist:

Corollary 4 *(a priori bounds) Let $\gamma > 3/2$, $\mu(\rho)$ and $\lambda(\rho)$ verify (9) and (10) with $\nu(\rho) \geq \rho^{\beta}$ for some $\beta > 4\gamma/3$. Then any suitable solution (ρ, u) of (15), verifies:*

$$u \in L^{2+\log_2(1/\kappa)}_{\text{loc}}((0,T] \times \mathbb{R}^3).$$

Note that in this corollary no assumption needs to be made on ρ. Indeed we take advantage of the fact that the condition on G depends on ν: We have

$$\begin{aligned} \rho^{1+\alpha}\frac{|G|^2}{\nu} &= \frac{|\rho^{\gamma}|^2}{\rho^{(1-\alpha)}\nu} \\ &\leq \rho^{2\gamma-1+\alpha-\beta} \end{aligned}$$

which is dominated by a power of ρ slightly better than $2/3\gamma$. So we can apply the theorem, using the fact that the usual entropy inequality gives ρ bounded in $L^{\infty}(L^{\gamma})$.

When considering a ρ-dependent viscosity coefficient $\mu(\rho)$, another interesting problem arises. Indeed, the actual form of the first viscosity term should then be:

$$\text{div}(\mu(\rho)D(u)),$$

where $D(u) = \nabla u + (\nabla u)^T$ is the symmetric part of the gradient of u. Note that when μ is constant this remark is not relevant since we have:

$$\text{div}(\mu D(u)) = \text{div}(\mu\nabla u) + \mu\sum_i \partial_{ij}u = \text{div}(\mu\nabla u) + \nabla(\mu\text{div}u).$$

For our purpose, working with $D(u)$ instead of ∇u would be far more tedious, since it would add another pure system term. Finding a equivalent of Theorem 2 with such a term is therefore a very interesting and challenging open problem.

To conclude this remarks, let us mention that the same type of corollaries (with the same restrictions) can be applied to the Navier-stokes equation with temperature. For instance, in the recent existence result of Feireisl [6], the viscosity coefficients are allowed to depend on the temperature and are required to verify the condition (3). Let us also mention the result of Bresch and Desjardin [1] in which density dependent viscosity coefficients were considered. However, in this last paper, the coefficients have to vanish on vacuum, so our results do not apply.

1.2 Idea of the Proof

In this subsection we want to describe the main idea of the proof in a simplified framework. We thus focus on the first statement of Theorem 1 and assume that $G = 0$ and $\rho^{1/3}F$ is bounded.

We introduce a sequence of functions ϕ_k:

$$\phi_k(y) = [|y| - C_k]_+^2,$$

where $[z]_+ = \sup(0, z)$ and C_k is an increasing sequence of number defined (in this particular example) by:

$$C_k = K(1 - 2^{-k}),$$

where K will be chosen later. (Note that C_k converges to K when $k \to \infty$). With these notations, we set:

$$U_k = \sup_{0\le t\le T}\left(\int_{\mathbb{R}^3} \rho\phi_k(\theta)\, dx\right) + \int_0^T \int_{\mathbb{R}^3} \mu\phi_k''(\theta)|\nabla\theta|^2\, dx\, dt.$$

If we think of $\int \rho\theta^2\, dx$ as an energy (as in the case of compressible Navier-Stokes equation), then we can think of $\int \rho\phi_k(\theta)\, dx$ as a level set of the energy, namely the energy corresponding to the values of $|\theta|$ that are greater than C_k. The quantity U_k is thus the sum of the supremum of the k-level set of energy on all the times $0 \le t \le T$ and the viscous dissipation of this k-level set of energy over the same time interval. Our goal is now to determine how the quantity U_k depends on the previous $(k-1)$-level set quantity U_{k-1}.

This is done by using the inequality (4) with $\phi = \phi_k$ and integrating it over $[0, t]$ for every $0 \le t \le T$. Noticing that if $K > 2\|\theta_0\|_{L^\infty}$, then

$$\int_{\mathbb{R}^3} \rho_0\phi_k(\theta_0)\, dx = 0 \qquad \text{for all } k > 1,$$

we deduce:

$$\begin{aligned} U_k &\le 2\int_0^T \int_{\mathbb{R}^3} |\rho F\phi_k'(\theta)|\, dx\, dt \\ &\le C\int_0^T \int_{\mathbb{R}^3} \rho^{2/3}|\phi_k'(\theta)|\, dx\, dt. \end{aligned}$$

The next step is to control the right-hand side in terms of U_{k-1}. The main ingredients are Sobolev embedding and Tchebychev inequality. We denote:

$$\theta_k = (|\theta| - C_k)_+.$$

Since U_{k-1} controls the square of the $L^\infty(0,T;L^2(\mathbb{R}^3))$ norm of $\rho^{1/2}\theta_{k-1}$ together with the square of the $L^2([0,T]\times\mathbb{R}^3)$ norm of the gradient in x of θ_{k-1}, Sobolev embeddings and Hölder inequalities yield:

$$\|\rho^{1/5}\theta_{k-1}\|^2_{L^{10/3}([0,T]\times\mathbb{R}^3)} \leq CU_{k-1}. \tag{17}$$

Next, using (17) and the fact that when $|\theta| \geq C_k$, we have $\theta_{k-1} \geq C_k - C_{k-1}$ and so $1_{\{\theta_k>0\}} \leq \frac{\theta_{k-1}}{C_k - C_{k-1}}$, we get:

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \rho^{2/3} |\phi_k'(\theta)| \, dx \, dt \\
&= 2 \int_0^T \int_{\mathbb{R}^3} \rho^{2/3} \theta_k \mathbf{1}_{\{|\theta| \geq C_k\}} \, dx \, dt \\
&\leq 2 \int_0^T \int_{\mathbb{R}^3} \rho^{2/3} \theta_{k-1} \mathbf{1}_{\{\theta_{k-1} \geq C_k - C_{k-1}\}} \, dx \, dt \\
&\leq 2 \int_0^T \int_{\mathbb{R}^3} \rho^{2/3} \theta_{k-1} \frac{\theta_{k-1}^{7/3}}{(C_k - C_{k-1})^{7/3}} \mathbf{1}_{\{\theta_{k-1} \geq C_k - C_{k-1}\}} \, dx \, dt \\
&\leq \frac{C}{(C_k - C_{k-1})^{7/3}} U_{k-1}^{5/3},
\end{aligned}$$

which leads to:

$$U_k \leq \frac{C 2^{7k/3}}{K^{7/3}} U_{k-1}^{5/3}.$$

Note that in more general cases we will get an estimate of the form:

$$U_k \leq \frac{C 2^{\alpha k}}{K^{\underline{\gamma}}} U_{k-1}^{\beta}.$$

What is important in this inequality is to have $\beta > 1$ and $\underline{\gamma} > 0$. As a matter of fact, we can then prove (see Lemma 16) that for any given U_1, there exists a K large enough for which any sequence U_k satisfying this induction inequality will converge to 0. The definition of U_k then yields that $\nabla\theta_\infty$ is zero, with $\theta_\infty = (\theta - K)_+$. This implies that θ_∞ is constant and this constant has to be 0 since $\theta_\infty \in L^2(L^6)$. It follows that $\theta \leq K$ almost everywhere.

Note that the "non-linearisation" process, which gives a $\beta > 1$, is primordial to counter fight the growth of $2^{\alpha k}$. This is provided by the double action of the (linear) Sobolev embedding which gives L^p-norms with $p > 2$ and the non-linear Tchebychev type inequality which can be used thanks to the fact that we consider a previous energy level set. This is the key idea of De Giorgi's method.

When considering an initial value which is not bounded, we need to "non-linearize" the contribution of $\int \rho\theta^2\,dx$ near $t=0$. This can be done introducing increasing time T_k and integrating (4) on $[T_k, T]$.

The difficulties considering the system case are more serious. The problem is to control the viscous term of the right-hand side part of (11). This can be done only in a linear way. This forces to take a sequence C_k converging to infinity. The L^p norm will then depends on the rate of decreasing of U_k from a level set to the next.

2 Consistency of the notion of suitable solutions

In this short section we check that the notion of suitable solutions is consistent with the differential equations. This is quite important for us since the method relies entirely on the estimates (4) and (11). We will show the two following lemmas:

Lemma 5 *Let θ be a regular solution to (1) (2), then θ satisfies (4) with an equality.*

Lemma 6 *Let u be a regular solution to (7) (2), then u satisfies (11) for any ϕ verifying (12).*

Proof of Lemma 5: Using (1) and (2), we find:

$$\rho\partial_t\theta + \rho v\cdot\nabla\theta - \mathrm{div}(\mu\nabla\theta) = \rho F + \mathrm{div}(\rho G).$$

Multiplying by $\phi'(\theta)$ and integrating in x, we find:

$$\int_{\mathbb{R}^3} \rho\partial_t\phi(\theta)\,dx + \int_{\mathbb{R}^3} \rho v\cdot\nabla\phi(\theta)\,dx + \int_{\mathbb{R}^3} \mu\phi''(\theta)|\nabla\theta|^2\,dx$$
$$= \int_{\mathbb{R}^3} \rho F\phi'(\theta)\,dx + \int_{\mathbb{R}^3} \phi'(\theta)\mathrm{div}(\rho G)\,dx.$$

Using again (2) gives the result. $\square$

Proof of Lemma 6: ¿From (7) and (2) we find:

$$\rho\partial_t u + \rho v\cdot\nabla u - \mathrm{div}(2\mu\nabla u) - \nabla(\lambda\mathrm{div}u) = \rho F + \mathrm{div}(\rho G).$$

Multiplying it by $\phi'(|u|)u/|u|$ and integrating with respect to x we find:

$$\begin{aligned}&\int_{\mathbb{R}^3} \rho\partial_t(\phi(|u|))\,dx + \int_{\mathbb{R}^3} \rho v\cdot\nabla(\phi(|u|))\,dx + \int_{\mathbb{R}^3} 2\mu\frac{\phi'(|u|)}{|u|}|\nabla u|^2\,dx \\ &+ \int_{\mathbb{R}^3} \lambda\frac{\phi'(|u|)}{|u|}|\mathrm{div}u|^2\,dx + \int_{\mathbb{R}^3} 2\mu\left[\phi''(|u|) - \frac{\phi'(|u|)}{|u|}\right]|\nabla|u||^2\,dx \\ &\leq -\int_{\mathbb{R}^3}\lambda\left[\phi''(|u|) - \frac{\phi'(|u|)}{|u|}\right]\sum_{ij}\partial_i u_j\frac{u_iu_j}{|u|^2}(\mathrm{div}u)\,dx + \int_{\mathbb{R}^3}\phi'(|u|)\frac{u}{|u|}\cdot(\rho F)\,dx \\ &+ \int_{\mathbb{R}^3}\phi'(|u|)\sum_{ij}\frac{u_j}{|u|}\partial_i(\rho G_{ij})\,dx.\end{aligned}$$

We conclude noticing that, from the definition of ν in (9), we have:

$$\begin{aligned}&\int_{\mathbb{R}^3}\mu\frac{\phi'(|u|)}{|u|}|\nabla u|^2\,dx + \int_{\mathbb{R}^3}\lambda\frac{\phi'(|u|)}{|u|}|\mathrm{div}u|^2\,dx \\ &\quad\geq \int_{\mathbb{R}^3}\nu\frac{\phi'(|u|)}{|u|}|\nabla u|^2\,dx.\end{aligned}$$

□

In order to simplify the presentation, we will prove both theorems in the framework of the second one. More precisely, we will show that if $\lambda = 0$ then $u \in L^\infty_{\mathrm{loc}}([0,T]\times\mathbb{R}^N)$, and that if $\lambda \neq 0$ then we get some L^p norms. This will include Theorem 1, by applying the result to $u = (\theta,\theta,\theta)$.

3 Main propositions

First, we introduce the function:

$$v_k = [|u| - C_k]_+,$$

where C_k is an increasing sequence of positive numbers (to be chosen later). Note that v_k^2 can be seen as a level set of energy since $v_k^2 = 0$ for $|u| < C_k$ and is of order $|u|^2$ for $|u| \gg C_k$. We also consider a non-decreasing sequence of time T_k.

Then, we define

$$U_k = \sup_{T_k<t<T}\left(\int_{\mathbb{R}^3}\rho(t,x)\frac{|v_k(t,x)|^2}{2}\,dx\right) + \int_{T_k}^T\int_{\mathbb{R}^3}\nu|d_k(t,x)|^2\,dx\,dt,$$

where:

$$d_k^2 = \frac{C_k\mathbf{1}_{\{|u|\geq C_k\}}}{|u|}|\nabla|u||^2 + \frac{v_k}{|u|}|\nabla u|^2.$$

Note that if we take $C_k = 0$, we get

$$U_k = \sup_{T_k<t<T}\left(\int_{\mathbb{R}^3}\rho(t,x)\frac{|u(t,x)|^2}{2}\,dx\right) + \int_{T_k}^T\int_{\mathbb{R}^3}\nu|\nabla u(t,x)|^2\,dx\,dt.$$

Our goal is to show that for an appropriate choice of C_k, the sequence U_k goes to zero as k goes to infinity. This will be a consequence of the following propositions:

Proposition 7 (scalar case) *Consider u solution to (11) with $\lambda = 0$ and let F and G verify (13)-(14). For a fixed $K > 0$, we define C_k by:*

$$C_k = K(1 - 2^{-k}).$$

Then there exists $A_p \geq 1$, $\beta_{1p} > 1$, $\beta_{2p} > 1$, and $\gamma_p > 0$, depending only on p such that the following statements hold:

- *If $u_0 \in L^\infty(\mathbb{R}^3)$, then, taking $T_k = 0$ for all k and $K > 2\|u_0\|_{L^\infty}$, we have:*

$$U_k \leq \frac{A_p^k}{K^{\gamma_p}} \left(U_{k-1}^{\beta_{1p}} + U_{k-1}^{\beta_{2p}} \right) \qquad \text{for all } k \geq 2.$$

- *If $\rho \in L^\infty(0, T; L^r(\mathbb{R}^3))$ for some $r > 3/2$, then for every $t_0 > 0$ we define $T_k = t_0(1 - 2^{-k})$, and there exists $A'_r \geq 1$, $\beta'_r > 1$, and $\gamma'_r > 0$ such that for any $K > 0$:*

$$U_k \leq \frac{A_p^k}{K^{\gamma_p}} U_{k-1}^{\beta_{1p}} + \frac{{A'_r}^k}{t_0 K^{\gamma'_r}} U_{k-1}^{\beta'_r} \qquad \text{for all } k \geq 2.$$

Proposition 8 (system case) *Consider u solution to (11) where F and G verify (13)-(14), and assume that μ and λ verify (9) and (10). For a fixed $K > 0$, we define C_k by:*

$$C_k = K2^k.$$

Then there exists $0 < \varepsilon < 1$, $\beta_{1p} > 1$, $\beta_{2p} > 1$, and $\gamma_p > 0$, depending only on p such that the following holds true:

- *When $u_0 \in L^\infty(\mathbb{R}^3)$, we take $T_k = 0$ for all k. Then we have, for any $K > \|u_0\|_{L^\infty}$:*

$$U_k \leq \frac{U_{k-1}^{\beta_{1p}} + U_{k-1}^{\beta_{2p}}}{K^{\gamma_p}} + \varepsilon\kappa U_{k-1} \qquad \text{for all } k \geq 2.$$

- *When $\rho \in L^\infty(0, T; L^r(\mathbb{R}^3))$ for some $r > 3/2$, then for every $t_0 > 0$ we define $T_k = t_0(1 - \eta^{-k})$. Then there exists $\beta'_r > 1$, $\gamma'_r > 0$ and $0 < \eta < 1$ such that for any $K > 0$ we have:*

$$U_k \leq \frac{U_{k-1}^{\beta_p}}{K^{\gamma_p}} + \frac{U_{k-1}^{\beta'_r}}{(1-\eta) t_0 K^{\gamma'_r}} + \varepsilon\kappa U_{k-1} \qquad \text{for all } k \geq 2.$$

The next section is dedicated to the proofs of those two propositions. In the following one we will show how those propositions indeed imply Theorem 1 and Theorem 2.

4 Proof of Propositions 7 and 8

This section is devoted to the proof of Propositions 7 and 8. The proof is split into several steps.

Step 1: Evolution of $\int_{\mathbb{R}^3} \rho v_k^2\, dx$. The following lemma gives the inequality satisfied by the energy of the level set function v_k:

Lemma 9 *Let u be a solution of (11) in $Q =]0,T[\times\mathbb{R}^3$, then we have:*

$$\frac{d}{dt}\int_{\mathbb{R}^3} \rho \frac{v_k^2}{2}\, dx + \int_{\mathbb{R}^3} \nu d_k^2\, dx$$
$$\leq -\int_{\mathbb{R}^3} \lambda r_k\, dx + \int_{\mathbb{R}^3} \frac{v_k}{|u|} u \cdot (\rho F + \mathrm{div}(\rho G))\, dx, \tag{18}$$

where:

$$r_k = (\mathrm{div} u) u \cdot \nabla |u| \frac{C_k}{|u|^2} \mathbf{1}_{\{|u|\geq C_k\}}.$$

Proof. This lemma follows from (11) with

$$\phi(y) = \frac{1}{2}(y - C_k)_+^2,$$

and using the fact that

$$\phi''(|u|) - \frac{\phi'(|u|)}{|u|} = \frac{C_k \mathbf{1}_{\{|u|\geq C_k\}}}{|u|}.$$

□

Step 2: First estimates on U_k. Integrating (18) on $[\sigma, t]\times\Omega$, with $T_{k-1} \leq \sigma \leq T_k \leq t \leq T$ we get

$$\int_{\mathbb{R}^3} \rho \frac{|v_k(t,x)|^2}{2}\, dx + \int_\sigma^t \int \nu d_k^2(s,x)\, dx\, ds$$
$$\leq \int \rho \frac{|v_k(\sigma,x)|^2}{2}\, dx - \int_\sigma^t \int \lambda r_k\, dx\, dt$$
$$+ \int_\sigma^t \int \frac{v_k}{|u|} u \cdot (\rho F + \mathrm{div}(\rho G))\, dx\, dt.$$

When $T_k = 0$ and $C_k > \|u_0\|_{L^\infty}$ we have (with $\sigma = T_k = 0$):

$$\int \rho \frac{|v_k(T_k,x)|^2}{2}\, dx$$
$$= \int \rho \frac{|v_k(0,x)|^2}{2}\, dx$$
$$= \int \rho \frac{(|u_0| - C_k)_+^2}{2}\, dx = 0.$$

When $T_k = t_0(1-\eta^{-k})$, then integrating with respect to σ between T_{k-1} and T_k and dividing by $T_{k-1} - T_k = t_0\eta^{k-1}(1-\eta)$, we get

$$\begin{aligned}
U_k = & \sup_{t\in[T_k,T]} \left(\int \rho \frac{|v_k(t,x)|^2}{2}\, dx + \int_{T_k}^{t} \int_{\mathbb{R}^3} \nu d_k^2(s,x)\, dx\, ds \right) \\
\le & \frac{1}{t_0\eta^{k-1}(1-\eta)} \int_{T_{k-1}}^{T_k} \int \rho \frac{|v_k(\sigma,x)|^2}{2}\, dx\, d\sigma \\
& + \int_{T_{k-1}}^{T} \int \lambda(\rho) r_k\, dx\, dt \\
& + \int_{T_{k-1}}^{T} \int \frac{v_k}{|u|} u \cdot (\rho F + \operatorname{div}(\rho G))\, dx\, dt.
\end{aligned}$$

The following lemma follows:

Lemma 10

(i) When $u_0 \in L^\infty(\mathbb{R}^3)$, we take $T_k = 0$ for all k and $C_k > \|u_0\|_{L^\infty}$, then we have:

$$\begin{aligned}
U_k \le & \int_0^T \int \lambda r_k\, dx\, dt \\
& + \int_0^T \int \left(\frac{v_k}{|u|}\right) u \cdot (\rho F + \operatorname{div}(\rho G))\, dx\, dt.
\end{aligned} \tag{19}$$

(ii) When u_0 is only bounded in $L^2(\rho_0)$, we take $T_k = t_0(1-\eta^{-k})$, then we have:

$$\begin{aligned}
U_k \le & \frac{1}{t_0\eta^{k-1}(1-\eta)} \int_{T_{k-1}}^{T_k} \int \rho \frac{|v_k(\sigma,x)|^2}{2}\, dx\, d\sigma \\
& + \int_{T_{k-1}}^{T} \int \lambda r_k\, dx\, dt \\
& + \int_{T_{k-1}}^{T} \int \left(\frac{v_k}{|u|}\right) u \cdot (\rho F + \operatorname{div}(\rho G))\, dx\, dt.
\end{aligned} \tag{20}$$

Step 3: Some useful lemmas. We now want to show that the right-hand side in (19) and (20) can be controled by terms of the form U_{k-1}^{β} with $\beta > 1$. This is the corner stone of De Giorgi's method for the regularity of elliptic equation and a key step in this paper.

We start by giving the following technical lemma, which provides some useful inequalities for the rest of the paper and the proof of which quite straightforward (and is given in the appendix for the comfort of the reader):

Lemma 11 *The function u can be split in the following way:*

$$u = u\frac{v_k}{|u|} + u\left(1 - \frac{v_k}{|u|}\right),$$

where:

$$\left| u\left(1-\frac{v_k}{|u|}\right)\right| \le C_k.$$

Moreover the following bounds hold:

$$\begin{aligned}
&\frac{v_k}{|u|}|\nabla u| \le d_k,\\
&\mathbf{1}_{\{|u|\ge C_k\}}|\nabla |u|| \le d_k,\\
&|\nabla v_k| \le d_k,\\
&\left|\nabla \frac{u v_k}{|u|}\right| \le 3 d_k.
\end{aligned}$$

The next Lemma will be crucial in what follows:

Lemma 12 *For every nonnegative numbers* $0<\alpha<1$, $\beta>1$ *satisfying:*

$$\begin{aligned}
p_1 &= \frac{1}{\beta-\alpha} \ge 1\\
q_1 &= \frac{3}{2\alpha+\beta} \ge 1,
\end{aligned}$$

we have:

$$\|\rho^\alpha v_k^{2\beta}\|_{L^{p_1}(T_k,T;L^{q_1}(\mathbb{R}^3))} \le U_k^\beta.$$

Proof. We obviously have:

$$\|\rho^\alpha v_k^{2\beta}\|_{L^{p_1}(T_k,T;L^{q_1}(\mathbb{R}^3))} = \|\rho^{\alpha/\beta} v_k^2\|^\beta_{L^{p_1\beta}(T_k,T;L^{q_1\beta}(\mathbb{R}^3))}.$$

Next, we note that:

$$\rho^{\alpha/\beta} v_k^2 = (\rho v_k^2)^{\alpha/\beta} v_k^{2(1-\frac{\alpha}{\beta})},$$

and:

$$\begin{aligned}
&\|(\rho v_k^2)^{\alpha/\beta}\|_{L^\infty(T_k,T;L^{\beta/\alpha}(\mathbb{R}^3))} \le U_k^{\alpha/\beta},\\
&\|v_k^{2(1-\frac{\alpha}{\beta})}\|_{L^{\frac{1}{1-\alpha/\beta}}(T_k,T;L^{\frac{3}{1-\alpha/\beta}}(\mathbb{R}^3))} \le \|\nabla v_k\|^{2(1-\alpha/\beta)}_{L^2(]T_k,T[\times\mathbb{R}^3)} \le U_k^{1-\alpha/\beta}.
\end{aligned}$$

(where we have used Lemma 11 for the last inequality). It is now readily seen that Hölder's inequalities give the result. □

Step 4: Terms involving F and G.

Lemma 13 *If* F *and* G *satisfy conditions (13)-(14), then there exists some* $\beta>1$ *such that*

$$\left| -\int_{T_{k-1}}^T \int_{\mathbb{R}^3} \frac{v_k}{|u|} u\cdot \mathrm{div}(\rho G)\, dx\, d\sigma \right| \le C(C_k - C_{k-1})^{-\beta} U_{k-1}^{(1+\beta)/2} \qquad (21)$$

and

$$\left| - \int_{T_{k-1}}^{T} \int_{\mathbb{R}^3} \frac{v_k}{|u|} u \cdot \rho F \, dx \, d\sigma \right| \leq C(C_k - C_{k-1})^{-2\beta+1} U_{k-1}^{\beta}. \qquad (22)$$

Proof. First, we write

$$\begin{aligned}
& - \int_{\mathbb{R}^3} \frac{v_k}{|u|} u \cdot \operatorname{div}(\rho G) \, dx \\
= \ & \int_{\mathbb{R}^3} \rho G : \left(\nabla u (1 - \frac{C_k}{|u|})_+ + (u \otimes \nabla)|u| \frac{C_k}{|u|^2} \mathbf{1}_{\{|u| \geq C_k\}} \right) dx \\
\leq \ & C \left(\int_{\mathbb{R}^3} \rho^2 \frac{|G|^2}{\nu} \mathbf{1}_{\{|u| \geq C_k\}} \, dx \right)^{1/2} \\
& \quad \times \left(\int_{\mathbb{R}^3} \nu (|\nabla u|^2 (1 - \frac{C_k}{|u|})_+^2 + \nu |\nabla |u||^2 \frac{C_k^2}{|u|^2} \mathbf{1}_{\{|u| \geq C_k\}}) \, dx \right)^{1/2} \\
\leq \ & C \left(\int_{\mathbb{R}^3} \frac{\rho^2 |G|^2}{\nu} \mathbf{1}_{\{|u| \geq C_k\}} \, dx \right)^{1/2} \left(\int_{\mathbb{R}^3} \nu |d_k|^2 \, dx \right)^{1/2}.
\end{aligned}$$

Moreover, we have:

$$\begin{aligned}
& \int_{T_{k-1}}^{T} \int_{\Omega} \frac{\rho^2 |G|^2}{\nu} \mathbf{1}_{\{|u| \geq C_k\}} \, dx \, d\sigma \\
& \leq (C_k - C_{k-1})^{-2\beta} \int_{T_{k-1}}^{T} \int_{\mathbb{R}^3} \rho^{1-\alpha} v_{k-1}^{2\beta} \frac{\rho^{1+\alpha} G^2}{\nu} \, dx \, d\sigma \\
& \leq (C_k - C_{k-1})^{-2\beta} \left\| \frac{\rho^{1+\alpha} |G|^2}{\nu} \right\|_{L^{p_1'}(T_{k-1},T;L^{q_1'}(\mathbb{R}^3))} \|\rho^{1-\alpha} v_k^{2\beta}\|_{L^{p_1}(T_{k-1},T;L^{q_1}(\mathbb{R}^3))} \\
& \leq (C_k - C_{k-1})^{-2\beta} \left\| \frac{\rho^{1+\alpha} |G|^2}{\nu} \right\|_{L^{p_1'}(T_{k-1},T;L^{q_1'}(\mathbb{R}^3))} U_{k-1}^{\beta}
\end{aligned}$$

where we used Lemma 12. In order to have $\beta > 1$, we need to take p_1 and q_1 such that

$$\begin{aligned}
p_1 &< \frac{1}{\alpha} \\
q_1 &> \frac{3}{3 - 2\alpha}
\end{aligned}$$

(using Lemma 12 with $1 - \alpha$ instead of α). This leads to the following condition on p_1' and q_1' (conjugate of p_1 and q_1):

$$\begin{aligned}
p_1' &> \frac{1}{1 - \alpha} \\
q_1' &> \frac{3}{2\alpha}.
\end{aligned}$$

It is readily seen that such coefficients will satisfy (14), thus, using (13), we deduce

$$\left| -\int_{T_{k-1}}^{T} \int_{\mathbb{R}^3} \frac{v_k}{|u|} u \cdot \mathrm{div}(\rho G)\, dx\, d\sigma \right| \leq C(C_k - C_{k-1})^{-\beta} U_{k-1}^{(1+\beta)/2}.$$

Next, we consider the term involving F. We have:

$$\begin{aligned}
&\left| -\int_{T_{k-1}}^{T} \int_{\mathbb{R}^3} \frac{v_k}{|u|} u \cdot \rho F\, dx\, d\sigma \right| \\
\leq\ & \int_{T_{k-1}}^{T} \int_{\mathbb{R}^3} v_k \rho |F|\, dx\, d\sigma \\
\leq\ & (C_k - C_{k-1})^{-(2\beta-1)} \int_{T_{k-1}}^{T} \int_{\mathbb{R}^3} \rho^{1-\alpha} v_{k-1}^{2\beta} \frac{\rho^\alpha F}{\rho^\alpha}\, dx\, d\sigma \\
\leq\ & (C_k - C_{k-1})^{-2\beta+1} \left\|\rho^\alpha |F|\right\|_{L^{p_1'}(T_{k-1},T;L^{q_1'}(\mathbb{R}^3))} \|\rho^{1-\alpha} v_k^{2\beta}\|_{L^{p_1}(T_{k-1},T;L^{q_1}(\mathbb{R}^3))} \\
\leq\ & (C_k - C_{k-1})^{-2\beta+1} \left\|\rho^\alpha |F|\right\|_{L^{p_1'}(T_{k-1},T;L^{q_1'}(\mathbb{R}^3))} U_{k-1}^{\beta}
\end{aligned}$$

where we used lemma 12 as before (with the same conditions on p_1 and q_1). Using (13), we deduce

$$\left| -\int_{T_{k-1}}^{T} \int_{\mathbb{R}^3} \frac{v_k}{|u|} u \cdot \rho F\, dx\, d\sigma \right| \leq C(C_k - C_{k-1})^{-2\beta+1} U_{k-1}^{\beta}.$$

Remark: Note that when $\lambda = 0$ and $u_0 \in L^\infty(\mathbb{R}^3)$, and if we take $K > 2\|u_0\|_{L^\infty}$, $T_k = 0$ and $C_k = K(1 - 2^{-k})$, then (19), (21) and (22) yield

$$U_k \leq C(C_k - C_{k-1})^{-\gamma'} U_{k-1}^{\beta'}$$

with $\gamma' = \sup(\beta, 2\beta - 1)$ and $\beta' = \inf(\beta, (1+\beta)/2) > 1$. Since $C_k - C_{k-1} = K2^{-k}$, we deduce

$$U_k \leq CK^{-\gamma'} 2^{\gamma' k} U_{k-1}^{\beta'}.$$

This gives the first part of Proposition 7. In the next step we will show how to deal with unbounded initial data.

Step 5: Control of the time layer (case $T_k \neq 0$).

Lemma 14 *If ρ is bounded in $L^\infty(0,T;L^r(\mathbb{R}^3))$ for some $r > 3/2$, then we have:*

$$\begin{aligned}
&\frac{1}{t_0 \eta^{k-1}(1-\eta)} \int_{T_{k-1}}^{T_k} \int \rho \frac{|v_k(\sigma,x)|^2}{2}\, dx\, d\sigma \\
&\leq (t_0 \eta^{k-1}(1-\eta))^{-1} (C_k - C_{k-1})^{-\frac{2\alpha}{3}} ||\rho||_{L^\infty(0,T;L^r(\mathbb{R}^3))}^{\frac{3-\alpha}{3}} U_{k-1}^{1+\alpha/3}
\end{aligned}$$

with

$$\alpha = \frac{2r-3}{r-1} > 0$$

Proof. First, using Hölder and Sobolev inequalities, we have:

$$\begin{aligned}
&\int_{T_{k-1}}^{T_k} \int \rho \frac{|v_k(\sigma,x)|^2}{2}\,dx\,d\sigma \\
&\leq \left[\int_{T_{k-1}}^{T_k} \left(\int |v_k(\sigma,x)|^6\,dx \right)^{1/3} d\sigma \right] \sup_{t\in[T_{k-1},T_k]} \left(\int \rho^{3/2}(x,t) 1_{\{v_k>0\}}\,dx \right)^{2/3} \\
&\leq \left[\int_{T_{k-1}}^{T} \int |\nabla v_k(\sigma,x)|^2\,dx\,d\sigma \right] \sup_{t\in[T_{k-1},T_k]} \left(\int \rho^{3/2}(x,t) 1_{\{v_k>0\}}\,dx \right)^{2/3} \\
&\leq C U_{k-1} \sup_{t\in[T_{k-1},T_k]} \left(\int \rho^{3/2}(x,t) 1_{\{v_k>0\}}\,dx \right)^{2/3}.
\end{aligned}$$

Next, we note that for all x such that $v_k(x) > 0$, we have $|u(x)| > C_k$ and so

$$\begin{aligned}
v_{k-1}(x) &= [|u(x)| - C_{k-1}]_+ \\
&= [|u(x)| - C_k + (C_k - C_{k-1})]_+ \\
&> C_k - C_{k-1},
\end{aligned}$$

which yields

$$1_{\{v_k>0\}} \leq (C_k - C_{k-1})^{-1} v_{k-1}.$$

It follows that

$$\begin{aligned}
&\left(\int \rho^{\frac{3}{2}}(x,t) 1_{\{v_k>0\}}\,dx \right)^{\frac{2}{3}} \\
&\leq (C_k - C_{k-1})^{-\frac{2\alpha}{3}} \left(\int \rho^{\frac{3-\alpha}{2}}(x,t) \rho^{\frac{\alpha}{2}}(x,t) |v_{k-1}|^\alpha\,dx \right)^{\frac{2}{3}} \\
&\leq (C_k - C_{k-1})^{-\frac{2\alpha}{3}} \left(\int \rho^{\frac{3-\alpha}{2-\alpha}}(x,t)\,dx \right)^{\frac{2-\alpha}{3}} \left(\int \rho(x,t) |v_{k-1}|^2\,dx \right)^{\frac{\alpha}{3}}
\end{aligned}$$

With $\alpha = \frac{2r-3}{r-1}$ (so that $\frac{3-\alpha}{2-\alpha} = r$), we deduce

$$\begin{aligned}
&\sup_{t\in[T_{k-1},T_k]} \left(\int \rho^{\frac{3}{2}}(x,t) 1_{\{v_k>0\}}\,dx \right)^{\frac{2}{3}} \\
&\leq (C_k - C_{k-1})^{-2\alpha/3} ||\rho||_{L^\infty(0,T;L^r(\mathbb{R}^3))}^{(3-\alpha)/3} \left(\sup_{t\in[T_{k-1},T_k]} \int \rho(x,t) |v_{k-1}|^2\,dx \right)^{\frac{\alpha}{3}} \\
&\leq (C_k - C_{k-1})^{-2\alpha/3} ||\rho||_{L^\infty(0,T;L^r(\mathbb{R}^3))}^{\frac{r}{3(r-1)}} U_{k-1}^{\frac{2r-3}{3(r-1)}}
\end{aligned}$$

□

Remark: When $\lambda = 0$, we can take $T_k = 1 - 2^{-k}$ and $C_k = K(1 - 2^{-k})$. Proceeding as before, (20), (21), (22) and Lemma 14 give the second part of Proposition 7.

Step 6: The second viscosity term. It only remains to control the term corresponding to the second viscosity coefficient. This is achieved by the following lemma:

Lemma 15 *Under the assumptions (9) and (10), we have*

$$\left|\int_{T_{k-1}}^{T}\int_{\mathbb{R}^3} \lambda r_k \, dx\, dt\right| \leq \kappa \left(\frac{1}{3}\frac{C_k}{C_k - C_{k-1}}\right)^{1/2} U_{k-1}.$$

Proof. We have

$$\begin{aligned}
&\left|\int_{T_{k-1}}^{T}\int_{\mathbb{R}^3} \lambda r_k \, dx\, dt\right| \\
&= \left|\int_{T_{k-1}}^{T}\int_{\mathbb{R}^3} \lambda(\mathrm{div} u) u\cdot\nabla|u| \frac{C_k}{|u|^2}\mathbf{1}_{\{|u|\geq C_k\}}\, dx\, dt\right| \\
&\leq \left(\int_{T_{k-1}}^{T}\int_{\mathbb{R}^3} |\lambda|\frac{v_{k-1}}{|u|}|\mathrm{div} u|^2\, dx\, dt\right)^{1/2} \left(\int_{T_{k-1}}^{T}\int_{\mathbb{R}^3} |\lambda|\frac{C_k^2}{|u| v_{k-1}}|\nabla|u||^2 \mathbf{1}_{\{|u|\geq C_k\}}\right)^{1/2} \\
&\leq \left(\kappa\int_{T_{k-1}}^{T}\int_{\mathbb{R}^3} \nu d_k^2\, dx\, dt\right)^{1/2} \left(\int_{T_{k-1}}^{T}\int_{\mathbb{R}^3} \frac{|\lambda|}{\nu}\frac{\nu C_k}{v_{k-1}}\mathbf{1}_{\{|u|\geq C_k\}}|d_k|^2\, dx\, dt\right)^{1/2}.
\end{aligned}$$

Noticing that

$$\frac{C_k}{v_{k-1}}\mathbf{1}_{\{|u|\geq C_k\}}\frac{|\lambda|}{\nu} \leq \frac{\kappa}{3}\frac{C_k}{C_k - C_{k-1}},$$

we deduce

$$\left|\int_{T_{k-1}}^{T}\int_{\mathbb{R}^3} \lambda r_k \, dx\, dt\right| \leq \kappa \left(\frac{1}{3}\frac{C_k}{C_k - C_{k-1}}\right)^{1/2} U_{k-1}.$$

Step 7: Conclusion. We have already proven Proposition 7, which corresponds to $\lambda = 0$ (the first part was completed at the end of Step 4, and the second part at the end of Step 5). In order to control the term due to the second viscosity coefficient, and prove Proposition 8, we need the quantity $\frac{C_k}{C_k - C_{k-1}}$ to be bounded. We thus take $C_k = K2^k$. Then we have

$$\frac{C_k}{C_k - C_{k-1}} \leq 2,$$

and so:

$$\left| \int_{T_{k-1}}^{T} \int_{\mathbb{R}^3} \lambda r_k \, dx \, dt \right| \leq \varepsilon \kappa U_{k-1}, \tag{23}$$

where $\varepsilon = \sqrt{2/3} < 1$. The first part of Proposition 8 now follows using (19), (22), (21), and (23).

The second part follows (20), (22), (21), Lemma 14, (23) and the fact that since

$$C_k - C_{k-1} = K 2^{k-1} > 1,$$

we can always choose $\eta < 1$ such that

$$\eta^{-k+1}(C_k - C_{k-1})^{-\frac{2\alpha}{3}} \leq 1 \qquad \text{for all } k \geq 2.$$

5 Proofs of the theorems

We can now complete the proofs of the theorems. We have to show that the sequences U_k constructed in the previous section converge to zero as k goes to infinity for an appropriate choice of constant K. It will be a consequence of the following lemmas:

Lemma 16 *Let U_k be a sequence satisfying*

$$0 \leq U_0 \leq C,$$
$$0 \leq U_k \leq \frac{A^k}{K}(U_{k-1}^{\beta_1} + U_{k-1}^{\beta_2}), \qquad \forall k \geq 1,$$

for some constants $A \geq 1$, $1 < \beta_1 < \beta_2$ and $C > 0$.

Then there exists K_0 such that for every $K > K_0$ the sequence U_k converges to 0 when k goes to infinity.

Lemma 17 *Let U_k be a sequence satisfying:*

$$0 \leq U_0 \leq C,$$
$$0 \leq U_k \leq \frac{1}{K}(U_{k-1}^{\beta_1} + U_{k-1}^{\beta_2}) + \varepsilon \kappa U_{k-1}, \qquad \forall k \geq 1$$

for some constants $0 < \varepsilon < 1$, $0 < \kappa < 1$, $1 < \beta_1 < \beta_2$ and $C > 0$. Then there exists K_0 such that for every $K > K_0$ the sequence U_k converges to 0 when k goes to infinity. Moreover there exists $0 < \varepsilon_1 < 1$ and $C_K > 0$ such that:

$$U_k \leq C_K (\varepsilon_1 \kappa)^k.$$

Proof of Lemma 16: We introduce

$$\overline{U}_k = \frac{U_k}{K^{\frac{1}{\beta_1 - 1}}}.$$

As long as $K > 1$ we get for every k:

$$\begin{aligned}\overline{U}_k &\leq A^k \left(\overline{U}_{k-1}^{\beta_1} + \frac{\overline{U}_{k-1}^{\beta_2}}{K^{\frac{\beta_2 - \beta_1}{\beta_1 - 1}}} \right) \\ &\leq A^k \left(\overline{U}_{k-1}^{\beta_1} + \overline{U}_{k-1}^{\beta_2} \right).\end{aligned}$$

Next, we consider the sequence

$$W_k = (2A)^k W_{k-1}^{\beta_1},$$

and we claim that if W_0 is small enough then $W_k < 1$ for every k and W_k converges to 0. Indeed, introducing

$$\overline{W}_k = (2A)^{\frac{k}{\beta_1 - 1}} (2A)^{\frac{1}{(\beta_1 - 1)^2}} W_k,$$

we get

$$0 \leq \overline{W}_{k+1} \leq \overline{W}_k^{\beta_1}.$$

So if $W_0 \leq C_0^* = (2A)^{-1/(\beta-1)^2}$, we have $\overline{W}_0 \leq 1$ and by induction $\overline{W}_k \leq 1$ for every k. This gives:

$$W_k \leq (2A)^{\frac{-k}{\beta - 1}} C^{\frac{-1}{(\beta-1)^2}}.$$

Since $2A > 1$, this shows that W_k converges to 0 when k goes to infinity. Therefore, for K big enough, $\overline{U}_0 \leq W_0$ and as long as $\overline{U}_{k-1} < 1$, we have $\overline{U}_k \leq W_k$. So by induction we show that this is valid for every k and so $\overline{U}_k$ converges to 0. This implies that U_k converges to 0. □

Proof of Lemma 17: Take ε_1 such that:

$$\varepsilon < \varepsilon_1 < 1$$

and K big enough such that

$$\frac{C^{\beta_1 - 1} + C^{\beta_2 - 1}}{K} \leq (\varepsilon_1 - \varepsilon)\kappa.$$

Then $U_1 \leq \varepsilon_1 \kappa U_0$ and we can show by induction that

$$U_k \leq (\varepsilon_1 \kappa)^k C.$$

□

Theorem 1 (respectively Theorem 2) is now a straighforward consequence of Proposition 7 (resp. Proposition 8) and Lemma 16 (resp. Lemma 17).

Proof of Theorem 1: Let $u = (\theta, \theta, \theta)$. Then u verifies (11) with $\lambda = 0$, so we can use Proposition 7. We can then use Lemma 16 with $K = \inf(K^{\gamma_p}, K^{\gamma'_p})$ and $A = \sup(A_p, A'_q)$. If follows that for K big enough U_k converges to 0.

Next, we observe that

$$U_k \quad \geq \quad \int_0^\infty \int_{\mathbb{R}^3} \mathbf{1}_{\{t \in (T_k, T)\}} d_k^2 \, dx \, dt.$$

So when $T_k = 0$, we get that $d_k^2 \leq 2|\nabla u|^2$ and d_k converges almost everywhere to d_∞ defined with $C_\infty = K$. By Lebesgue's dominated convergence theorem and Lemma 11 we deduce

$$\int_0^T \int_{\mathbb{R}^3} |\nabla(|u| - K)_+|^2 \, dx \, dt \leq \int_0^T \int_{\mathbb{R}^3} d_\infty^2 \, dx \, dt = 0,$$

so $(|u| - K)_+ = 0$ and $|\theta| \leq K/3$. If $T_k = t_0(1 - 2^{-k})$, we proceed in the same way and we find that:

$$\int_{t_0}^T \int_{\mathbb{R}^3} |\nabla(|u| - K)_+|^2 \, dx \, dt \leq \int_0^T \int_{\mathbb{R}^3} d_\infty^2 \, dx \, dt = 0.$$

It follows that θ is bounded on $(t_0, T) \times \mathbb{R}^3$, and this is true for any $t_0 > 0$. $\square$

Proof of Theorem 2: We now use Proposition 8 and Lemma 17 with $K = \inf(K^{\gamma_p}, K^{\gamma'_p})$. It follows that for K big enough we get

$$U_k \leq C_K (\varepsilon_1 \kappa)^k.$$

¿From Lemma 11, we have:

$$U_k \geq \|v_k\|^2_{L^2(\sup T_k, T; L^6(\mathbb{R}^3))} \geq \|v_k\|^2_{L^2(\sup T_k, T; L^2_{\text{loc}}(\mathbb{R}^3))},$$

so using Tchebichev's inequality, we find that for every $R > 0$ we have

$$\begin{aligned} \mathcal{L}(\{|u| \geq 2K2^k\} \cap \{(\sup T_k, T) \times B(R)\}) \quad &\leq \quad \mathcal{L}(\{|u| \geq 2C_k\} \cap \{(\sup T_k, T) \times B(R)\}) \\ &\leq \quad \frac{U_k}{C_k^2} \\ &\leq \quad C_K 2^{-2k} (\varepsilon_1 \kappa)^k \\ &\leq \quad C_K 2^{-2k - pk} \end{aligned}$$

for some $p > \log_2 \frac{1}{\kappa}$. This implies that u lies in $L^{2+p,*}_{\text{loc}}$, and thus is bounded in $L^{2+\log_2 \frac{1}{\kappa}}_{\text{loc}}$. $\square$

A Proof of Lemma 11

Proof. The function $(1 - v_k/|u|)$ is Lipshitz and equal to:

$$
\begin{aligned}
1 - \frac{v_k}{|u|} &= 1 \qquad \text{if} \quad |u| \leq C_k \\
&= \frac{C_k}{|u|} \qquad \text{if} \quad |u| \geq C_k.
\end{aligned}
$$

Therefore:

$$\left| u \left(1 - \frac{v_k}{|u|} \right) \right| \leq C_k.$$

Let us first show that:

$$\frac{v_k}{|u|} |\nabla u| \leq d_k \tag{24}$$

$$\mathbf{1}_{\{|u| \geq C_k\}} |\nabla |u|| \leq d_k. \tag{25}$$

Statement (24) comes from the definition of d_k and the fact that $v_k \leq |u|$:

$$d_k^2 \geq \frac{v_k}{|u|} |\nabla u|^2 \geq \left(\frac{v_k}{|u|} |\nabla u| \right)^2.$$

To show (25), notice that:

$$|\nabla |u||^2 = \left| \frac{u}{|u|} \nabla u \right|^2 \leq |\nabla u|^2.$$

So:

$$d_k^2 \geq \frac{(C_k) \mathbf{1}_{\{|u| \geq (C_k)\}} + v_k}{|u|} |\nabla |u||^2,$$

with:

$$((C_k) + v_k) \mathbf{1}_{\{|u| \geq (C_k)\}} = |u| \mathbf{1}_{\{|u| \geq (C_k)\}}.$$

So:

$$d_k^2 \geq \mathbf{1}_{\{|u| \geq (C_k)\}} |\nabla |u||^2.$$

Then the bound on ∇v_k follows (25) since:

$$|\nabla v_k| = |\nabla |u|| \mathbf{1}_{\{|u| \geq (C_k)\}}.$$

To find the last inequality we fist write:

$$\nabla \left(\frac{u v_k}{|u|} \right) = \frac{u}{|u|} \nabla v_k + v_k \nabla \left(\frac{u}{|u|} \right).$$

The first term can be bounded by:

$$\left| \frac{u}{|u|} \nabla v_k \right| \leq |\nabla v_k| \leq d_k.$$

The second one can be rewritten in the following way:

$$v_k \nabla \left(\frac{u}{|u|} \right) = \frac{v_k}{|u|} \nabla u - \frac{v_k u}{|u|^2} \nabla |u|.$$

So, thanks to (24) and (25):

$$\begin{aligned} \left| v_k \nabla \left(\frac{u}{|u|} \right) \right| &\leq \frac{v_k}{|u|} |\nabla u| + \mathbf{1}_{\{|u| \geq (C_k)\}} |\nabla |u|| \\ &\leq 2d_k. \end{aligned}$$

This gives:

$$\left| \nabla \left(\frac{u v_k}{|u|} \right) \right| \leq 3d_k.$$

This ends the proof of the lemma. □

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