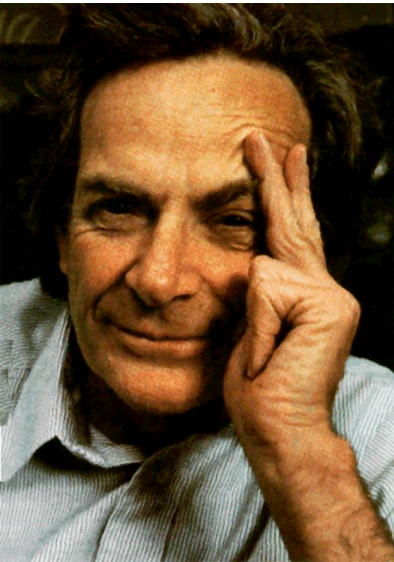


# The role of Chern Simons theory in solving the fractional quantum Hall effect mystery

The Chern-Simons theory unifies the fractional quantum Hall with the well understood integral quantum Hall effect, in the process leading to the prediction and the subsequent discovery of a new class of topological fermions in nature.



# Richard Feynman's last blackboard



1918-1988

What I cannot create,  
I do not understand.

Know how to solve every  
problem that has been solved.

Why const  $\times$   $\$0.01$  . PG

TO LEARN:  
Bethe Ansatz Probs.  
Kondo  
2-D Hall  
Nuclei Temp  
Non linear Chemical Hydro

(A)  $f = U(r, a)$   
 $g = 4(r-z) u(r, z)$

(B)  $f = 2|r \cdot a| (u \cdot a)$

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# The Nobel Prize in Physics 1985

"for the discovery of the quantized Hall effect"



**Klaus von Klitzing**

Federal Republic of Germany

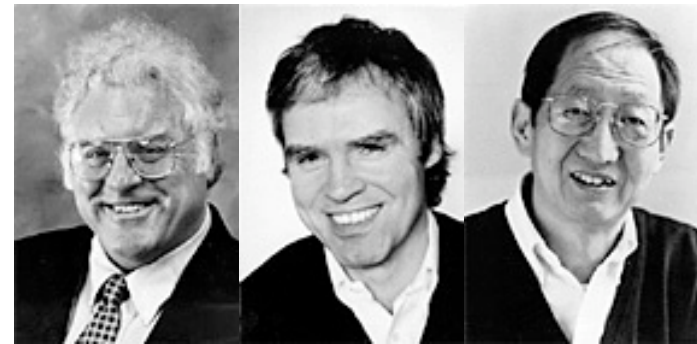
Max-Planck-Institute for Solid State Research  
Stuttgart, Federal Republic of Germany

b.1943



# The Nobel Prize in Physics 1998

"for their discovery of a new form of quantum fluid with fractionally charged excitations"



**Robert B. Laughlin**

1/3 of the prize

USA

Stanford University  
Stanford, CA, USA

b.1950

**Horst L. Störmer**

1/3 of the prize

Federal Republic of  
Germany

Columbia University  
New York, NY, USA

b.1949

**Daniel C. Tsui**

1/3 of the prize

USA

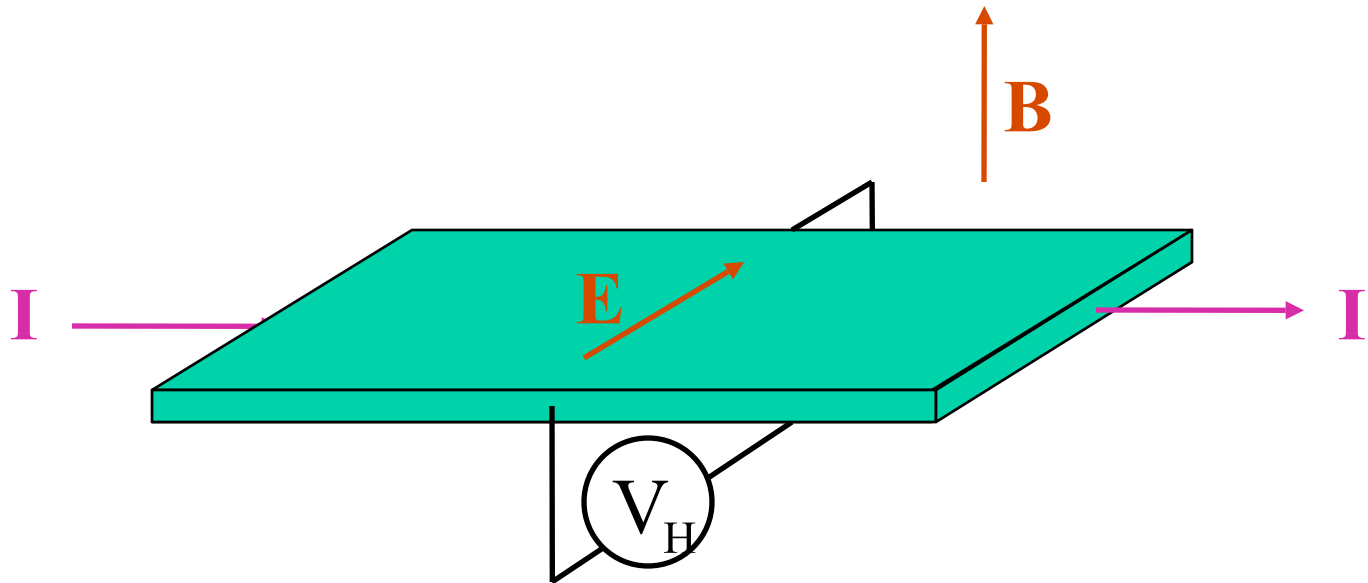
Princeton University  
Princeton, NJ, USA

b.1939  
(in Henan, China)

## Outline

- **The “fractional Hall effect” mystery**
- **Chern-Simons theory: prediction of composite fermions, a new class of topological fermions in physics**
- **Verifications / observations of composite fermions**
- **What is special about them**

## Hall effect (1879)



- In a magnetic field, electrons move perpendicular to the electric field due to the Lorentz force.

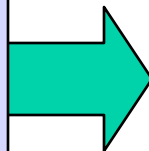
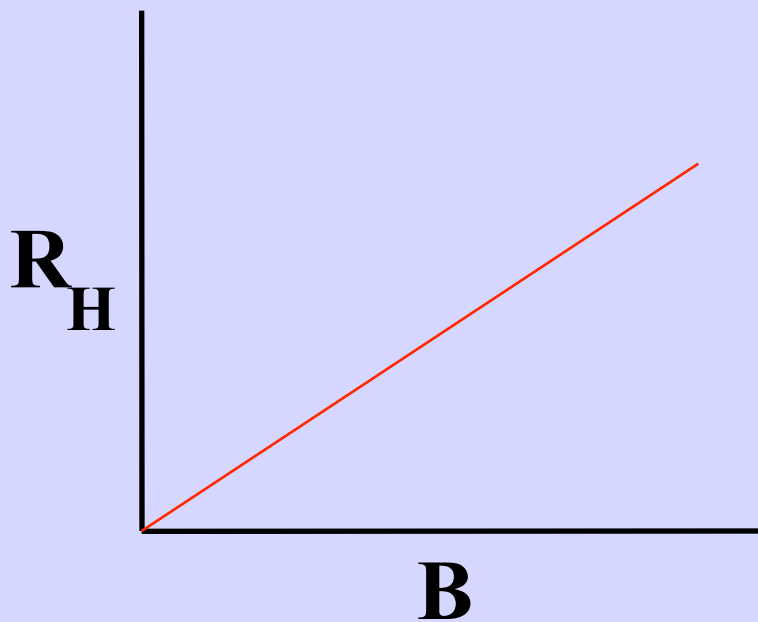
- A new resistance can be defined:

$$R_H \equiv \frac{V_H}{I} = \frac{B}{\rho e c}$$

classical result

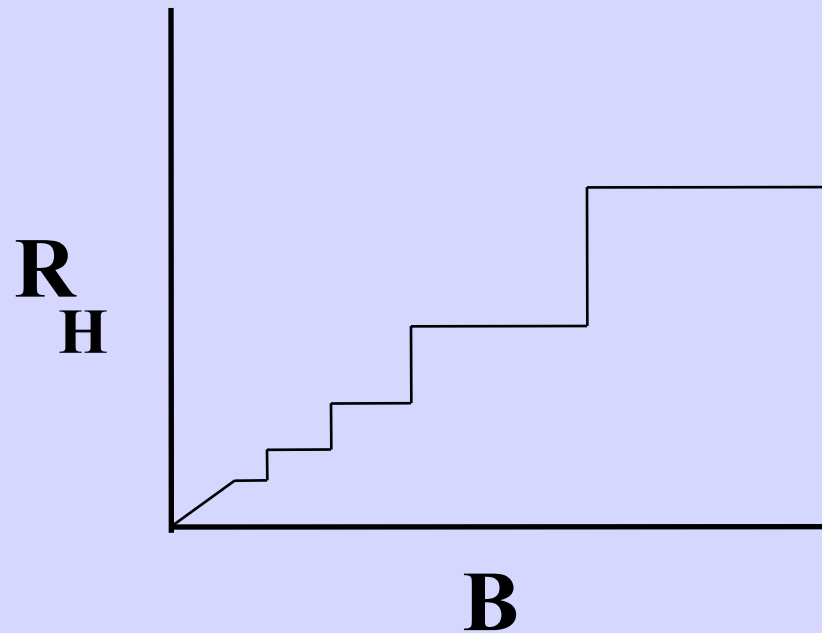
**3D**

Hall Effect



**2D**

Quantum Hall Effect



## New Method for High-Accuracy Determination of the Fine-Structure Constant Based on Quantized Hall Resistance

K. v. Klitzing

*Physikalisches Institut der Universität Würzburg, D-8700 Würzburg, Federal Republic of Germany, and  
Hochfeld-Magnetlabor des Max-Planck-Instituts für Festkörperforschung, F-38042 Grenoble, France*

and

G. Dorda

*Forschungslaboratorien der Siemens AG, D-8000 Mün.*

and

M. Pepper

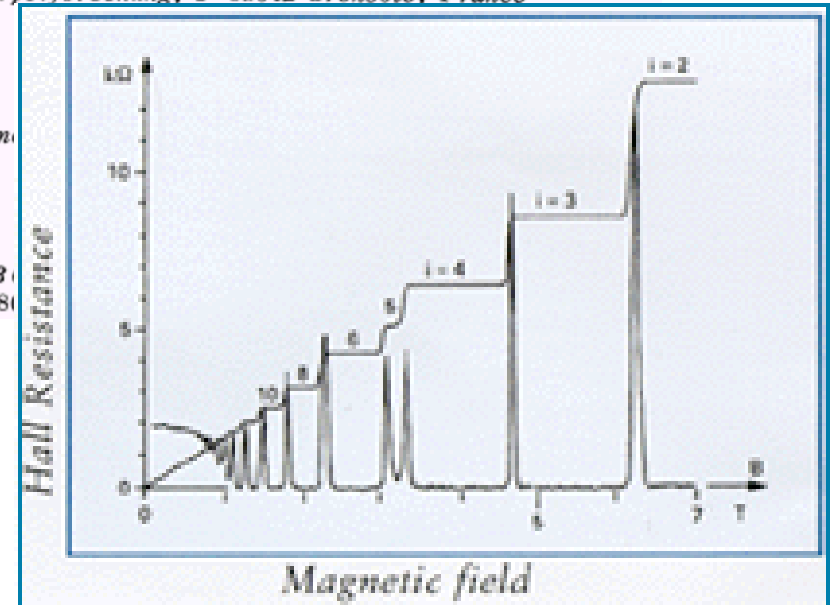
*Cavendish Laboratory, Cambridge CB3*

(Received 30 May 1980)

$$\alpha = e^2 / \hbar c$$

## Integral quantum Hall effect

$$R_H = \frac{h}{ie^2}$$



- **Relative accuracy of quantization: 3 parts in 10 billion. (One of the most accurate measurements of the fine structure constant.)**
- **Universal effect, independent of sample type, geometry, or disorder.**
- **Quantum effect.**

$$\alpha^{-1} = \frac{\hbar c}{e^2} = 137.03600300(270)$$

# Quantum Hall Effect

$$R_H = \frac{h}{ie^2}$$

“Quantum”

Consider a single electron in a magnetic field

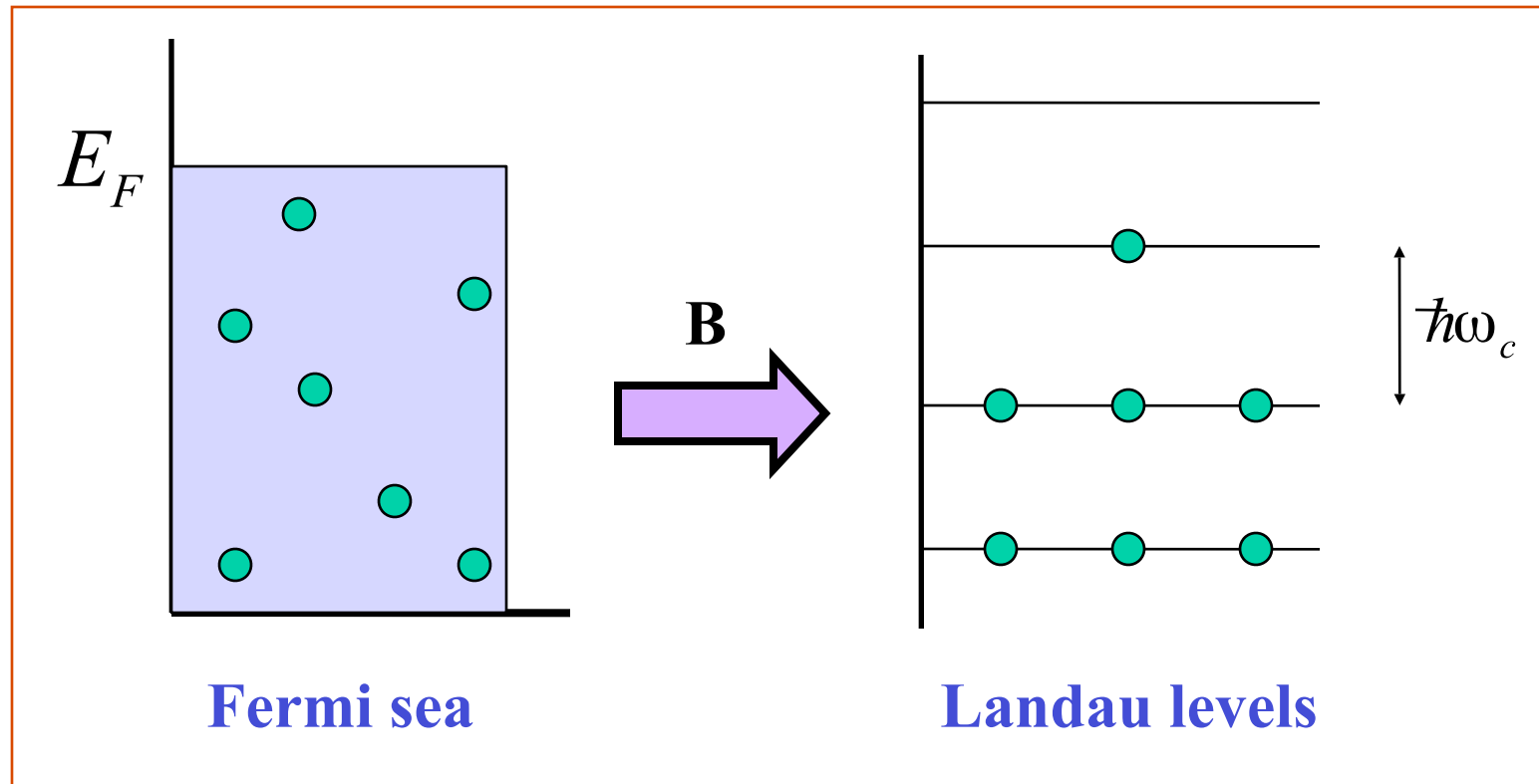
$$H = \frac{1}{2m} \left( \vec{p} + \frac{e}{c} \vec{A} \right)^2$$

**Solution: Kinetic energy is quantized into Landau levels.**

$$E = \left( n + \frac{1}{2} \right) \hbar \omega_c$$



## Landau levels (2D)



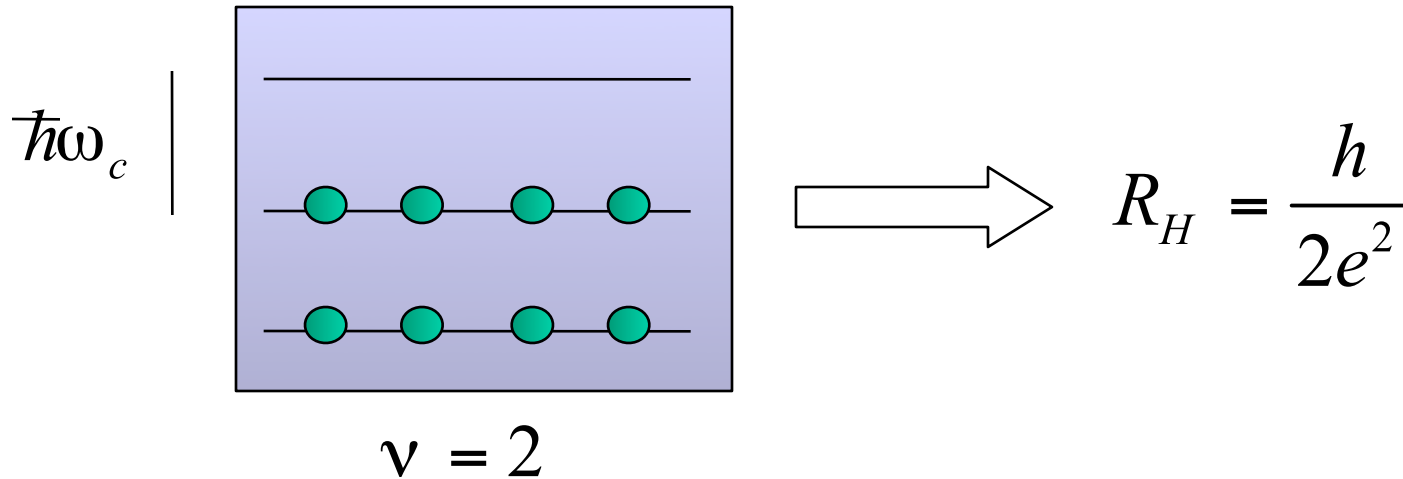
$$\text{Landau level degeneracy per unit area} = \frac{B}{\phi_0}, \quad \phi_0 = \frac{hc}{e}$$

$$\text{Filling factor} = \nu = \frac{\rho}{\text{degeneracy}} = \frac{\rho\phi_0}{B}$$

# Origin of the IQHE

(Laughlin, 1981)

The origin of the IQHE lies in the fact that the ground state at integral fillings is unique.



A giant “filled shell” atom of electrons.

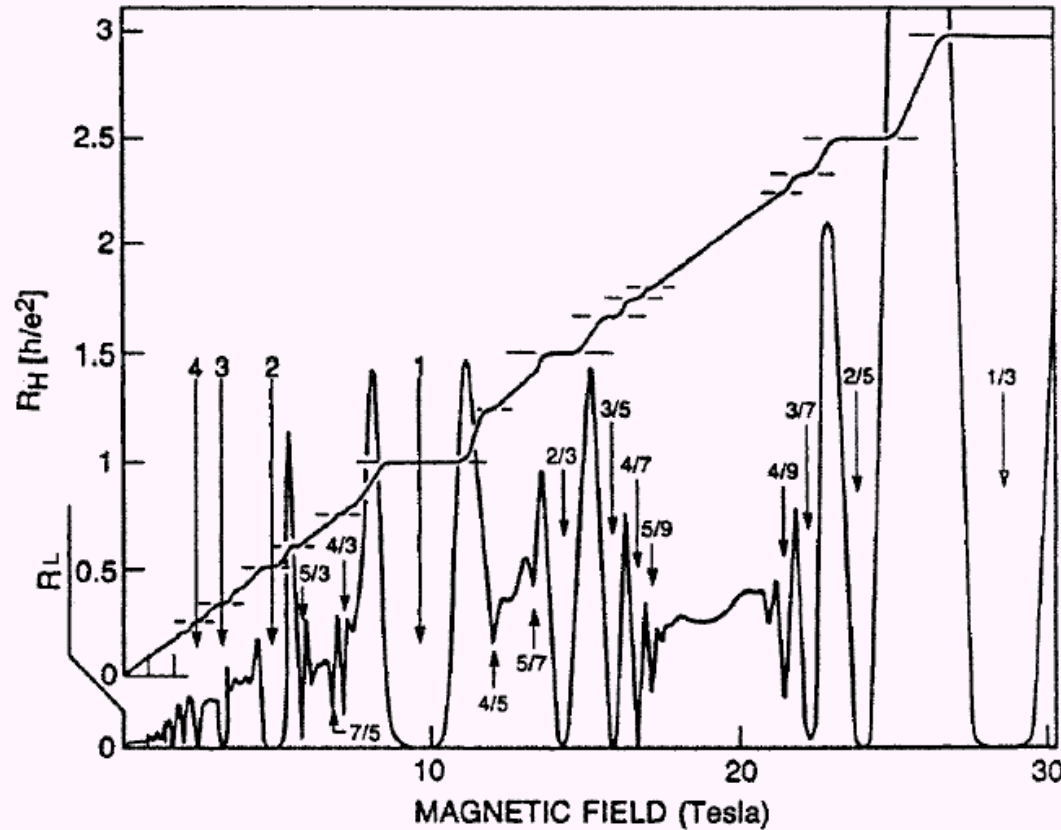
It can be understood in terms of non-interacting electrons.  
(Some disorder is also needed for plateaus.)

## Two-Dimensional Magnetotransport in the Extreme Quantum Limit

D. C. Tsui,<sup>(a), (b)</sup> H. L. Stormer,<sup>(a)</sup> and A. C. Gossard

*Bell Laboratories, Murray Hill, New Jersey 07974*

(Received 5 March 1982)



$$R_H = \frac{h}{fe^2}$$

**Fractional  
Quantum Hall  
Effect**

Willett et al. 1987

# The Great FQHE Mystery



Nature is telling us that there is something unique about the state at certain fractional fillings. What underlying physical principle causes this uniqueness? What is the nature of correlations in this state? How can such a complicated many body system produce such stunningly simple behavior? Why is the Hall quantization exact?

# Statement of the problem

- Find the solutions for the quantum mechanical problem of interacting electrons in a magnetic field.

$$H\Psi = E\Psi$$

$$H = \frac{1}{2m_b} \sum_j \left( \frac{\hbar}{i} \vec{\nabla}_j + \frac{e}{c} \vec{A}(\vec{r}_j) \right)^2 + \frac{e^2}{\epsilon} \sum_{j<k} \frac{1}{r_{jk}} + g\mu_B \vec{B} \cdot \vec{S}$$

- In the limit of  $B \rightarrow \infty$  the kinetic and Zeeman energies are constant.

$$H = \sum_{j<k} \frac{1}{r_{jk}} \quad (\text{to be solved with the lowest LL restriction})$$

- No exact solution.
- No small parameter. Even approximate strategies are doomed by the absence of a small parameter. Usually the interaction is treated as a perturbation. Here interaction is the only energy in the problem. No perturbation theory is possible.
- A truly strongly correlated state.

# Why the problem is hard

- If we “switch off” the interaction between electrons, all lowest Landau level configurations are degenerate.

$$\begin{aligned} N = 10^9 &\Rightarrow 10^{7 \times 10^8} \text{ states} \\ N = 100 &\Rightarrow 10^{72} \text{ states} \end{aligned} \quad (\nu = 0.4)$$

- The repulsive Coulomb interaction between electrons picks out one of these states as the ground state. Our goal is to find this state.
- On purely theoretical grounds, with no small parameter to guide us, we have no idea where to start. The problem appears hopelessly intractable.

# How to proceed?

## The method of analogy

**As more and more information is gathered, the situation first becomes more confusing, but eventually some patterns begin to be noticed. These patterns suggest an analogy that takes us to a deeper level of understanding.**

**Approach: Identify patterns.  
Look for an analogy.**

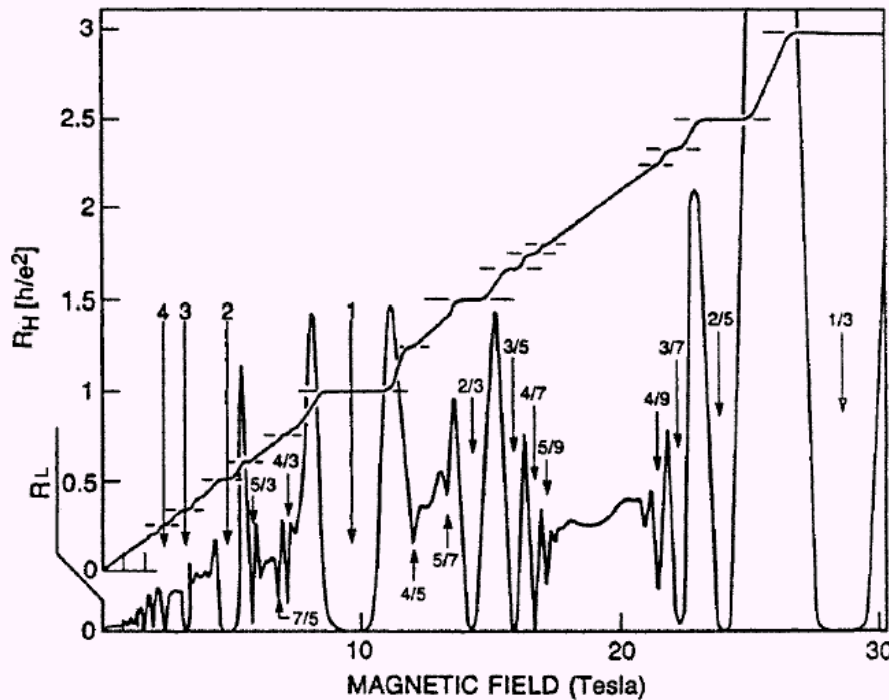
# Patterns

- **No even denominator fractions are seen. For example, the simplest fraction,  $\frac{1}{2}$ , is not seen.**
- **The fractions belong to certain sequences.**

$$v = \frac{n}{2n+1} = \frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \dots, \frac{10}{21}$$



# THE KEY ANALOGY



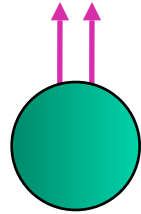
The fractions are qualitatively indistinguishable from the integers!!

- **Given the strong similarity between the IQHE and the FQHE, it is tempting to postulate that the FQHE is some kind of IQHE in disguise.**
- **But of what is it the IQHE? What are to FQHE as electrons are to the IQHE?**
- **It is natural to expect that these objects are some kind of fermions.**
- **This line of reasoning points inescapably to the existence of a new kind of fermions in the FQHE.**
- **Bound state of an odd number of electrons? No.**

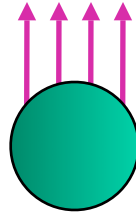
- **The fermions we are looking for are a new class of fermions, called “composite fermions.”**
- **Composite fermions are extremely complicated objects, with each composite fermion being the bound state of all electrons.**
- **Through a Chern-Simons mean field theory, composite fermions unify the fractional and the integral quantum Hall effects.**

# Composite fermions

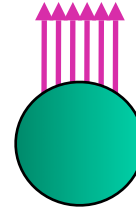
Jain, 1989



${}^2\text{CF}$



${}^4\text{CF}$



${}^6\text{CF}$

**Composite fermion = electron +  $2p$  quantized vortices**

A vortex is a topological object: the geometrical Berry phase for a closed loop around it is quantized, independent of the details. The composite fermion is therefore also a topological entity.

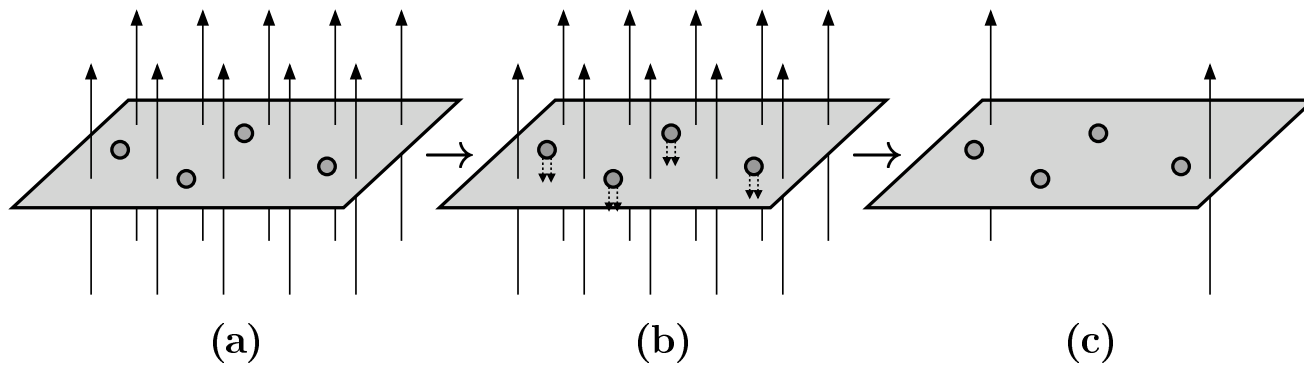
A vortex is often represented as a flux quantum. A composite fermion is sometimes thought of (somewhat inaccurately) as:

**Composite fermion = electron +  $2p$  flux quanta**

**In general: electron + flux = anyon (Leinaas & Myrheim; Wilczek)**

# Relating the FQHE to the IQHE

## The basic idea



$$B^* = B - 2p\rho\phi_0 \quad (\phi_0 = hc/e)$$

$$\nu = \frac{\nu^*}{2p\nu^* \pm 1}$$

# Chern Simons singular gauge transformation

$$H\Psi = E\Psi$$

$$H = \sum_j \frac{1}{2m_b} \left[ \frac{\hbar}{i} \nabla_j + \frac{e}{c} \mathbf{A}(\mathbf{r}_j) \right]^2 + \frac{e^2}{\epsilon} \sum_{j < k} \frac{1}{|\mathbf{r}_j - \mathbf{r}_k|}$$

$$\Psi = \prod_{j < k} \left( \frac{z_j - z_k}{|z_j - z_k|} \right)^{2p} \Psi' = \exp^{-i \sum_{j < k} \phi_{jk}} \Psi' \quad \phi_{jk} = i \ln \frac{z_j - z_k}{|z_j - z_k|}$$

$$H'\Psi' = E\Psi'$$

$$H' = \left[ \frac{1}{2m_b} \sum_i \left( \vec{p}_i + \frac{e}{c} \vec{A}(\vec{r}_i) - \frac{e}{c} \vec{a}(\vec{r}_i) \right)^2 + V \right]$$

$$\vec{a}(\vec{r}_i) = \frac{2p}{2\pi} \phi_0 \sum_j \vec{\nabla}_i \phi_{ij}$$

$$\vec{b}_i = \vec{\nabla}_i \times \vec{a}(\vec{r}_i) = 2p\phi_0 \sum_l \delta^2(\vec{r}_i - \vec{r}_l)$$

The Chern-Simons transformation attaches  $2p$  point “flux quanta” to each electron to convert it into a composite fermion.

Why make this transformation?

# Mean field approximation

So far, everything is exact. Now we make a mean-field approximation by spreading the point flux on each electron into a uniform magnetic field.

$$\vec{A} - \vec{a} = \vec{A} - \langle \vec{a} \rangle - \delta \vec{a} \equiv \vec{A}^* - \delta \vec{a}$$

$$\vec{\nabla} \times \vec{A}^* = B^* \hat{z}$$

$$H' = \frac{1}{2m_b} \sum_i \left( \vec{p}_i + \frac{e}{c} \vec{A}^*(\vec{r}_i) \right)^2 + V + V' = H'_0 + V + V'$$

The first term has known solutions (non-interacting particles in an effective field). The other two are to be treated perturbatively (assuming that the perturbation theory converges). The “unperturbed” mean-field solution is given by:

$$\Psi'_{MF} = \Phi_n$$
$$\Psi_{MF} = \Phi_n \prod_{j < k} \left( \frac{z_j - z_k}{|z_j - z_k|} \right)^{2p}$$

Because of the absence of a small parameter, it is not possible to obtain quantitative information from the Chern-Simons approach. But it suggests good variational wave functions.

# Functional integral formulation

Zero temperature partition function

$$\mathcal{Z} = \int \mathcal{D}\psi \mathcal{D}\psi^* \mathcal{D}\vec{a} \exp\left(\frac{i}{\hbar} \mathcal{S}\right)$$

$$\mathcal{S} = \int d^2\vec{r} \int dt \mathcal{L}$$

$$\mathcal{L} = \psi^* (i\partial_t - a_0) \psi + \frac{1}{2m_b} \left| \left( -i\hbar \vec{\nabla} + \frac{e}{c} \vec{A} - \frac{e}{c} \vec{a} \right) \psi \right|^2 + \frac{1}{2p\phi_0} a_0 \vec{\nabla} \times \vec{a} + \int d^2\vec{r}' \rho(\vec{r}) V(\vec{r} - \vec{r}') \rho(\vec{r}')$$

The action is written in terms of grassmann variables. The flux attachment is introduced through a Lagrange multiplier, which can be integrated to produce

$$\vec{\nabla} \times \vec{a}(\vec{r}) = 2p\phi_0 \rho(\vec{r}) = 2p\phi_0 \psi^*(\vec{r}) \psi(\vec{r})$$

$$\mathcal{L}_{CS} \sim \epsilon^{\mu\nu\lambda} A_\mu F_{\nu\lambda} = 2\epsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$a_0 \vec{\nabla} \times \vec{a}$  is precisely the CS Lagrangian in the Coulomb gauge.

$$\mathcal{L}_{CS} = \frac{1}{4p\phi_0} \epsilon^{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda = \frac{1}{2p\phi_0} \epsilon^{ij} a_0 \partial_i a_j - \frac{1}{4p\phi_0} \epsilon^{ij} a_i \partial_0 a_j$$



## The quantum mechanical solution

$$\nu = \frac{n}{2pn \pm 1}$$

$$\Psi_{MF} = \Phi_n \prod_{j < k} \left( \frac{z_j - z_k}{|z_j - z_k|} \right)^{2p}$$

No good. It does not have good correlations, and has much amplitude in higher Landau levels.

Try

$$\Psi_\nu = \mathcal{P}_{LLL} \Phi_n \prod_{j < k} (z_j - z_k)^{2p}$$
$$z_j = x_j - iy_j$$

# Microscopic meaning of composite fermion

$$\Psi_v = \prod_{j < k} (z_j - z_k)^{2p} \Phi_n$$

$$v = \frac{n}{2pn \pm 1}$$

$$z_j = x_j - iy_j$$

- **The Jastrow factor binds  $2p$  vortices to each electron.**
- **The bound state of an electron and  $2p$  vortices is interpreted as a particle, called the composite fermion.**

**(It is often useful to envision a composite fermion as the bound states of an electron and magnetic flux quanta. But that is only an intuitive picture. In reality, the magnetic field is uniform.)**

- **The Berry phase due to vortices partly cancels the AB phase to produce an effective magnetic field.**

$$\Phi^* = -2\pi \frac{BA}{\phi_0} + 2\pi 2p N_{\text{enc}} \equiv -2\pi \frac{B^* A}{\phi_0}$$

# The composite fermion theory in a nutshell

- Strongly interacting electrons transform into (approximately) free composite fermions by capturing quantized vortices.
- Composite fermions experience a reduced effective magnetic field.
- The solutions for the interacting problem at  $B$  have a one-to-one correspondence with the known solutions of the non-interacting problem at  $B^*$ .

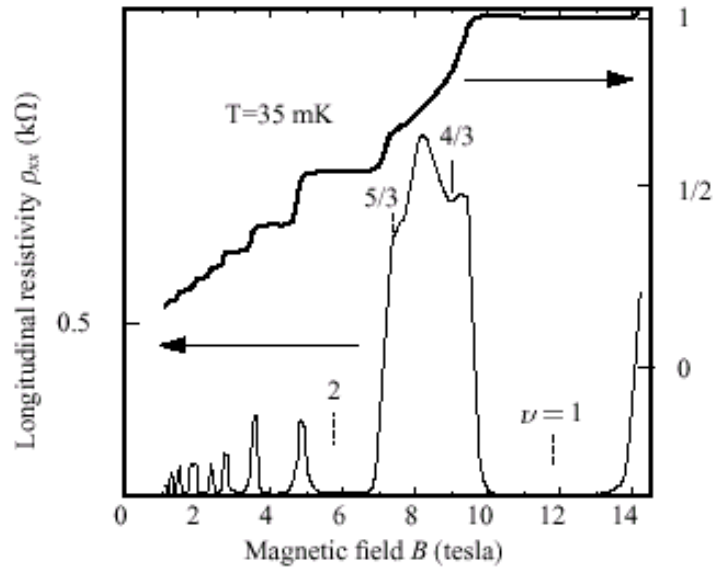
$$B^* = B - 2p\rho\phi_0 \quad \left( \phi_0 = \frac{hc}{e} \right)$$

$$\nu = \frac{\nu^*}{2p\nu^* \pm 1}$$

$$\Psi_\nu = \mathcal{P}_{LLL} \Phi_{\nu^*} \prod_{j < k} (z_j - z_k)^{2p}$$

Why do we believe in  
composite fermions?

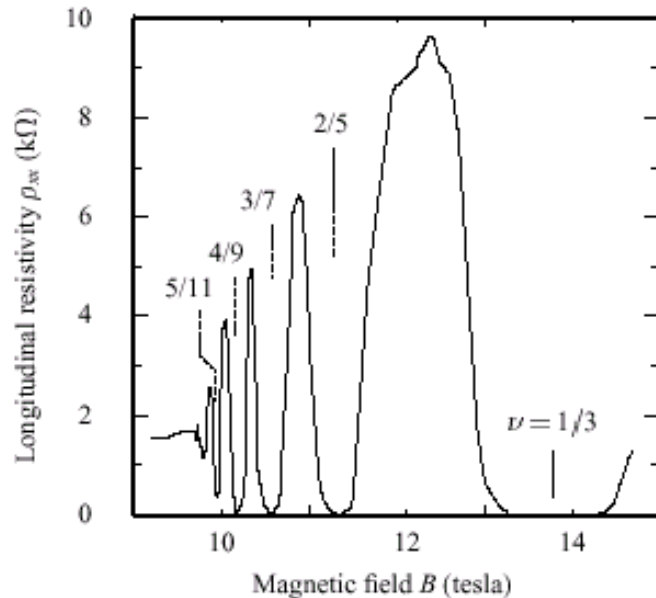
# Analogy between the IQHE and the FQHE



Hall resistance  $\rho_{xy}$  ( $h/e^2$ )

← IQHE

The dynamics of interacting electrons at  $B$  resembles that of non-interacting fermions at  $B^*$ .



← FQHE

Source: Clark et al.; Tsui and Stormer

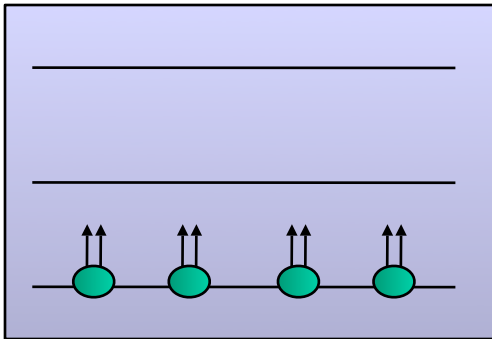
Comparing  $B$  and  $B^*$

# The magic of composite fermions

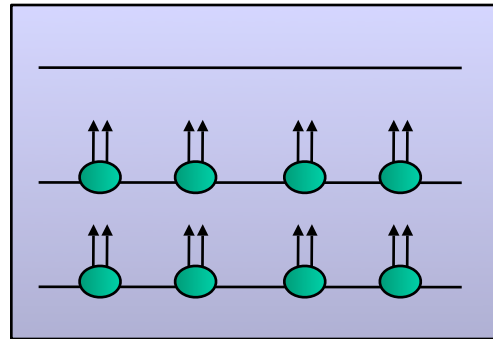
Consider composite fermions at  $\nu^* = n$

Here

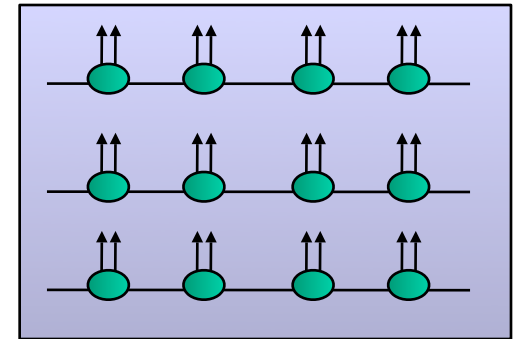
$$\nu = \frac{\nu^*}{2p\nu^* \pm 1} = \frac{n}{2pn \pm 1}$$



$$\frac{1}{3} \Leftrightarrow 1$$



$$\frac{2}{5} \Leftrightarrow 2$$

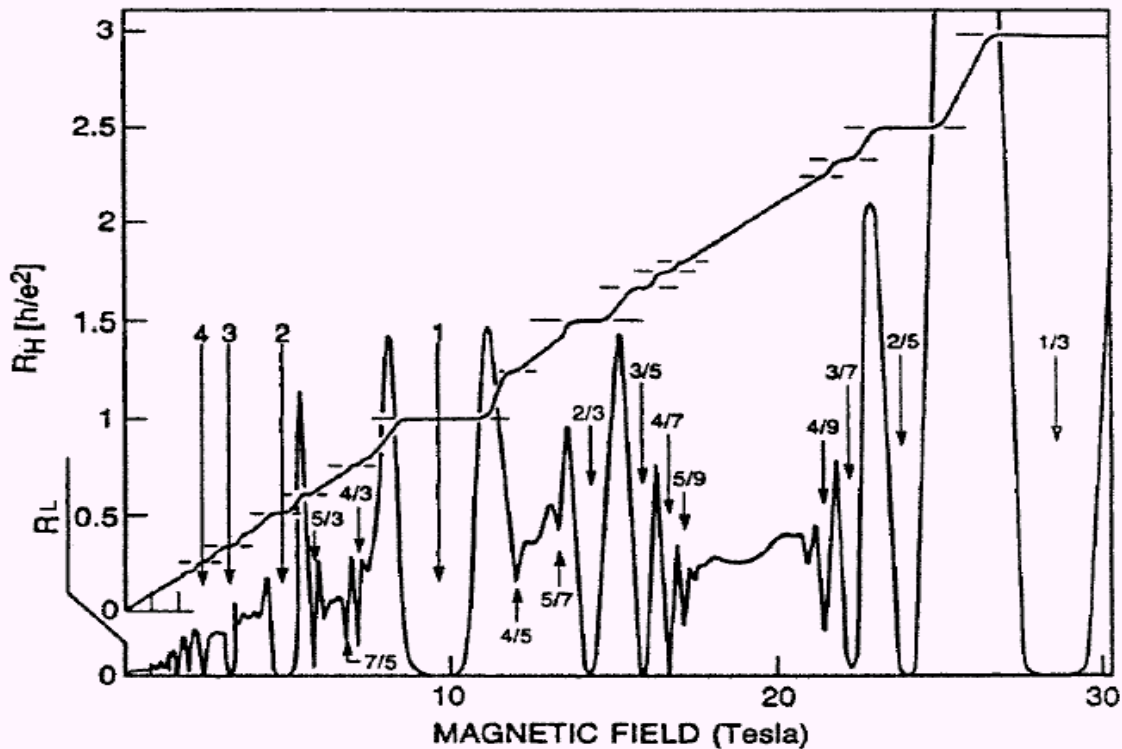


$$\frac{3}{7} \Leftrightarrow 3$$

At these filling factors, when electrons transform into composite fermions, the enormous ambiguity of the electron problem completely disappears to produce a unique state !!

The FQHE is the IQHE of composite fermions. The FQHE states are “filled shell / noble atoms of composite fermions.”

$$\nu = \frac{n}{2pn \pm 1} \quad \text{are precisely the observed fractions!}$$



$$\nu = \frac{n}{2n+1} = \frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \dots, \frac{10}{21}$$

$$\nu = \frac{n}{2n-1} = \frac{2}{3}, \frac{3}{5}, \frac{4}{7}, \dots, \frac{10}{19}$$

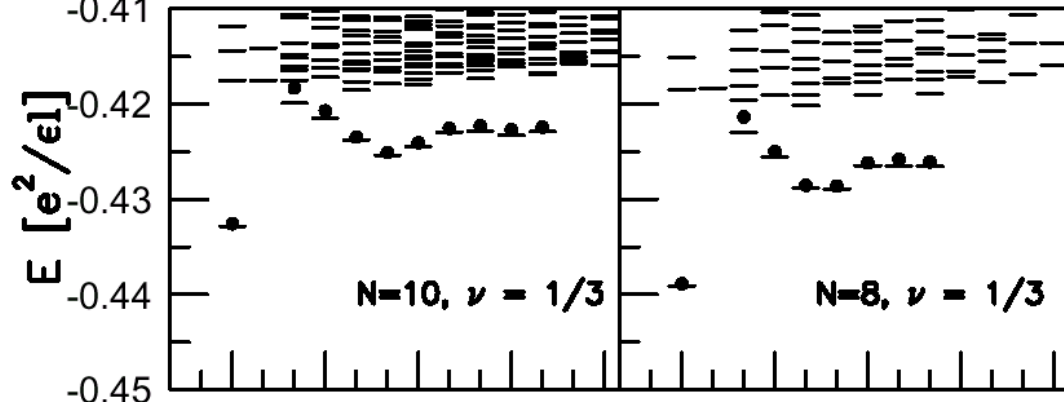
$$\nu = \frac{n}{4n+1} = \frac{1}{5}, \frac{2}{9}, \dots, \frac{6}{25}$$

$$\nu = \frac{n}{4n-1} = \frac{2}{7}, \frac{3}{11}, \dots, \frac{6}{23}$$

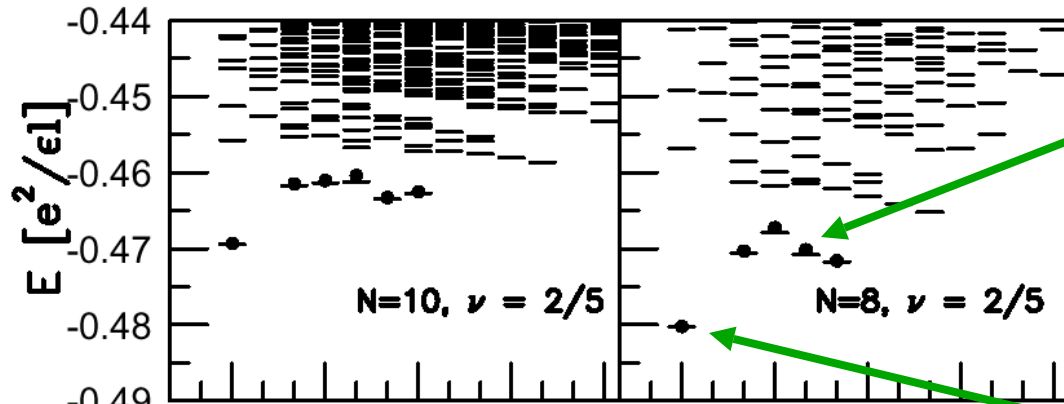
$$\nu' = 1 - \nu$$

- No even denominators appear.
- Sequences are analogous to the integer sequence.
- Robustness of the Hall quantization is also explained. It has a topological origin.

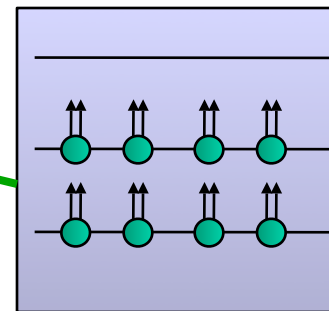
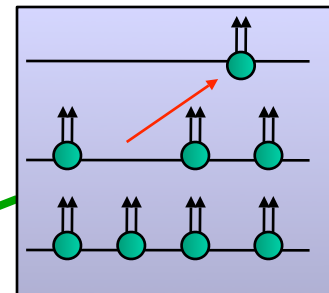
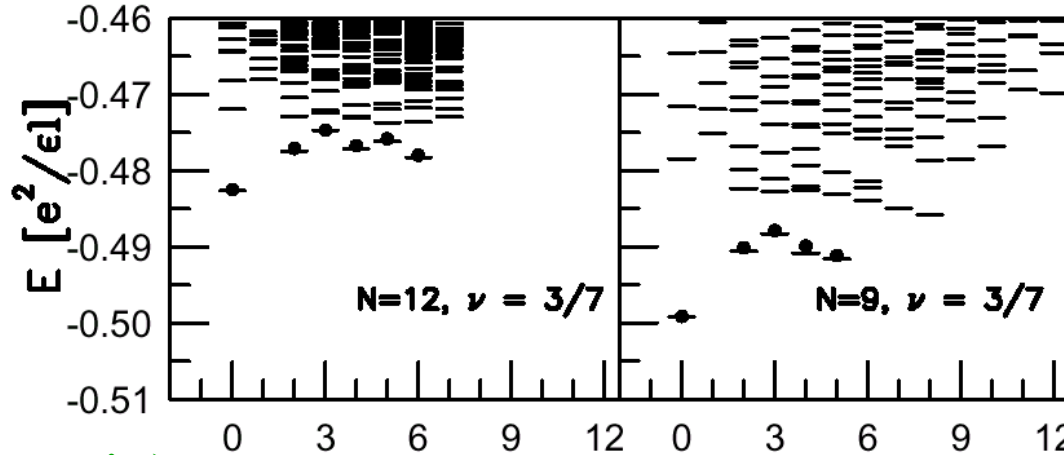
$$\nu = \frac{1}{3}$$



$$\nu = \frac{2}{5}$$



$$\nu = \frac{3}{7}$$



Dev, Kamilla, Jain (CF energies)  
He, Xie, Zhang (exact diagonalization)

L

L



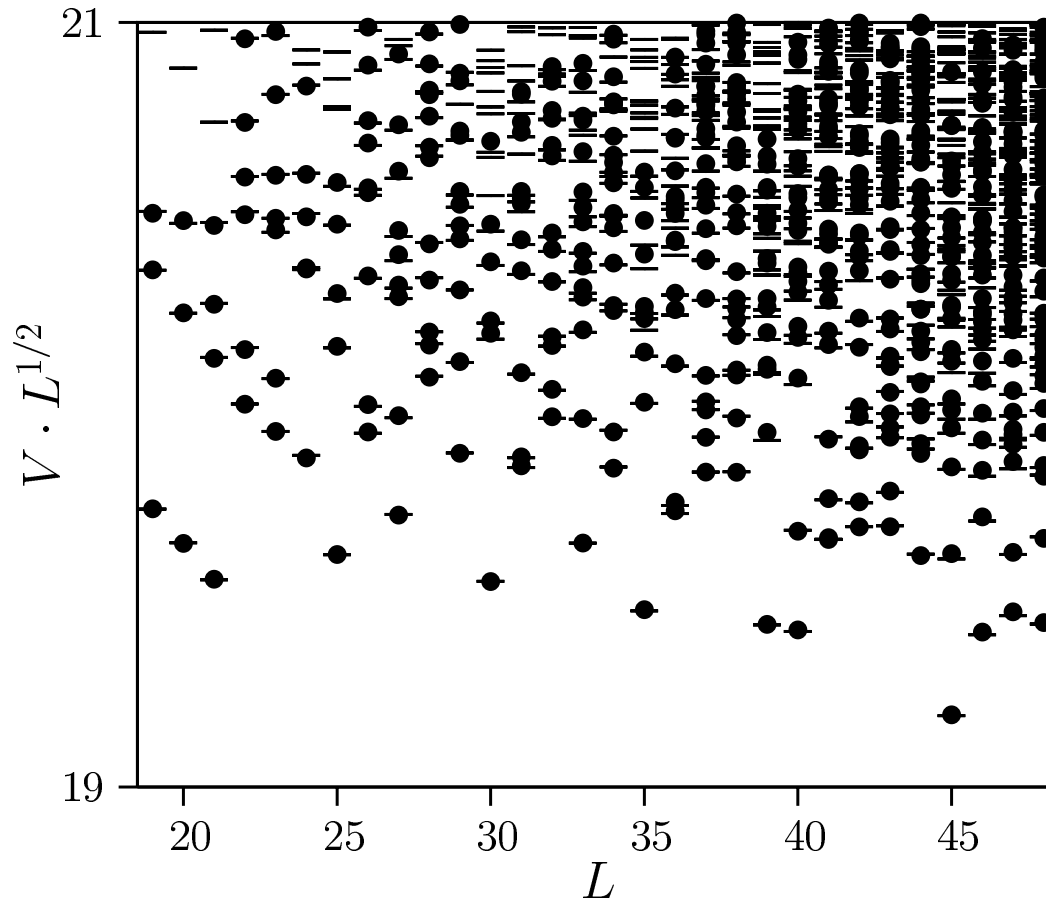
$\nu$	$N$	ground state		excited state ★	
		composite fermion	exact	composite fermion	exact
1/3	8	-0.4389	-0.4391	-0.4261	-0.4266
	10	-0.4326	-0.4328	-0.4224	-0.4229
2/5	8	-0.4802	-0.4802	-0.4714	-0.4717
	10	-0.4693	-0.4694	-0.4625	-0.4627
3/7	9	-0.4991	-0.4992	-0.4915	-0.4916
	12	-0.4825	-0.4826	-0.4782	-0.4783

(★ CF particle and hole at maximum separation)

**0.05% accuracy with no adjustable parameters!**

- The lack of free parameters turns out to be one of the strongest assets!
- Composite fermions not only provide a “physical picture,” but an accurate “microscopic theory.”

# Disk geometry



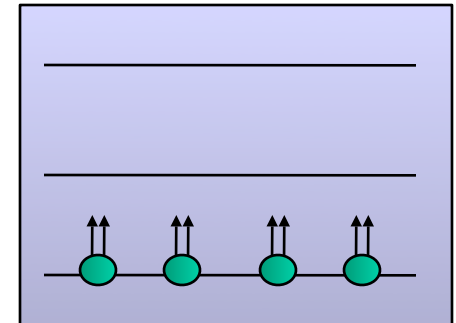
# What about Laughlin's theory of 1/m fractions?

**Laughlin:** 
$$\Psi_{1/3} = \prod_{j < k} (z_j - z_k)^3 \exp\left[-\frac{1}{4} \sum_l |z_l|^2\right]$$

**CF:** 
$$\Psi_{1/3} = \prod_{j < k} (z_j - z_k)^2 \Phi_1$$

The two wave functions are identical. But the CF theory provides a derivation of the wave function from a completely different principle.

- It gives a new physical understanding. The 1/3 state is seen as **one** filled quasi-Landau level of composite fermions carrying **two** vortices.
- It clarifies that this state belongs to a more general structure with an immense amount of other physics in it.
- It also takes us beyond wave functions! The phenomenology can be explained before we write a single wave function.



# This was only the beginning!

- **Composite fermions explain the fractional quantum Hall effect beautifully and convincingly.**
- **Once you take them seriously, you realize that they have numerous other, at-first-sight-bizarre consequences, which you would never believe, or even think of, if you did not know of composite fermions.**
- **A large number of such effects have been verified in experimental and theoretical studies carried out over the last 1.5 decades.**
- **Composite fermions have been directly observed. The existence of composite fermions carrying two, four, and six vortices has been established.**

# THE WORLD OF COMPOSITE FERMIONS



Reviews

“Composite fermions,”  
J.K. Jain, in preparation

"Composite Fermions" ed. O. Heinonen, 1998

"Perspectives in Quantum Hall Effects" ed. S. Das Sarma and A. Pinczuk, 1997

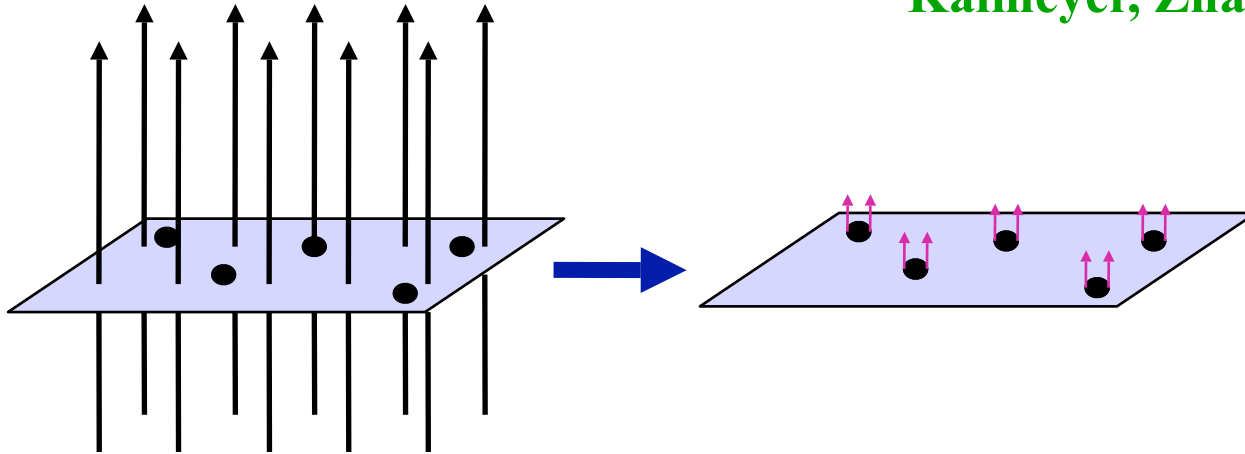
"Composite Fermion: A Quantum Particle and Its Quantum Fluids"

J.K. Jain, Physics Today p. 39, vol. 53(4), 2000.

\*Mostly Theoretical (So Far!)

## CF Fermi sea

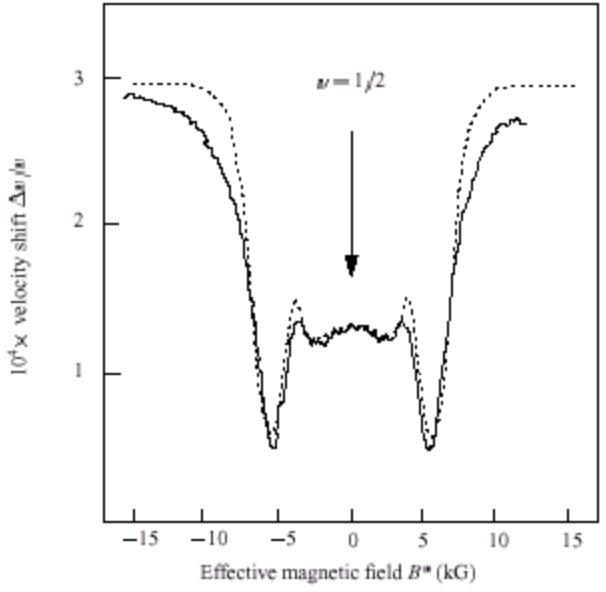
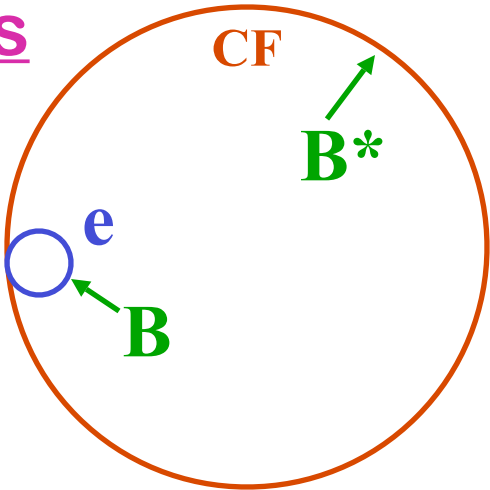
Halperin, Lee, Read, 1993  
Kalmeyer, Zhang, 1992



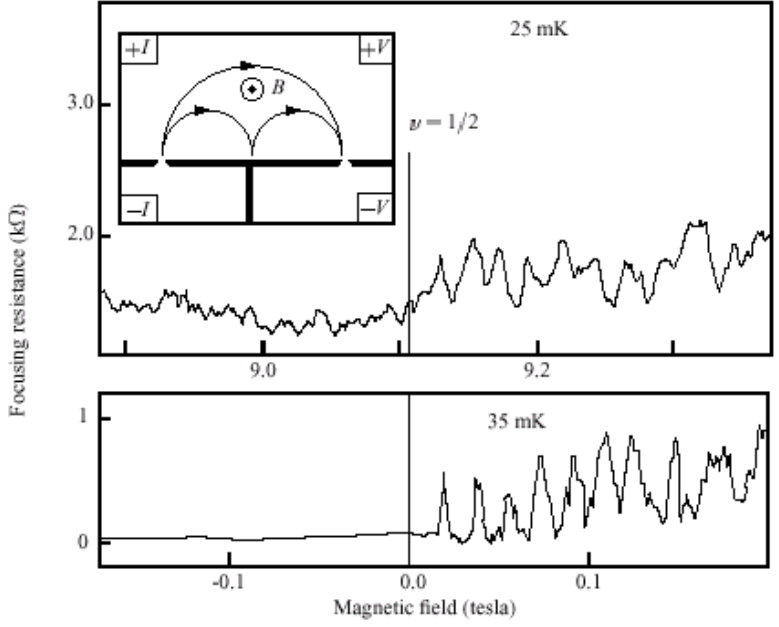
- At the half filled Landau level, composite fermions absorb all magnetic flux, giving  $B^*=0$ .
- Composite fermions form a Fermi sea here.
- No gap  $\Rightarrow$  no FQHE. Non-trivial physics behind the lack of FQHE at even denominator fractions.

# Cyclotron radius

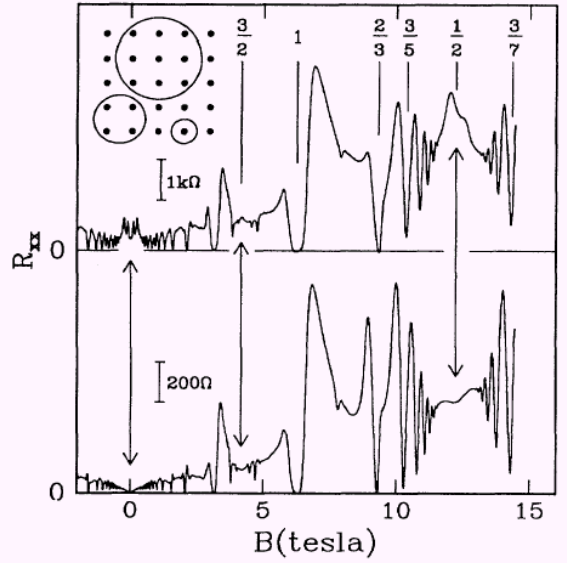
$$R^* = \frac{\hbar k_F}{eB^*}$$



SAW (Willett, 1993)



Magnetic focusing (Goldman, 1994)



Antidot resonances (Kang, 1993)

# An Exotic Particle

- **A new class of fermions:** Previous particles were bound states of elementary particles (quarks and leptons). A composite fermion is the bound state of an electron and quantum mechanical phases (quantized vortices).
- **Collective particle:** Even a single composite fermion is an enormously complicated, collective entity from the vantage point of the electrons, because a quantized vortex, one of its constituents, is made up of all electrons.
- **Quantum mechanical particle:** Ordinary particles are described by quantum mechanics. Composite fermions are produced by quantum mechanics. They would not exist in a purely classical world.
- **Topological particle:** The vortices, a constituent of the composite fermion, is a topological entity, which is responsible for most of the dramatic phenomenology of this quantum fluid.
  - It is astonishing that composite fermions behave as ordinary fermions to such an extent.
  - But it must not be forgotten that the state that appears simple in terms of composite fermions is an extremely complex, strongly correlated state of electrons.



# A new paradigm for collective behavior

## Superconductivity

- Elementary particles  
Cooper pairs
- Basic physics  
Bose condensation of  
Cooper pairs
- Microscopic theory

$$\Psi_{\text{BCS}} = A[\Phi_0(\mathbf{r}_{1,2}) \dots \Phi_0(\mathbf{r}_{N-1,N})]$$

## FQHE

- Elementary particles  
Composite fermions
- Basic physics  
IQHE of composite fermions
- Microscopic theory

$$\Psi_{\nu} = \prod_{j < k} (z_j - z_k)^{2p} \Phi_n$$

The quantum mechanical phases enter into the physics of this quantum fluid in the form of quantized vortices bound to electrons, producing a new class of topological fermions. An entirely new language is required to describe this state.

# An Ideal Theory

**Unification**: Explains all fractions in a unified manner. Unifies the FQHE and the IQHE. The FQHE is simply the IQHE, but for different fermions. The FQHE is only a part of the superstructure described by composite fermions. Everything follows from a single principle!

**Uniqueness**: A detailed microscopic description of a complex many body state with no adjustable parameters.

**Simplicity**: A simple intuitive understanding of the physics.

**Falsifiability**: Makes numerous predictions, both qualitative and quantitative.

**Accuracy**: 1-2% accuracy with no free parameters.

**New physics**: A quantum fluid with topological structure. Not a Bose-Einstein condensate. Requires new concepts and a new language to describe some of the most beautiful structures found in physics.

**Dramatic phenomenology: precise quantizations, effective magnetic field, fractional local charge, fractional winding statistics, Luttinger liquid, non-abelian statistics ...**

**Not an effective theory: Condensed matter theories are often approximate solutions to approximate models, hoping they capture the essential physics. Here, we have an accurate solution of the actual quantum mechanical problem. (cf. quantum chemistry)**

**New particles: A new class of fermions. Previous fermions or bosons were bound states of elementary particles (quarks and leptons); a composite fermion is the bound state of an electron and quantum mechanical phases (quantized vortices).**

**It would be a terrible waste if this beautiful physical structure did not find applications elsewhere in physics.**