Trace formula for the Sturm-Liouville eigenvalue problem with its applications to n-body problem

Xijun Hu
Department of Mathematics
Shandong University, P.R.C.
Based on works
with Yuwei Ou and Penghui Wang

Outline

- Krein-type Trace formula

- Applications to Estimate non-degenerate and Relative Morse index

- Estimate the stability of Lagrangian orbits in planar three body problems

- Hill-type formula

Krein-type Trace formula

- M.G.Krein 1950's, Krein considers the following system

$$\dot{z}(t) = \lambda J D(t) z(t) \quad (1)$$

$$z(0) = -z(T), \quad (2)$$

where $D \geq 0$ and $\int_0^T D(t)dt > 0$, $J = \begin{pmatrix} 0_n & -I_n \\ I_n & 0_n \end{pmatrix}$.

- Let λ_j are the eigenvalues for the system (1-2), and $\begin{pmatrix} A_{11} & A_{12} \\ A_{12} & A_{22} \end{pmatrix} = \frac{1}{T}\int_0^T D(t)dt$

Krein's Trace formula

- Krein

$$\lim_{r\to\infty} \sum_{|\lambda_j|<r} \frac{1}{\lambda_j} = 0, \tag{3}$$

- Krein

$$\sum \frac{1}{\lambda_j^2} = \frac{T^2}{2} Tr(A_{11}A_{22} - A_{12}^2). \tag{4}$$

Krein's Trace formula

- $$y'' + \lambda R(t)y = 0, \;\; y(0) + y(T) = y'(0) + y'(T) = 0, \qquad (5)$$

  where $R(t)$ is continuous path of real symmetric matrices on $\mathbb{R}^n$.

- set $R_{ave} = \frac{1}{T}\int_0^T R(t)dt$ and

  $$X(t) = \int_0^t (R(s) - R_{ave})ds + C, \qquad (6)$$

  where C is a constant matrix which is chosen such that $X_{ave} = 0$.

Krein's Trace formula

- Krein

$$\sum \frac{1}{\lambda_j} = \frac{T}{4}\int_0^T Tr(R(t))dt, \tag{7}$$

- Krein

$$\sum \frac{1}{\lambda_j^2} = \frac{T}{2}\int_0^T Tr(X^2(t))dt + \frac{T^2}{48} Tr[(\int_0^T R(t)dt)^2]. \tag{8}$$

Krein's Stability criteria

- Let $\gamma(t)$ be the fundamental solution.

- The system is called stable if $\sigma(\gamma(T)) \in \mathbb{U}$, hyperbolic if $\sigma(\gamma(T)) \cap \mathbb{U} = \emptyset$

- Krein gives an interesting stability criteria under the condition $D \geq 0$ and $\int_0^T D(t)dt > 0$,

$$\frac{T^2}{2} Tr(A_{11}A_{22} - A_{12}^2) < 1.$$

Notation for Trace formula

- $$\dot{z}(t) = J(B(t) + \lambda D(t))z(t) \quad (9)$$
  $$z(0) = Sz(T). \quad (10)$$

  S is symplectic orthogonal matrix.

- Let $\gamma_0(t)$ be the fundamental solution $\dot{\gamma}_0(t) = JB(t)\gamma_0(t)$ with $\gamma_0(0) = I_{2n}$.

- $M = S\gamma_0(T)$. $\hat{D}(t) = \gamma_0^T(t)D(t)\gamma_0(t)$.

- For $k \in \mathbb{N}$,

$$M_k = \int_0^T J\widehat{D}(t_1) \cdots \int_0^{t_{k-1}} J\widehat{D}(t_k) dt_k \cdots dt_2 dt_1,$$

- $$G_k = (M - e^{\nu T} I_{2n})^{-1} M M_k.$$

Trace formula

- Theorem. Let $\nu \in \mathbb{C}$ satisfying $-J\frac{d}{dt} - B - \nu J$ is invertible. $\forall m$,

$$Tr\Big(\Big[D(-J\frac{d}{dt} - B - \nu J)^{-1}\Big]^m\Big)$$
$$= m\sum_{k=1}^{m} \frac{(-1)^{m+k}}{k}\Big[\sum_{j_1+\cdots+j_k=m} Tr(G_{j_1}\cdots G_{j_k})\Big].$$

- Krein's work on $B = 0, \nu = 0, S = -I_{2n}, D > 0$, $m = 1, 2$, motivation for our stability criteria

- for large m, the right hand side of trace formula is a little complicated, we can compute clear if we needed.

Conditional Trace for $m = 1$

- For $N \in \mathbb{N}$, let

$$W_N = \bigoplus_{\nu \in \sigma(-J\frac{d}{dt}), |\nu| \le N} \ker(-J\frac{d}{dt} - \nu),$$

and denote by P_N the orthogonal projection onto W_N.

- Let

$$Tr(D(-J\frac{d}{dt} - B - \nu J)^{-1})$$
$$= \lim_{N\to\infty} Tr(P_N D(-J\frac{d}{dt} - B - \nu J)^{-1} P_N)$$

Trace formula

- $m = 1$

$$Tr[D(-J\frac{d}{dt} - B - \nu J)^{-1}]$$
$$= Tr\Big(J\int_0^T \gamma_0^T(t)D(t)\gamma_0(t)dt \cdot M(M - e^{\nu T}I_{2n})^{-1}\Big).$$

- $m = 2$

$$\begin{aligned} & Tr([D(-J\tfrac{d}{dt} - B - \nu J)^{-1}]^2) \\ = \quad & 2Tr(J\textstyle\int_0^T \gamma_0^T(t)D(t)\gamma_0(t)J\int_0^s \gamma_0^T(s)D(s)\gamma_0(s)dsdt) \\ & \cdot M(M - \lambda I_{2n})^{-1}) \\ & -Tr([J\textstyle\int_0^T \gamma_0^T(t)D(t)\gamma_0(t)dt \cdot M(M - e^{\nu T}I_{2n})^{-1}]^2). \end{aligned}$$

Trace formula

- If $MJ = JM$, $M^T = M$ then

$$
\begin{aligned}
Tr\Big(\big(D(A-\nu J-B)^{-1}\big)^2\Big) \;=\;& Tr\Big(\Big[J\int_0^T \hat{D}(s)ds\cdot M(M-\omega I_{2n})^{-1}\Big]^2\Big) \\
& -Tr\Big(\Big[J\int_0^T \hat{D}(s)ds\Big]^2 M(M-\omega I_{2n})^{-1}\Big).
\end{aligned}
$$

- In the case that $M = \pm I_{2n}$, set $\omega = e^{\nu T}$, then

$$
Tr\Big((D(-J\frac{d}{dt}-\nu J-B)^{-1})^2\Big)
$$

$$
= \frac{\pm\omega}{(1\mp\omega)^2} Tr((J\int_0^T \gamma_0^T(s)D(s)\gamma_0(s)ds)^2).
$$

Trace formula

- $$Tr((D(-J\frac{d}{dt} - \nu J - B)^{-1})^m) = \sum_{j=1}^{\infty} \frac{1}{\lambda_j^m}, \quad m \geq 2.$$

  λ_j is general eigenvalue of $-J\frac{d}{dt} - \nu J - B - \lambda D$, that is

  $$\ker(-J\frac{d}{dt} - \nu J - B - \lambda_j D) \neq 0$$

- $$\sum_{j=1}^{\infty} \frac{1}{\lambda_j^m} = m \sum_{k=1}^{m} \frac{(-1)^{m+k}}{k} \Big[\sum_{j_1+\cdots+j_k=m} Tr(G_{j_1} \cdots G_{j_k}) \Big], \quad m \geq 2.$$

Trace formula for Sturm-liouville system

- Consider the eigenvalue problems of Sturm-Liouville systems
  $-(P\dot{y}+Qy)^{\cdot}+Q^T\dot{y}+(R+\lambda R_1)y=0, y(T)=\bar{S}y(0), \dot{y}(T)=\bar{S}\dot{y}(0),$
  where $P(t)=P(t)^T$ is invertible, $\bar{S}\in O(n)$, $\bar{S}P(t+T)=P(t)\bar{S}$ and $\bar{S}Q(t+T)=Q(t)\bar{S}$. $S=\begin{pmatrix} \bar{S} & 0_n \\ 0_n & \bar{S} \end{pmatrix}$.

- Corresponds to the linear Hamiltonian system,
  $$\dot{z}=JB_\lambda(t)z, \quad z\in\mathbb{R}^{2n}, \qquad (11)$$
  with
  $$B(t)=\begin{pmatrix} P^{-1}(t) & -P^{-1}Q(t) \\ -Q(t)^T P^{-1}(t) & Q(t)^T P^{-1}(t)Q(t)-R(t)+\lambda R_1(t) \end{pmatrix}.$$

Trace formula for Sturm-liouville system

- Let $\tilde{\gamma}_\lambda(T)$ be the fundamental solution of (11) and $\tilde{M} = S\tilde{\gamma}_0(T)$.

- Theorem. For $m \in \mathbb{N}$

$$\sum_{j=1}^{\infty} \frac{1}{\lambda_j^m} = (-1)^m m \sum_{k=1}^{m} \frac{(-1)^{m+k}}{k} \Big[\sum_{j_1+\cdots+j_k=m} Tr(\tilde{G}_{j_1} \cdots \tilde{G}_{j_k}) \Big], \forall m \in \mathbb{N},$$

especially for $m = 1$,

$$\sum_{j=1}^{\infty} \frac{1}{\lambda_j} = -Tr\Big(\int_0^T \tilde{\gamma}^{-1}(t) D_1(t) \tilde{\gamma}(t) dt \cdot \tilde{M}(\tilde{M} - I_{2n})^{-1} \Big).$$

Applications: Bound of non-degenerate

- For a Hamiltonian system, the property of non-degeneration is invariant under the small perturbation. A natural question will be arisen: can we give an upper bound for the perturbation, such that, under the smaller perturbation, the systems preserve the non-degenerate property?

- Theorem. Suppose $-J\frac{d}{dt} - B - \nu J$ is non-degenerate and for D, D_1, D_2 such that $D_1 \leq D \leq D_2$, and $D_1 < 0$, $D_2 > 0$, if there exists $k \in 2\mathbb{N}$, such that

$$Tr\Big((D_j(-J\frac{d}{dt} - B - \nu J)^{-1})^k\Big) \leq 1$$

for $j = 1, 2$, then $A - B - D - \nu J$ is non-degenerate.

Applications: Estimate the relative Morse index

- Relative Morse index $I(-J\frac{d}{dt} - B - \nu J, -J\frac{d}{dt} - B - D - \nu J)$ is well defined, Maslov-type index

- Theorem. Suppose $-J\frac{d}{dt} - B - \nu J$ is non-degenerate and $D_1 \le D \le D_2$, where $D_1 < 0$, $D_2 > 0$. Let
$$m^- = \inf\{[Tr((D_1(-J\frac{d}{dt} - B - \nu J)^{-1})^k)], k \in 2\mathbb{N}\}$$
$$m^+ = \inf\{[Tr((D_2(-J\frac{d}{dt} - B - \nu J)^{-1})^k)], k \in 2\mathbb{N}\},$$
then
$$-m^- < I(-J\frac{d}{dt} - B - \nu J, -J\frac{d}{dt} - B - D - \nu J) < m^+.$$

Applications: stability criteria

- For $\omega \in \mathbb{U}$ s.t. $\omega = e^{i\theta_0}$ with $\theta_0 \in [0, \pi]$, let $\mathbb{U}_\omega = \{e^{i\theta}, \theta \in [-\theta_0, \theta_0]\}$, denote $e_\omega(M)$ by the total algebraic multiplicities of all eigenvalues of M on $\mathbb{U}_\omega$.

- Denote by $e(M)$ the total algebraic multiplicities of all eigenvalues of M on $\mathbb{U}$. Obviously

$$e(M) \geq e_\omega(M), \quad \forall \omega \in \mathbb{U}.$$

- For simplification, we consider the case $S = I_{2n}$, and the general case is similar.

- We denote $\tilde{\gamma}$ be the fundamental solution with respect to $B+D$, and $\widetilde{M}=\tilde{\gamma}(T)$.

- The next theorem is motivated by Krein's work.

  Theorem. *Suppose* $M=I_{2n}$, $\frac{\omega}{(1-\omega)^2}Tr((J\int_0^T \hat{D}(s)ds)^2)\leq 1$ *and* $D>0$. *Then*

$$e_\omega(\widetilde{M})/2=n. \tag{12}$$

- Generation of Krein's stability Criteria

- The proof is based on the Maslov-type index theory (Y.Long's book) and the Trace formula.

Identity related to the zeta function

- For a self-adjoint and elliptic operators $\mathcal{A}$, Atiyah, Patodi, Singer defined the zeta function by

$$\eta_{\mathcal{A}}(s) = \sum_{\lambda \neq 0} (sign \lambda)|\lambda|^{-s}$$

for $Re(s)$ large, where $\lambda \in \sigma(\mathcal{A})$. It can be extends meromorphically to the whole s-plane. In our paper, let $\mathcal{A} = -J\frac{d}{dt} - B - \nu J$, then $\eta_{\mathcal{A}}(s)$ is well defined. Our trace formula could compute $\eta_{\mathcal{A}}(2k+1)$ for $k \in \mathbb{N}$.

- The most simple case $B = 0$

$$\sum_{k \in \mathbb{Z}} \frac{1}{(2k\pi + \nu)^2} = \frac{1 + \cos\nu}{2\sin^2\nu}.$$

Applications to Lagrangian orbits in planar three body problem

- (1772 Lagrange) three bodies form an equilateral triangle, each body travels along a specific Keplerian orbit

- Sun-Jupiter-Trojan asteroids system

- The stability had studied by many authors: Gascheau, Routh, Danby, Roberts, Meyer, Schmidt, Martínez, Samà,

- linear stability depend on $\beta = \frac{27(m_1m_2+m_1m_3+m_2m_3)}{(m_1+m_2+m_3)^2}$ and eccentricity $e \in [0,1)$

Stability analysis for Lagrangian orbits

- Linear stable if $\beta < 1$, eccentricity $e = 0$. Numerical for general

- Recently, Long, Sun and Hu introduced Maslov-type index and operator theory in studying the stability in n-body problem, give analytic proof for stability bifurcation diagram of Lagrangian orbits.

- not estimate the stability and hyperbolic region

Stability analysis for Lagrangian orbits via Trace formula

- We estimate the stability region and hyperbolic region.

- $\beta > \beta_1 = 9 - (\frac{27-12\sqrt{2}}{7})^2 \approx 6.9472$ is hyperbolic

- The elliptic relative equilibrium of square central configurations is hyperbolic with any eccentricity.

- The Trace formula is from Hill-type formula.

History of Hill-type formula

- In study of the motion of lunar perigee, Hill (1877) consider

$$\ddot{x}(t) + \theta(t)x(t) = 0, \tag{13}$$

where $\theta(t) = \sum_{j\in\mathbb{Z}} \theta_j e^{2j\sqrt{-1}t}$ with $\theta_0 \neq 0$ is a real π-periodic function.

- Let $\gamma(t)$ be the fundamental solution, that is,

$$\begin{aligned} \dot{\gamma}(t) &= \begin{pmatrix} 0 & -\theta(t) \\ 1 & 0 \end{pmatrix} \gamma(t), \\ \gamma(0) &= I_2. \end{aligned}$$

Hill's formula

- Suppose $\rho = e^{c\sqrt{-1}\pi}$, $\rho^{-1} = e^{-c\sqrt{-1}\pi}$ are the eigenvalues of the monodromy matrix $\gamma(\pi)$

- In order to compute c, Hill obtain

$$\frac{\sin^2(\frac{\pi}{2}c)}{\sin^2(\frac{\pi}{2}\theta_0)} = \det((-\frac{d^2}{dt^2} - \theta_0)^{-1}(-\frac{d^2}{dt^2} - \theta)), \tag{14}$$

where the right hand side of (14) is the Fredholm determinant.

Hill-type formula

- The right hand side of the original formula of Hill is a determinant of an infinite matrix. Hill did not prove the convergence of the infinite determinant, and the convergence was proved by Poincaré.

- Hill formula for a periodic solution of an arbitrary Lagrangian system on manifold was given by Bolotin 1988

- Periodic orbits for ODE by Denk 1998

Hill-type formula for Hamiltonian systems

- consider the operator

$$(-J\frac{d}{dt} - B)(-J\frac{d}{dt} + P_0)^{-1} = id - (B + P_0)(-J\frac{d}{dt} + P_0)^{-1},$$

  where P_0 is the orthogonal projection onto $\ker -J\frac{d}{dt}$

- Let

$$\det((-J\frac{d}{dt} - B)(-J\frac{d}{dt} + P_0)^{-1})$$
$$= \lim_{N\to\infty} \det(P_N(-J\frac{d}{dt} - B)P_N(-J\frac{d}{dt} + P_0)^{-1}P_N)$$

Hill-type formula for Hamiltonian systems

- HU and Wang

$$\det((-J\frac{d}{dt} - B - \nu J)(-J\frac{d}{dt} + P_0)^{-1})$$
$$= C(S)e^{-\frac{1}{2}\int_0^T Tr(JB(t))dt}\lambda^{-n}\det(S\gamma(T) - \lambda I_{2n}),$$

where $\lambda = e^{\nu T}$ and $C(S) > 0$ is constant depending on S.

- For periodic boundary condition, $C(I_{2n}) = \frac{1}{T^{2n}}$.

$$\det((-J\frac{d}{dt} - B)(-J\frac{d}{dt} + P_0)^{-1}) = \frac{1}{T^{2n}}\det(\gamma(T) - I_{2n}).$$

Hill-type formula for indefinite Lagrangian systems

- consider the eigenvalue problem of the Sturm-Liouville systems:

$$-(P\dot{y}+Qy)^{\cdot}+Q^T\dot{y}+(R+\lambda R_1)y=0. \qquad (15)$$

$P(t)$ is invertible, $\bar{S}$ boundary condition

- Trace formula

$$\prod_{\lambda_j}(1-\frac{1}{\lambda_j})=\det(S\gamma_1(T)-I_{2n})\cdot\det(S\gamma_0(T)-I_{2n})^{-1},$$

Where $\tilde{\gamma}_\lambda(T)$ be the fundamental solution of (11).

Applications of Hill-type formula

- numerical computation of monodromy matrix, Hill, Denk
- Relation with relative Morse index, Maslov-type index theory
- **We get Trace formula based on it**

Reference

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Thank you