

NORMAL CYCLIC POLYTOPES AND NON-VERY AMPLE CYCLIC POLYTOPES

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ABSTRACT. Let d and n be positive integers with $n \geq d + 1$ and τ_1, \dots, τ_n integers with $\tau_1 < \dots < \tau_n$. Let $C_d(\tau_1, \dots, \tau_n) \subset \mathbb{R}^d$ denote the cyclic polytope of dimension d with n vertices $(\tau_1, \tau_1^2, \dots, \tau_1^d), \dots, (\tau_n, \tau_n^2, \dots, \tau_n^d)$. We are interested in finding the smallest integer γ_d such that if $\tau_{i+1} - \tau_i \geq \gamma_d$ for $1 \leq i < n$, then $C_d(\tau_1, \dots, \tau_n)$ is normal. One of the known results is $\gamma_d \leq d(d+1)$. In the present paper a new inequality $\gamma_d \leq d^2 - 1$ is proved. Moreover, it is shown that if $d \geq 4$ with $\tau_3 - \tau_2 = 1$, then $C_d(\tau_1, \dots, \tau_n)$ is non-very ample.

INTRODUCTION

The cyclic polytope is one of the most distinguished polytopes and played the essential role in the classical theory of convex polytopes ([3]). Let d and n be positive integers with $n \geq d + 1$ and τ_1, \dots, τ_n real numbers with $\tau_1 < \dots < \tau_n$. The convex polytope $C_d(\tau_1, \dots, \tau_n)$ which is the convex hull of the finite set

$$\{(\tau_1, \tau_1^2, \dots, \tau_1^d), \dots, (\tau_n, \tau_n^2, \dots, \tau_n^d)\} \subset \mathbb{R}^d$$

is called a *cyclic polytope*. It is known that $C_d(\tau_1, \dots, \tau_n)$ is a simplicial polytope of dimension d with n vertices. The combinatorial type of $C_d(\tau_1, \dots, \tau_n)$ is independent of the particular choice of real numbers τ_1, \dots, τ_n .

The present paper is devoted to the study on *integral* cyclic polytopes. A convex polytope is called integral if all of its vertices have integer coordinate. The integral convex polytope has established an active area lying between combinatorics and commutative algebra ([5, 10]).

Let, in general, $\mathcal{P} \subset \mathbb{R}^N$ be an integral convex polytope, define $\mathcal{P}^* \subset \mathbb{R}^{N+1}$ to be the convex hull of all points $(1, \alpha) \in \mathbb{R}^{N+1}$ with $\alpha \in \mathcal{P}$ and let $\mathcal{A}_{\mathcal{P}} = \mathcal{P}^* \cap \mathbb{Z}^{N+1}$ denote the set of integer points in \mathcal{P}^* . Let $\mathbb{Z}_{\geq 0}$ denote the set of nonnegative integers and $\mathbb{Q}_{\geq 0}$ the set of nonnegative rational numbers.

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We say that \mathcal{P} is *normal* if one has

$$\mathbb{Z}_{\geq 0}\mathcal{A}_{\mathcal{P}} = \mathbb{Z}\mathcal{A}_{\mathcal{P}} \cap \mathbb{Q}_{\geq 0}\mathcal{A}_{\mathcal{P}}.$$

Moreover, \mathcal{P} is called *very ample* if the set

$$(\mathbb{Z}\mathcal{A}_{\mathcal{P}} \cap \mathbb{Q}_{\geq 0}\mathcal{A}_{\mathcal{P}}) \setminus \mathbb{Z}_{\geq 0}\mathcal{A}_{\mathcal{P}}$$

is finite. One of the most fundamental questions on integral convex polytopes is to determine if given an integral convex polytope is normal ([7]).

On the other hand, we say that an integral convex polytope $\mathcal{P} \subset \mathbb{R}^N$ has the *integer decomposition property* if, for each $m = 1, 2, \dots$ and for each $\alpha \in m\mathcal{P} \cap \mathbb{Z}^N$, there exist $\alpha_1, \dots, \alpha_m$ belonging to $\mathcal{P} \cap \mathbb{Z}^N$ such that $\alpha = \alpha_1 + \dots + \alpha_m$. Here $m\mathcal{P} = \{m\alpha : \alpha \in \mathcal{P}\}$. If \mathcal{P} has the integer decomposition property, then \mathcal{P} is normal. However, the converse is false. For example, the tetrahedron $\mathcal{T}_3 \subset \mathbb{R}^3$ with the vertices $(0, 0, 0)$, $(1, 1, 0)$, $(1, 0, 1)$ and $(0, 1, 1)$ is normal, but cannot have the integer decomposition property because $(1, 1, 1) \in 2\mathcal{T}_3$. If $\mathcal{P} \subset \mathbb{R}^d$ is an integral convex polytope of dimension d with $\mathbb{Z}(\mathcal{P}^* \cap \mathbb{Z}^{d+1}) = \mathbb{Z}^{d+1}$, then \mathcal{P} has the integer decomposition property if and only if \mathcal{P} is normal. Lemma 1.6 says that every integral cyclic polytope $\mathcal{P} \subset \mathbb{R}^d$ satisfies $\mathbb{Z}(\mathcal{P}^* \cap \mathbb{Z}^{d+1}) = \mathbb{Z}^{d+1}$. In particular it follows that an integral cyclic polytope is normal if and only if it has the integer decomposition property.

Let, as before, d and n be positive integers with $n \geq d + 1$. Given integers τ_1, \dots, τ_n with $\tau_1 < \dots < \tau_n$, we wish to examine whether $C_d(\tau_1, \dots, \tau_n)$ is normal or not. Thus our final goal is to classify the integers τ_1, \dots, τ_n with $\tau_1 < \dots < \tau_n$ for which $C_d(\tau_1, \dots, \tau_n)$ is normal. Even though to find a complete classification seems to be rather difficult, many fascinating problems arise in the natural way. As a first step toward our goal, we are interested in finding the smallest integer γ_d such that if $\tau_{i+1} - \tau_i \geq \gamma_d$ for $1 \leq i < n$, then $C_d(\tau_1, \dots, \tau_n)$ is normal. It follows immediately from [4, Theorem 1.3 (b)] that one has $\gamma_d \leq d(d + 1)$. In the present paper a new inequality $\gamma_d \leq d^2 - 1$ is proved (Theorem 2.1). Moreover, it is shown that if $d \geq 4$ with $\tau_3 - \tau_2 = 1$, then $C_d(\tau_1, \dots, \tau_n)$ is non-very ample (Theorem 3.1).

A brief overview of the present paper is as follows. After preparing notation, terminologies together with several lemmata in Section 1, a proof of Theorem 2.1 is achieved in Section 2. Moreover, Section 3 is devoted to showing Theorem 3.1.

Finally the study on an algebraic aspect of integral cyclic polytopes including toric rings of integral cyclic polytopes will be done in the forthcoming paper [6].

1. PRELIMINARIES

In this section, we prepare notation and lemmata for our main theorem.

First of all, we will review some fundamental facts on cyclic polytopes. Let d and n be positive integers with $n \geq d + 1$. It is convenient to work with a homogeneous version of the cyclic polytopes, hence, throughout the present paper, we consider

$C_d^*(\tau_1, \dots, \tau_n)$ instead of $C_d(\tau_1, \dots, \tau_n)$. For n real numbers τ_1, \dots, τ_n with $\tau_1 < \dots < \tau_n$, we set

$$v_i := (1, \tau_i, \tau_i^2, \dots, \tau_i^d) \in \mathbb{R}^{d+1} \text{ for } 1 \leq i \leq n.$$

In other words, $C_d^*(\tau_1, \dots, \tau_n) = \text{conv}(\{v_i : 1 \leq i \leq n\}) \subset \mathbb{R}^{d+1}$. Unless stated otherwise, we will always assume the indices are ordered like $\tau_1 < \dots < \tau_n$. See [11, Chapter 0] for some basic properties of cyclic polytopes. We will use a well-known characterization of their facets. (See, e.g., [11, Theorem 0.7]).

Let $[n] := \{1, \dots, n\}$ and let us say that a set $S \subset [n]$ forms a facet of $C_d^*(\tau_1, \dots, \tau_n)$ if $\text{conv}(\{v_i : i \in S\})$ is its facet.

Proposition 1.1 (Gale's evenness condition). *A set $S \subset [n]$ with d elements forms a facet of $C_d^*(\tau_1, \dots, \tau_n)$ if and only if S satisfies the following condition: If i and j with $i < j$ are not in S , then the number of elements of S between i and j is even. In other words,*

$$2 \mid \#\{k \in S \mid i < k < j\},$$

where $\#X$ stands for the number of elements contained in a finite set X .

Hereafter, we will assume that τ_1, \dots, τ_n are integers.

Let $\Delta_{ij} := \tau_j - \tau_i$ for $i, j \in [n]$. The proof of Proposition 1.1 yields a description of the inequality of the supporting hyperplane defining each facet. Let $S = \{k_1, \dots, k_d\} \subset [n]$ and consider the polynomial

$$\sum_{i=0}^d c_{S,i} t^i := \prod_{i \in S} (t - \tau_i).$$

Then all d vectors v_{k_1}, \dots, v_{k_d} vanish by the linear form

$$\sigma_S : \mathbb{R}^{d+1} \ni (w_0, w_1, \dots, w_d) \mapsto \sum_{i=0}^d c_{S,i} w_i \in \mathbb{R},$$

thus it defines the hyperplane spanned by them. Note that we index the first coordinate by 0. Hence, if the set S forms a facet \mathcal{F} of $\mathcal{P}^* = C_d^*(\tau_1, \dots, \tau_n)$, then σ_S is the linear form defining \mathcal{F} , which means that $\sigma_S(x) \geq 0$ if x is in \mathcal{P}^* and $\sigma_S(x) = 0$ if x is in \mathcal{F} . For every $j \in [n] \setminus S$, it holds $\sigma_S(v_j) = \prod_{i \in S} \Delta_{ij}$. This has a useful implication, that is, if we write a vector $x \in \mathbb{Z}^{d+1}$ as $x = \sum_{i \in S} \lambda_i v_i + \lambda_j v_j$ with rational coefficients λ_i , then the denominator of λ_j is a divisor of $\prod_{i \in S} \Delta_{ij}$, because $\sigma_S(x) = \lambda_j \prod_{i \in S} \Delta_{ij}$ is an integer.

We introduce a special representation of cyclic polytopes which is sometimes helpful. Write the vectors v_1, \dots, v_n as row vectors into a matrix, namely,

$$(1) \quad \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} 1 & \tau_1 & \tau_1^2 & \dots & \tau_1^d \\ 1 & \tau_2 & \tau_2^2 & \dots & \tau_2^d \\ \vdots & \vdots & & & \vdots \\ 1 & \tau_n & \tau_n^2 & \dots & \tau_n^d \end{pmatrix}.$$

Lemma 1.2. *The aforementioned matrix can be transformed to the following matrix by using a unimodular transformation:*

$$(2) \quad \begin{pmatrix} 1 & 0 & \dots & \dots & 0 \\ 1 & \Delta_{12} & 0 & \ddots & \vdots \\ 1 & \Delta_{13} & \Delta_{13}\Delta_{23} & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 1 & \Delta_{1,d+1} & \Delta_{1,d+1}\Delta_{2,d+1} & \dots & \prod_{k=1}^d \Delta_{k,d+1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \Delta_{1,n} & \Delta_{1,n}\Delta_{2,n} & \dots & \prod_{k=1}^d \Delta_{k,n} \end{pmatrix}.$$

In particular, the convex hull of the row vectors of this matrix is unimodularly equivalent to $C_d^*(\tau_1, \dots, \tau_n)$.

A proof of the above lemma is essentially the same as a proof of the well-known Vandermonde determinant. Note that Lemma 1.2 is valid for any ordering of the parameters τ_1, \dots, τ_n , i.e., any ordering of v_1, \dots, v_n .

Let us identify a special case where the polytopes are indeed unimodularly equivalent.

Lemma 1.3. *An integral cyclic polytope $C_d^*(\tau_1, \dots, \tau_d)$ is unimodularly equivalent to $C_d^*(-\tau_n, \dots, -\tau_1)$. Moreover, for any integer m , $C_d^*(\tau_1, \dots, \tau_d)$ is unimodularly equivalent to $C_d^*(\tau_1 + m, \dots, \tau_n + m)$.*

Proof. The replacement $\tau_i \mapsto -\tau_i$ corresponds to a multiplication with -1 in every column of (1) with an odd exponent. This is a unimodular transformation. The second statement is immediate from Lemma 1.2, because the matrix (2) depends only on the differences $\Delta_{ij} = \tau_j - \tau_i$. \square

We define a certain class of vectors which we will use in the sequel. Let $S = \{i_1, \dots, i_q\} \subset [n]$ be a non-empty set, where $i_1 < \dots < i_q$. Then we define

$$b_S := \sum_{i \in S} \frac{1}{\prod_{j \in S \setminus \{i\}} \Delta_{ij}} v_i = \sum_{k=1}^q \frac{(-1)^{k+1}}{\prod_{j \in S \setminus \{i_k\}} |\Delta_{i_k j}|} v_{i_k},$$

where $b_S = v_{i_1}$ when $q = 1$, i.e., $\#S = 1$. If S is small, we will sometimes omit the brackets around the elements, thus we write, for example, $b_{ij} = b_{\{i,j\}}$. However, the vector does not depend on the order of the indices.

Example 1.4. Let us write down b_S 's for small sets S . Assume $1 \leq i < j < k < l \leq n$. Then

$$\begin{aligned} b_i &= v_i, \\ b_{ij} &= \frac{1}{\Delta_{ij}}v_i - \frac{1}{\Delta_{ij}}v_j, \\ b_{ijk} &= \frac{1}{\Delta_{ij}\Delta_{ik}}v_i - \frac{1}{\Delta_{ij}\Delta_{jk}}v_j + \frac{1}{\Delta_{ik}\Delta_{jk}}v_k, \\ b_{ijkl} &= \frac{1}{\Delta_{ij}\Delta_{ik}\Delta_{il}}v_i - \frac{1}{\Delta_{ij}\Delta_{jk}\Delta_{jl}}v_j + \frac{1}{\Delta_{ik}\Delta_{jk}\Delta_{kl}}v_k - \frac{1}{\Delta_{il}\Delta_{jl}\Delta_{kl}}v_l. \end{aligned}$$

The sign changes are due to a reordering of the indices since $\Delta_{ij} = -\Delta_{ji}$. If v_i, v_j, v_k, v_l are given in the form (2), i.e., if

$$\begin{pmatrix} v_i \\ v_j \\ v_k \\ v_l \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ 1 & \Delta_{ij} & 0 & \ddots & \cdots & \cdots & \vdots \\ 1 & \Delta_{ik} & \Delta_{ik}\Delta_{jk} & \ddots & \cdots & \cdots & \vdots \\ 1 & \Delta_{il} & \Delta_{il}\Delta_{jl} & \Delta_{il}\Delta_{jl}\Delta_{kl} & 0 & \cdots & 0 \end{pmatrix},$$

then $b_i = (1, 0, \dots, 0)$, $b_{ij} = (0, -1, 0, \dots, 0)$, $b_{ijk} = (0, 0, 1, 0, \dots, 0)$ and $b_{ijkl} = (0, 0, 0, -1, 0, \dots, 0)$. In general, $b_1, b_{12}, \dots, b_{12\dots d+1}$ look like $(0, \dots, 0, \pm 1, 0, \dots, 0)$ when v_1, \dots, v_{d+1} are of the form (2).

The following proposition collects the basic properties on these vectors.

Proposition 1.5. (1) For any non-empty set $S \subset [n]$, one has $b_S \in \mathbb{Z}^{d+1}$.

(2) Let $S \subset [n]$ and $a, b \in S$ with $a \neq b$. Then we have a recursion formula

$$b_S = \frac{1}{\Delta_{ba}}b_{S \setminus \{a\}} + \frac{1}{\Delta_{ab}}b_{S \setminus \{b\}}.$$

(3) For any distinct $d+1$ indices $i_1, \dots, i_{d+1} \in [n]$ (not necessarily ordered), the vectors

$$b_{i_1}, b_{i_1 i_2}, b_{i_1 i_2 i_3}, \dots, b_{i_1 \dots i_{d+1}}$$

form a \mathbb{Z} -basis for \mathbb{Z}^{d+1} .

(4) If $\#S \geq d+2$, then $b_S = 0$.

Proof. The second statement can be verified by elementary computations, using $\Delta_{ij} + \Delta_{jk} = \Delta_{ik}$ for $i, j, k \in [n]$. To prove the first statement, we consider the components of b_S as rational functions in $\tau_i, i \in S$. By induction on $\#S$, we prove the following statement. The components of b_S are symmetric polynomials in $\tau_i, i \in S$, and their coefficients depend only on $\#S$.

If $\#S = 1$, then $b_S = b_i = v_i = (1, \tau_i, \tau_i^2, \dots, \tau_i^d)$, thus the claim holds. Now consider a set S with at least two distinct elements a, b . Let

$$f_j(\tau_a, \tau_i, i \in S), f_j(\tau_b, \tau_i, i \in S)$$

be the j -th components of $b_{S \setminus b}$, $b_{S \setminus a}$, respectively. Then the difference between these polynomials is zero if we set $\tau_a = \tau_b$, hence the quotient

$$\frac{f_j(\tau_a, \tau_i, i \in S) - f_j(\tau_b, \tau_i, i \in S)}{\tau_a - \tau_b}$$

is a polynomial as claimed. It is obviously symmetric in a and b . Since we are free to choose any two elements of S , it is symmetric in all variables. The coefficients of the polynomial depend only on $\#S$, so the claim is proven. Note that the degree of the polynomial decreases by one by taking the quotient. Since the degree of the components of v_i is at most $d + 1$, we conclude that $b_S = 0$ for $\#S \geq d + 2$.

To prove the third statement, we first note that the vertices $v_{i_1}, \dots, v_{i_{d+1}}$ are linearly independent. Take an element $x \in \mathbb{Z}^{d+1}$ and write it as $x = \sum \lambda_j v_{i_j}$. By considering $\sigma_{\{i_1, \dots, i_d\}}(x)$, we can say that the coefficient $\lambda_{i_{d+1}}$ is of the form

$$\lambda_{i_{d+1}} = \frac{k}{\prod_{j=1}^d \Delta_{i_j i_{d+1}}}$$

for an integer k . Thus, $x + (-1)^d k b_{i_1 \dots i_{d+1}} \in \mathbb{Z}^{d+1}$ is a vector in the subspace spanned by v_{i_1}, \dots, v_{i_d} . These vectors define a $(d-1)$ -dimensional cyclic polytope again, so we can proceed by induction and obtain a representation of x as a \mathbb{Z} -linear combination of the $b_{i_1}, b_{i_1 i_2}, \dots, b_{i_1 \dots i_{d+1}}$. \square

We apply this construction to prove another useful fact on cyclic polytopes.

Lemma 1.6. *For an integral cyclic polytope $\mathcal{P} \subset \mathbb{R}^d$ of dimension d , one has*

$$\mathbb{Z}\mathcal{A}_{\mathcal{P}} = \mathbb{Z}^{d+1}.$$

Proof. First, we notice that $\mathbb{Z}\mathcal{A}_{\mathcal{P}} \subset \mathbb{Z}^{d+1}$ is obvious. To prove the other inclusion, we construct a basis of \mathbb{Z}^{d+1} from $d + 1$ points in $\mathcal{A}_{\mathcal{P}}$. We choose $d + 1$ vertices v_1, \dots, v_{d+1} of \mathcal{P}^* and consider the vectors

$$b_{i_{d+1}}, b_{i_{d+1}} + b_{i_d i_{d+1}}, b_{i_{d+1}} + b_{i_d i_{d+1}} + b_{i_{d-1} i_d i_{d+1}}, \dots, \sum_{l=1}^{d+1} b_{i_l \dots i_{d+1}}.$$

Let us denote them by $c_j := \sum_{l=j}^{d+1} b_{i_l \dots i_{d+1}}$ for $j = 1, \dots, d + 1$. By Proposition 1.5 (3), they constitute a \mathbb{Z} -basis of \mathbb{Z}^{d+1} . Hence, if each c_j is contained in \mathcal{P}^* , then our claim follows. For this, let us consider the coefficient of a vertex v_{i_k} in the sequence of

$$b_{i_d}, b_{i_d i_{d+1}}, b_{i_{d-1} i_d i_{d+1}}, \dots, b_{i_1 \dots i_{d+1}}.$$

The coefficient of v_{i_k} appears first in $b_{i_k \dots i_{d+1}}$, where it has a positive sign. After that, its sign is alternating and the absolute value is non-increasing since the denominators increase. Hence, the sum of those coefficients and thus the coefficient in c_j is nonnegative. So, c_j is a convex combination of the vertices of \mathcal{P}^* . \square

Finally, we discuss the normality of integral cyclic polytopes.

Lemma 1.7. *Let \mathcal{P} be an integral cyclic polytope of dimension d . If any simplex of dimension d whose vertices are chosen from those of \mathcal{P} is normal, then \mathcal{P} itself is also normal.*

Proof. Let v_1, \dots, v_n be the vertices of \mathcal{P}^* . A proof is a direct application of Carathéodory's Theorem (see, e.g., [9, Section 7]). Let $x \in \mathbb{Z}\mathcal{A}_{\mathcal{P}} \cap \mathbb{Q}_{\geq 0}\mathcal{A}_{\mathcal{P}}$. Now, Carathéodory's Theorem guarantees that there exist $d+1$ vertices $v_{i_1}, \dots, v_{i_{d+1}}$ of \mathcal{P}^* such that $x \in \mathbb{Z}\mathcal{A}_{\mathcal{Q}} \cap \mathbb{Q}_{\geq 0}\mathcal{A}_{\mathcal{Q}}$, where $\mathcal{Q} = \text{conv}(\{v_{i_1}, \dots, v_{i_{d+1}}\})$. Here we use that $\mathbb{Z}\mathcal{A}_{\mathcal{P}} = \mathbb{Z}^{d+1} = \mathbb{Z}\mathcal{A}_{\mathcal{Q}}$ by Lemma 1.6. If \mathcal{Q} is normal, then we have $x \in \mathbb{Z}_{\geq 0}\mathcal{A}_{\mathcal{Q}}$, in particular, $x \in \mathbb{Z}_{\geq 0}\mathcal{A}_{\mathcal{P}}$. This implies that \mathcal{P} is normal. \square

2. NORMAL CYCLIC POLYTOPES

Our goal of this section is to prove

Theorem 2.1. *Work with the same notations as in Section 1. If $\Delta_{i,i+1} \geq d^2 - 1$ for $1 \leq i \leq n-1$, then $\mathcal{P} = C_d(\tau_1, \dots, \tau_n)$ is normal. In particular, $\gamma_d \leq d^2 - 1$.*

Most parts of this section are devoted to proving the simplex case. In fact, once we know that \mathcal{P} is always normal when $n = d+1$ and $\Delta_{i,i+1} \geq d^2 - 1$ for $1 \leq i \leq d$, Theorem 2.1 follows immediately from Lemma 1.7.

Before giving a proof, we prepare two lemmata, Lemma 2.3 and Lemma 2.4. First, for Lemma 2.3, we start from proving

Proposition 2.2. *Let $(r_1, r_2, \dots, r_{d+1}) \in \mathbb{Q}^{d+1}$ satisfying*

$$0 \leq r_1 \leq r_2 \leq \dots \leq r_{d+1} \leq 1 \quad \text{and} \quad \sum_{i=1}^{d+1} r_i = m.$$

Then one has

$$(a) \sum_{i=1}^j r_i \leq \frac{jm}{d+1} \quad \text{and} \quad (b) \sum_{i=1}^j r_{d+2-i} \geq \frac{jm}{d+1}$$

for any integer j with $1 \leq j \leq d+1$.

Proof. We prove by induction on j .

First, we show $r_1 \leq \frac{m}{d+1}$. Suppose that $r_1 > \frac{m}{d+1}$. Then one has $r_i > \frac{m}{d+1}$ for all $1 \leq i \leq d+1$ by $r_1 \leq r_2 \leq \dots \leq r_{d+1}$. Thus, $m = \sum_{i=1}^{d+1} r_i > (d+1) \cdot \frac{m}{d+1} = m$, a contradiction. Similarly, we also have $r_{d+1} \geq \frac{m}{d+1}$.

Now, we assume that the assertions (a) and (b) hold for any integer j' with $1 \leq j' < j$, where j is some integer with $2 \leq j \leq d+1$. Let $d+1 = kj + q$, where k is a positive integer and $0 \leq q \leq j-1$, i.e., k (resp. q) is a quotient (resp. a remainder) of $d+1$ divided by j . Suppose that $\sum_{i=1}^j r_i > \frac{jm}{d+1}$. Then one has

$$\sum_{i=1}^j r_{(k-1)j+i} \geq \sum_{i=1}^j r_{(k-2)j+i} \geq \dots \geq \sum_{i=1}^j r_i > \frac{jm}{d+1}.$$

Moreover, by the hypothesis of induction, one also has $\sum_{i=kj+1}^{d+1} r_i = \sum_{i=1}^q r_{d+2-i} \geq \frac{mq}{d+1}$ when $q \neq 0$. Hence, we obtain

$$m = \sum_{i=1}^{d+1} r_i > k \cdot \frac{jm}{d+1} + \frac{mq}{d+1} = m \cdot \frac{kj+q}{d+1} = m,$$

a contradiction. Therefore, the assertion (a) also holds for j . Similarly, we also have the assertion (b) for j , as required. \square

Lemma 2.3. *Let d be a positive integer and $(r_1, r_2, \dots, r_{d+1}) \in \mathbb{Q}^{d+1}$ satisfying that $0 \leq r_1 \leq r_2 \leq \dots \leq r_{d+1} \leq 1$ and that $\sum_{i=1}^{d+1} r_i$ is an integer which is greater than 1. Then one has*

$$(3) \quad \max_{\substack{1 \leq i_1 < i_2 < \dots < i_l \leq d+1, \\ 2 \leq l \leq d}} \left\{ \sum_{j=1}^l r_{i_j} : \sum_{j=1}^{l-1} r_{i_j} \leq 1 \right\} \geq 1 + \frac{1}{d+1}.$$

Proof. Let $m = \sum_{i=1}^{d+1} r_i$. When $m > d$, it must be satisfied that $r_i = 1$ for $1 \leq i \leq d+1$ and $m = d+1$ by our assumption. Thus, we may assume that $2 \leq m \leq d$. Let M denote the value of the left-hand side of (3).

The first step. Assume that $m-1 > \lfloor \frac{d+1}{2} \rfloor$. Then, by Proposition 2.2, one has $r_d + r_{d+1} \geq \frac{2m}{d+1}$, while $r_d \leq 1$. Hence,

$$M \geq r_d + r_{d+1} \geq \frac{2m}{d+1} > \frac{2}{d+1} \left(\left\lfloor \frac{d+1}{2} \right\rfloor + 1 \right) \geq \frac{2}{d+1} \left(\frac{d}{2} + 1 \right) = 1 + \frac{1}{d+1}.$$

The second step. Assume that $m-1 \leq \lfloor \frac{d+1}{2} \rfloor$ and let $d+1 = km + q$, where k is a positive integer and $0 \leq q \leq m-1$, i.e., k (resp. q) is a quotient (resp. a remainder) of $d+1$ divided by m .

If we suppose that $\sum_{j=0}^{k-1} r_{jm+q+1} > 1$, then one has

$$1 < \sum_{j=0}^{k-1} r_{jm+q+1} \leq \sum_{j=0}^{k-1} r_{jm+q+2} \leq \dots \leq \sum_{j=0}^{k-1} r_{jm+q+m}.$$

Thus, $m = \sum_{i=1}^{d+1} r_i \geq \sum_{i=q+1}^{d+1} r_i > m$, a contradiction. Hence, we have

$$\sum_{j=0}^{k-1} r_{jm+q+1} \leq 1.$$

The third step. If we assume that $q \neq m-1$, that is, $0 \leq q \leq m-2$, then one has $\sum_{j=0}^{k-2} r_{jm+q+2} \leq \frac{d-q-m+1}{d-q}$. In fact, on the contrary, suppose that $\sum_{j=0}^{k-2} r_{jm+q+2} > \frac{d-q-m+1}{d-q}$. Then,

$$\frac{d-q-m+1}{d-q} < \sum_{j=0}^{k-2} r_{jm+q+2} \leq \sum_{j=0}^{k-2} r_{jm+q+3} \leq \dots \leq \sum_{j=0}^{k-2} r_{jm+q+m+1}.$$

Thus, $\sum_{i=q+2}^{(k-1)m+q+1} r_i > \frac{m(d-q-m+1)}{d-q}$. Moreover, since $\sum_{i=q+2}^{d+1} r_i = m - \sum_{i=1}^{q+1} r_i$, we also have $\sum_{i=(k-1)m+q+2}^{d+1} r_i \geq \frac{(m-1)(m-\sum_{i=1}^{q+1} r_i)}{d-q}$ by Proposition 2.2. Hence,

$$\begin{aligned} m - \sum_{i=1}^{q+1} r_i &= \sum_{i=q+2}^{d+1} r_i > \frac{m(d-q-m+1)}{d-q} + \frac{(m-1)(m-\sum_{i=1}^{q+1} r_i)}{d-q} \\ &= \frac{m(d-q)}{d-q} - \frac{(m-1)\sum_{i=1}^{q+1} r_i}{d-q} \geq m - \sum_{i=1}^{q+1} r_i, \end{aligned}$$

a contradiction. Here, since $m-1 \leq \lfloor \frac{d+1}{2} \rfloor \leq \frac{d+1}{2}$ and $0 \leq q \leq m-2 < d$, we have $m+q \leq 2m-2 \leq d+1$, which means that $\frac{m-1}{d-q} \leq 1$. Thus, one has

$$\sum_{j=0}^{k-2} r_{jm+q+2} \leq \frac{d-q-m+1}{d-q}.$$

Similarly, if we assume that $q = m-1$, then one has

$$\sum_{j=0}^{k-1} r_{jm+1} \leq \frac{d-m+2}{d+1}.$$

The fourth step. In this step, we prove that

$$\sum_{j=0}^{k-1} r_{jm+q+1} + r_{d+1} \geq 1 + \frac{1}{d+1}.$$

We assume that $0 \leq q \leq m-2$. Suppose, on the contrary, $\sum_{j=0}^{k-1} r_{jm+q+1} + r_{d+1} < 1 + \frac{1}{d+1}$. Then $\sum_{j=1}^{k-1} r_{jm+q+1} + r_{d+1} < 1 + \frac{1}{d+1} - r_{q+1} < 1 + \frac{1}{d-q} - r_{q+1}$. Thus,

$$\begin{aligned} 1 + \frac{1}{d-q} - r_{q+1} &> \sum_{j=1}^{k-1} r_{jm+q+1} + r_{km+q} \geq \sum_{j=1}^{k-1} r_{jm+q} + r_{km+q-1} \geq \cdots \\ &\geq \sum_{j=1}^{k-1} r_{jm+q+1-(m-2)} + r_{km+q-(m-2)} = \sum_{j=0}^{k-2} r_{jm+q+3} + r_{(k-1)m+q+2}. \end{aligned}$$

Moreover, by the third step, we also have $\sum_{j=0}^{k-2} r_{jm+q+2} \leq \frac{d-q-m+1}{d-q}$. Hence,

$$\begin{aligned} m - \sum_{i=1}^{q+1} r_i &= \sum_{i=q+2}^{d+1} r_i < m-1 + \frac{m-1}{d-q} - (m-1)r_{q+1} + \frac{d-q-m+1}{d-q} \\ &= m - (m-1)r_{q+1} \leq m - (q+1)r_{q+1} \leq m - \sum_{i=1}^{q+1} r_i, \end{aligned}$$

a contradiction. Similarly, when $q = m - 1$, if we suppose that $\sum_{j=1}^k r_{jm} + r_{km+m-1} < 1 + \frac{1}{d+1}$, then

$$1 + \frac{1}{d+1} > \sum_{j=1}^k r_{jm} + r_{km+m-1} \geq \sum_{j=1}^k r_{jm-1} + r_{km+m-2} \geq \cdots \geq \sum_{j=0}^{k-1} r_{jm+2} + r_{km+1}$$

and $\sum_{j=0}^{k-1} r_{jm+1} \leq \frac{d-m+2}{d+1}$ by the third step, so we obtain $m = \sum_{i=1}^{d+1} r_i < m - 1 + \frac{m-1}{d+1} + \frac{d-m+2}{d+1} = m$, a contradiction.

The fifth step. Thanks to the second and fourth steps, we have

$$M \geq \sum_{j=0}^{k-1} r_{jm+q+1} + r_{d+1} \geq 1 + \frac{1}{d+1},$$

as desired. \square

We also prepare another

Lemma 2.4. *Let l be an integer with $l \geq 2$ and i_1, \dots, i_l distinct integers. We set*

$$Z_l(j) = \frac{\prod_{k=1}^{j-1} \Delta_{i_k i_j}}{\prod_{1 \leq k \leq l, k \neq j} |\Delta_{i_k i_j}|} p_j + \frac{\prod_{k=1}^{j-1} \Delta_{i_k i_{j+1}}}{\prod_{1 \leq k \leq l, k \neq j+1} |\Delta_{i_k i_{j+1}}|} p_{j+1} + \cdots + \frac{\prod_{k=1}^{j-1} \Delta_{i_k i_l}}{\prod_{1 \leq k \leq l, k \neq l} |\Delta_{i_k i_l}|} p_l$$

for $2 \leq j \leq l$. Then, for any $2 \leq j \leq l - 1$, we have

$$\begin{aligned} Z_l(j) &= \frac{\prod_{k=1}^{j-1} \Delta_{i_k i_j}}{\prod_{1 \leq k \leq l, k \neq j} |\Delta_{i_k i_j}|} p_j + \frac{1}{\Delta_{i_j i_{j+1}}} Z_l(j+1) - \frac{1}{\Delta_{i_j i_{j+1}} \Delta_{i_j i_{j+2}}} Z_l(j+2) + \\ &\quad \cdots + (-1)^{l-j+1} \frac{1}{\prod_{k=j+1}^l \Delta_{i_j i_k}} Z_l(l). \end{aligned}$$

A proof is given by elementary computations.

Now, Lemma 2.4 says that if $Z_l(j+1), \dots, Z_l(l)$ are integers, then there exists an integer p_j such that $Z_l(j)$ becomes an integer. In fact, since

$$\frac{1}{\Delta_{i_j i_{j+1}}} Z_l(j+1) - \cdots + (-1)^{l-j+1} \frac{1}{\prod_{k=j+1}^l \Delta_{i_j i_k}} Z_l(l) = \frac{P}{C},$$

where P is some integer and $C = \prod_{k=j+1}^l |\Delta_{i_j i_k}|$, and the numerator (resp. the denominator) of $\frac{\prod_{k=1}^{j-1} \Delta_{i_k i_j}}{\prod_{1 \leq k \leq l, k \neq j} |\Delta_{i_k i_j}|}$ is either 1 or -1 (resp. C), it is obvious that there exists an integer p_j such that $Z_l(j)$ becomes an integer.

Let $\mathcal{Q} \subset \mathbb{R}^N$ be an integral convex polytope of dimension d . In general, when $\mathbb{Z}\mathcal{A}_{\mathcal{Q}} = \mathbb{Z}^{N+1}$, in order to prove that \mathcal{Q} is normal, it suffices to show that for any $\alpha = (m, \alpha_1, \dots, \alpha_N) \in \mathbb{Z}\mathcal{A}_{\mathcal{Q}} \cap \mathbb{Q}_{\geq 0}\mathcal{A}_{\mathcal{Q}} = \mathbb{Q}_{\geq 0}\mathcal{A}_{\mathcal{Q}} \cap \mathbb{Z}^{N+1}$ with $m \geq 2$, we find $\alpha' \in \mathcal{Q}^* \cap \mathbb{Z}^{N+1}$ and $\alpha'' \in \mathbb{Q}_{\geq 0}\mathcal{A}_{\mathcal{Q}} \cap \mathbb{Z}^{N+1}$ with $\alpha = \alpha' + \alpha''$. (This is equivalent to prove that \mathcal{Q} satisfies the integer decomposition property.) In particular, when

\mathcal{Q} is a simplex, since there exists a unique $(r_1, \dots, r_{d+1}) \in \mathbb{Q}^{d+1}$ such that $\alpha = \sum_{i=1}^{d+1} r_i u_i$ and $\sum_{i=1}^{d+1} r_i = m$, where u_1, \dots, u_{d+1} are the vertices of \mathcal{Q}^* , we may find $(r'_1, \dots, r'_{d+1}) \in \mathbb{Q}^{d+1}$ with $\sum_{i=1}^{d+1} r'_i u_i \in \mathcal{Q}^* \cap \mathbb{Z}^{N+1}$ and $(r''_1, \dots, r''_{d+1}) \in \mathbb{Q}^{d+1}$ with $\sum_{i=1}^{d+1} r''_i u_i \in \mathbb{Q}_{\geq 0} \mathcal{A}_{\mathcal{Q}} \cap \mathbb{Z}^{N+1}$ satisfying $r'_i + r''_i = r_i$ for $1 \leq i \leq d+1$.

Hence, it is enough to show that for any $\alpha = \sum_{i=1}^{d+1} r_i u_i \in \mathbb{Q}_{\geq 0} \mathcal{A}_{\mathcal{Q}} \cap \mathbb{Z}^{N+1}$ with $\sum_{i=1}^{d+1} r_i \geq 2$, there exists $(r'_1, \dots, r'_{d+1}) \in \mathbb{Q}^{d+1}$ such that

$$\sum_{i=1}^{d+1} r'_i = 1, \quad 0 \leq r'_i \leq r_i \quad \text{for } 1 \leq i \leq d+1 \quad \text{and} \quad \sum_{i=1}^{d+1} r'_i u_i \in \mathbb{Z}^{N+1}.$$

Now, we come to the position to verify the normality of integral cyclic polytopes in the case where $n = d+1$ and $\Delta_{i,i+1} \geq d^2 - 1$ for $1 \leq i \leq d$. Let \mathcal{P} be such cyclic polytope. Let m be an integer with $m \geq 2$ and α an element in $\mathbb{Z} \mathcal{A}_{\mathcal{P}} \cap \mathbb{Q}_{\geq 0} \mathcal{A}_{\mathcal{P}} = \mathbb{Q}_{> 0} \mathcal{A}_{\mathcal{P}} \cap \mathbb{Z}^{d+1}$ with the first coordinate m . Since \mathcal{P}^* is a simplex of dimension d , there exists a unique $(r_1, \dots, r_{d+1}) \in \mathbb{Q}^{d+1}$, where $\sum_{i=1}^{d+1} r_i = m$, such that $\alpha = \sum_{i=1}^{d+1} r_i v_i$. Then what we must do is to show that there exists $(r'_1, \dots, r'_{d+1}) \in \mathbb{Q}^{d+1}$ such that

$$(4) \quad \sum_{i=1}^{d+1} r'_i = 1, \quad 0 \leq r'_i \leq r_i \quad \text{for } 1 \leq i \leq d+1 \quad \text{and} \quad \sum_{i=1}^{d+1} r'_i v_i \in \mathbb{Z}^{d+1}.$$

The first step. If there exists r_i with $r_i \geq 1$, say, r_1 , then we may set $r'_1 = 1$ and $r'_2 = \dots = r'_{d+1} = 0$. Moreover, when $m \geq d+1$, since $\sum_{i=1}^{d+1} r_i = m$ and $r_i \geq 0$, there is at least one r_i with $r_i \geq 1$. Thus, we may assume that

$$2 \leq m \leq d \quad \text{and} \quad 0 \leq r_i \leq 1 \quad \text{for } 1 \leq i \leq d+1.$$

The second step. By Lemma 2.3, there exist r_{i_1}, \dots, r_{i_l} among (r_1, \dots, r_{d+1}) such that $\sum_{j=1}^l r_{i_j} \geq 1 + \frac{1}{d+1}$ and $\sum_{j=1}^{l-1} r_{i_j} \leq 1$, where $0 \leq r_{i_1} \leq \dots \leq r_{i_l} \leq 1$ and $2 \leq l \leq d$, although we do not know whether $1 \leq i_1 < \dots < i_l \leq d+1$. Let r_{i_1}, \dots, r_{i_l} be such ones. However, we assume that $0 \leq r_{i_l} \leq r_{i_{l-1}} \leq \dots \leq r_{i_1} \leq 1$, i.e., we have

$$\sum_{j=2}^l r_{i_j} \leq 1 \quad \text{and} \quad \sum_{j=1}^l r_{i_j} \geq 1 + \frac{1}{d+1}.$$

Let $D = d^2 - 1$. Thus, $|\Delta_{ij}| \geq D$ for any $1 \leq i \neq j \leq d+1$. Now, we set $\varepsilon(l) = \frac{l-1}{D}$ for $2 \leq l \leq d$. Then it is easy to see that $\varepsilon(l)$ enjoys the following properties:

$$(5) \quad \begin{aligned} \varepsilon(l) &\geq \sum_{a=2}^l \frac{1}{D^{a-1}}, \quad \frac{1}{d+1} = \varepsilon(d) > \varepsilon(d-1) > \dots > \varepsilon(2), \\ \varepsilon(l) - \frac{l-j+1}{D^{j-1}} &> \varepsilon(j-1) \quad \text{for } 3 \leq j \leq l. \end{aligned}$$

In the following two steps, by induction on l , we prove that if $\sum_{j=1}^l r_{i_j} \geq 1 + \varepsilon(l)$ and $\sum_{j=2}^l r_{i_j} \leq 1$, then there is $(r'_1, \dots, r'_{d+1}) \in \mathbb{Q}^{d+1}$ which satisfies (4). Once we know this, we obtain the required assertion from $2 \leq l \leq d$ and $\frac{1}{d+1} = \varepsilon(d) \geq \varepsilon(l)$.

The third step. Assume that $l = 2$, i.e., we have $r_{i_1} + r_{i_2} \geq 1 + \frac{1}{D}$, where $0 \leq r_{i_2} \leq r_{i_1} \leq 1$.

Let p be a nonnegative integer satisfying

$$\frac{p}{|\Delta_{i_1 i_2}|} \leq r_{i_2} < \frac{p+1}{|\Delta_{i_1 i_2}|}.$$

Then it is clear that there exists such a unique nonnegative integer p . Let $r'_{i_2} = \frac{p}{|\Delta_{i_1 i_2}|}$, $r'_{i_1} = 1 - r'_{i_2}$ and $r'_j = 0$ for any j with $j \in [d+1] \setminus \{i_1, i_2\}$. Thus, $\sum_{i=1}^{d+1} r'_i = 1$ and $0 \leq r'_{i_2} \leq r_{i_2}$. Moreover, since $r_{i_2} \leq 1$, we have $r'_{i_1} = 1 - r'_{i_2} \geq 1 - r_{i_2} \geq 0$. In addition, by $r_{i_1} + r_{i_2} \geq 1 + \frac{1}{D}$ and $|\Delta_{i_1 i_2}| \geq D$, we also have

$$r_{i_1} - r'_{i_1} = r_{i_1} - 1 + \frac{p}{|\Delta_{i_1 i_2}|} \geq \frac{1}{D} - r_{i_2} + \frac{p}{|\Delta_{i_1 i_2}|} \geq \frac{p+1}{|\Delta_{i_1 i_2}|} - r_{i_2} > 0.$$

On the other hand, by Proposition 1.2, we may consider v_{i_1} and v_{i_2} as $v_{i_1} = (1, 0, \dots, 0)$ and $v_{i_2} = (1, \Delta_{i_1 i_2}, 0, \dots, 0)$. Obviously, $\sum_{i=1}^{d+1} r'_i v_i \in \mathbb{Z}^{d+1}$.

The fourth step. Assume that $l \geq 3$. For each j with $2 \leq j \leq l$, we define each nonnegative integer p_j as follows. Let p_l be a nonnegative integer which satisfies

$$\frac{p_l}{\prod_{k=1}^{l-1} |\Delta_{i_k i_l}|} \leq r_{i_l} < \frac{p_l + 1}{\prod_{k=1}^{l-1} |\Delta_{i_k i_l}|},$$

and for $2 \leq j \leq l-1$, let p_j be an integer which satisfies $Z_l(j) \in \mathbb{Z}$ and

$$\frac{p_j}{\prod_{1 \leq k \leq l, k \neq j} |\Delta_{i_k i_j}|} \leq r_{i_j} < \frac{p_j + \prod_{k=j+1}^l |\Delta_{i_j i_k}|}{\prod_{1 \leq k \leq l, k \neq j} |\Delta_{i_k i_j}|},$$

where $Z_l(j)$ is as in Lemma 2.4. Thanks to Lemma 2.4, if $Z_l(j+1), \dots, Z_l(l) \in \mathbb{Z}$, then there exists an integer p_j with $Z_l(j) \in \mathbb{Z}$ and each p_j is uniquely determined by the above inequalities. Remark that we do not know whether p_j is nonnegative except for p_l . However, in our case, we may assume that p_2, \dots, p_{l-1} are all nonnegative because of the following discussions. In fact, on the contrary, suppose that there is j' with $p_{j'} < 0$. Let $q_{j'} \in \mathbb{Z}_{\geq 0}$ be a minimal nonnegative integer satisfying

$$\begin{aligned} \frac{\prod_{k=1}^{j'-1} \Delta_{i_k i_{j'}}}{\prod_{1 \leq k \leq l, k \neq j'} |\Delta_{i_k i_{j'}}|} q_{j'} + \frac{1}{\Delta_{i_{j'} i_{j'+1}}} Z_l(j'+1) - \frac{1}{\Delta_{i_{j'} i_{j'+1}} \Delta_{i_{j'} i_{j'+2}}} Z_l(j'+2) + \\ \dots + (-1)^{l-j'+1} \frac{1}{\prod_{k=j'+1}^l \Delta_{i_{j'} i_k}} Z_l(l) \in \mathbb{Z}. \end{aligned}$$

In particular, it follows from the minimality of $q_{j'}$ that $0 \leq q_{j'} < \prod_{k=j'+1}^l |\Delta_{i_j' i_k}|$. By our assumption, one has $\frac{q_{j'}}{\prod_{1 \leq k \leq l, k \neq j'} |\Delta_{i_k i_{j'}}|} > r_{i_{j'}}$. Thus,

$$r_{i_l} \leq \dots \leq r_{i_{j'}} < \frac{q_{j'}}{\prod_{1 \leq k \leq l, k \neq j'} |\Delta_{i_j' i_k}|} < \frac{\prod_{k=j'+1}^l |\Delta_{i_j' i_k}|}{\prod_{1 \leq k \leq l, k \neq j'} |\Delta_{i_k i_{j'}}|} = \frac{1}{\prod_{k=1}^{j'-1} |\Delta_{i_k i_{j'}}|} \leq \frac{1}{D^{j'-1}},$$

so one has $\sum_{j=j'}^l r_{i_j} < \frac{l-j'+1}{D^{j'-1}}$. From $\sum_{j=1}^l r_{i_j} \geq 1 + \varepsilon(l)$ and (5), we have

$$\sum_{j=1}^{j'-1} r_{i_j} > 1 + \varepsilon(l) - \frac{l-j'+1}{D^{j'-1}} > 1 + \varepsilon(j'-1)$$

when $j' \geq 3$. Hence, we may skip such case by the hypothesis of induction. When $j' = 2$, one has $r_{i_1} > 1 + \varepsilon(l) - \frac{l-1}{D} = 1$, a contradiction.

By using the above p_j 's, we define r'_1, \dots, r'_{d+1} by setting

$$r'_a = \begin{cases} \frac{p_j}{\prod_{1 \leq k \leq l, k \neq j} |\Delta_{i_k i_j}|}, & \text{if } a = i_j \in \{i_2, \dots, i_l\}, \\ 1 - \sum_{j=2}^l r'_{i_j}, & \text{if } a = i_1, \\ 0, & \text{otherwise.} \end{cases}$$

In particular, $\sum_{a=1}^{d+1} r'_a = 1$. By definition of $r'_{i_2}, \dots, r'_{i_l}$, we have $0 \leq r'_{i_j} \leq r_{i_j}$ for $2 \leq j \leq l$. Moreover, from $\sum_{j=2}^l r_{i_j} \leq 1$, we also have $r'_{i_1} = 1 - \sum_{j=2}^l r'_{i_j} \geq 1 - \sum_{j=2}^l r_{i_j} \geq 0$. In addition, from $\sum_{j=1}^l r_{i_j} \geq 1 + \varepsilon(l)$ and (5), we also have

$$\begin{aligned} r_{i_1} - r'_{i_1} &= r_{i_1} - 1 + \sum_{j=2}^l \frac{p_j}{\prod_{1 \leq k \leq l, k \neq j} |\Delta_{i_k i_j}|} \geq \varepsilon(l) - \sum_{j=2}^l r_{i_j} + \sum_{j=2}^l \frac{p_j}{\prod_{1 \leq k \leq l, k \neq j} |\Delta_{i_k i_j}|} \\ &\geq \sum_{j=2}^l \left(\frac{p_j}{\prod_{1 \leq k \leq l, k \neq j} |\Delta_{i_k i_j}|} + \frac{1}{D^{j-1}} - r_{i_j} \right) \geq \sum_{j=2}^l \left(\frac{p_j + \prod_{k=j+1}^l |\Delta_{i_j i_k}|}{\prod_{1 \leq k \leq l, k \neq j} |\Delta_{i_k i_j}|} - r_{i_j} \right) \\ &> 0. \end{aligned}$$

Finally, we verify that $\sum_{i=1}^{d+1} r'_i v_i \in \mathbb{Z}^{d+1}$. Again, by Proposition 1.2, we may consider v_{i_1}, \dots, v_{i_l} as follows:

$$\begin{pmatrix} v_{i_1} \\ v_{i_2} \\ \vdots \\ v_{i_l} \end{pmatrix} = \begin{pmatrix} 1 & 0 & \dots & \dots & \dots & 0 & \dots & 0 \\ 1 & \Delta_{i_1 i_2} & 0 & \ddots & \ddots & \vdots & \vdots \\ 1 & \Delta_{i_1 i_3} & \Delta_{i_1 i_3} \Delta_{i_2 i_3} & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 1 & \Delta_{i_1 i_l} & \Delta_{i_1 i_l} \Delta_{i_2 i_l} & \dots & \prod_{k=1}^{l-1} \Delta_{i_k i_l} & 0 & \dots & 0 \end{pmatrix}.$$

Hence, it is easy to check that

$$\sum_{i=1}^{d+1} r'_i v_i = \sum_{j=1}^l r'_{i_j} v_{i_j} = (1, Z_l(2), Z_l(3), \dots, Z_l(l), 0, \dots, 0) \in \mathbb{Z}^{d+1},$$

proving the assertion.

Remark 2.5. Since each lattice length of an edge $\text{conv}(\{v_i, v_j\})$ of \mathcal{P}^* coincides with Δ_{ij} , where $i < j$, it follows immediately from [4, Theorem 1.3 (b)] that \mathcal{P} is normal if $\Delta_{i,i+1} \geq d(d+1)$ for $1 \leq i \leq n-1$. (We are grateful to Gábor Hegedüs for informing us the result [4, Theorem 1.3 (b)].) Thus, our constraint $\Delta_{i,i+1} \geq d^2 - 1$ on integral cyclic polytopes is better than a general case, but this bound is still very rough. For example, $C_3(0, 1, 2, 3)$ is normal, while we have $\Delta_{12} = \Delta_{23} = \Delta_{34} = 1 < 8$. Similarly, $C_4(0, 1, 3, 5, 6)$ is also normal, although one has $\Delta_{12} = \Delta_{45} = 1$ and $\Delta_{23} = \Delta_{34} = 2$.

3. NON-VERY AMPLE CYCLIC POLYTOPES

Our goal of this section is to prove

Theorem 3.1. *Let d and n be positive integers satisfying $n \geq d + 1$ and $d \geq 4$. If $\Delta_{12} = 1$ or $\Delta_{n-2, n-1} = 1$, then $C_d(\tau_1, \dots, \tau_n)$ is not very ample.*

We obtain Theorem 3.1 as a conclusion of Proposition 3.2 and Corollary 3.3 below.

Proposition 3.2. *Let $\mathcal{P} = C_4(\tau_1, \dots, \tau_n)$. If $\Delta_{23} = 1$ or $\Delta_{n-2, n-1} = 1$, then \mathcal{P} is not very ample.*

Proof. Thanks to Lemma 1.3, by symmetry, we assume $\Delta_{23} = 1$. Consider the set

$$\mathcal{A}_{\mathcal{P},3} := \{x - v_3 : x \in \mathcal{P}^* \cap \mathbb{Z}^5\}.$$

We will prove that the monoid $\mathbb{Z}_{\geq 0}\mathcal{A}_{\mathcal{P},3}$ is not normal, thus there exists a vector $p \in \mathbb{Z}\mathcal{A}_{\mathcal{P},3} \cap \mathbb{Q}_{\geq 0}\mathcal{A}_{\mathcal{P},3} = \mathbb{Q}_{\geq 0}\mathcal{A}_{\mathcal{P},3} \cap \mathbb{Z}^5$ such that $p \notin \mathbb{Z}_{\geq 0}\mathcal{A}_{\mathcal{P},3}$. Then, for every integer $k \geq 1$, it holds that $kv_3 + p \in (\mathbb{Z}\mathcal{A}_{\mathcal{P}} \cap \mathbb{Q}_{\geq 0}\mathcal{A}_{\mathcal{P}}) \setminus \mathbb{Z}_{\geq 0}\mathcal{A}_{\mathcal{P}}$, see [1, Exercice 2.23]. Hence, \mathcal{P} is not very ample.

In the sequel, we denote the facet of \mathcal{P}^* spanned by the vertices v_i, v_j, v_k and v_l with \mathcal{F}_{ijkl} . Moreover, we denote the corresponding linear form with σ_{ijkl} . Note that every facet of \mathcal{P}^* containing v_3 defines also a facet of $\mathbb{Q}_{\geq 0}\mathcal{A}_{\mathcal{P},3}$.

The following vector has the required properties:

$$\begin{aligned} p &:= b_{23} + b_{134} + b_{12345} \\ &= \frac{\Delta_{12}\Delta_{15} + 1}{\Delta_{12}\Delta_{13}\Delta_{14}\Delta_{15}}v_1 + \frac{1}{\Delta_{23}} \left(1 - \frac{1}{\Delta_{12}\Delta_{24}\Delta_{25}}\right)v_2 - \frac{1}{\Delta_{23}} \left(1 + \frac{\Delta_{23}\Delta_{35} - 1}{\Delta_{13}\Delta_{34}\Delta_{35}}\right)v_3 \\ &\quad + \frac{\Delta_{24}\Delta_{45} - 1}{\Delta_{14}\Delta_{24}\Delta_{34}\Delta_{45}}v_4 + \frac{1}{\Delta_{15}\Delta_{25}\Delta_{35}\Delta_{45}}v_5. \end{aligned}$$

First, one has $p \in \mathbb{Z}^5$ from Proposition 1.5 (1). Then, by the second representation of p , it is a positive linear combination of the vectors $v_1 - v_3, v_2 - v_3, v_4 - v_3$ and

$v_5 - v_3$. Thus, $p \in \mathbb{Q}_{\geq 0}\mathcal{A}_{\mathcal{P},3}$. Moreover, since we assume $\Delta_{23} = 1$, the coefficient of v_3 is less than -1 . Hence, p lies beyond the facet \mathcal{F}_{1245} which is a facet of \mathcal{P}^* by Gale's evenness condition (Proposition 1.1). Thus, we have $p \notin \mathcal{A}_{\mathcal{P},3}$.

It remains to show that p cannot be written as a sum $\sum w_j$ with $w_j \in \mathcal{A}_{\mathcal{P},3}$. Suppose that we have such a representation. Then we remark that p has at least two summands. Consider a facet \mathcal{F}_{1234} . Then $\sigma_{1234}(p) = \frac{1}{\Delta_{15}\Delta_{25}\Delta_{35}\Delta_{45}}\sigma_{1234}(v_5) = 1$. Since $\sigma_{1234}(w_j) \geq 0$, $\sigma_{1234}(w_j) = 0$ for every summand w_j except one. Choose one $w_j \neq 0$ with $\sigma_{1234}(w_j) = 0$ and denote it by w . Further, we set $w' := p - w \in \mathbb{Z}_{\geq 0}\mathcal{A}_{\mathcal{P},3}$ the remaining sum. By Carathéodory's Theorem, there exist vertices v_{i_1}, \dots, v_{i_4} of \mathcal{P}^* and nonnegative numbers $\lambda_j \geq 0$, such that $w' = \sum_{j=1}^4 \lambda_j(v_{i_j} - v_3)$. Let i_4 be the greatest of those indices. Since $\sigma_{1234}(w') = 1$ and $\sigma_{1234}(v_{i_4}) = \Delta_{1i_4}\Delta_{2i_4}\Delta_{3i_4}\Delta_{4i_4}$, we conclude that

$$\lambda_4 \leq \frac{1}{\Delta_{1i_4}\Delta_{2i_4}\Delta_{3i_4}\Delta_{4i_4}}.$$

But the vertices v_{i_1}, \dots, v_{i_4} and v_3 define an integral cyclic polytope, thus the denominator of the coefficient of v_{i_4} has to be a divisor of $\Delta_{i_1i_4}\Delta_{i_2i_4}\Delta_{i_3i_4}\Delta_{4i_4}$. This is only possible if $\{i_1, i_2, i_3\} = \{1, 2, 4\}$. Thus, w' lies in the cone generated by $v_1 - v_3, v_2 - v_3, v_4 - v_3$ and $v_{i_4} - v_3$. Note that $\sigma_{1234}(w) = 0$ implies that w lies in the cone generated by $v_1 - v_3, v_2 - v_3$ and $v_4 - v_3$. Thus we can replace the polytope \mathcal{P}^* by the polytope \mathcal{Q}^* whose vertices are v_1, \dots, v_5 and v_{i_4} . The reason for doing this is that we know the facets of \mathcal{Q}^* . Here, $i_4 = 5$ is possible.

We consider the representation

$$w = a_1b_3 + a_2b_{23} + a_3b_{123} + a_4b_{1234}$$

with integer coefficients a_1, a_2, a_3, a_4 . This is possible from Proposition 1.5 (3). Since w is in the cone generated by $v_1 - v_3, v_2 - v_3$ and $v_4 - v_3$, we have $a_1 = 0$. Now consider a facet \mathcal{F}_{123i_4} of \mathcal{Q}^* . We compute

$$\sigma_{123i_4}(p) = \frac{1}{\Delta_{45}}(\Delta_{24}\Delta_{45} - 1)\Delta_{4i_4} + \frac{1}{\Delta_{45}}\Delta_{5i_4} = \Delta_{24}\Delta_{4i_4} - 1.$$

Moreover, $\sigma_{123i_4}(w) = -a_4\Delta_{4i_4}$. From $0 \leq \sigma_{123i_4}(w) \leq \sigma_{123i_4}(p)$, we conclude $0 \leq -a_4 \leq \Delta_{24} - 1$. Here we used that a_4 is an integer. Next, consider a facet \mathcal{F}_{2345} . We compute $\sigma_{2345}(w) = a_3\Delta_{14}\Delta_{15} + a_4\Delta_{15}$ and $\sigma_{2345}(p) = \Delta_{12}\Delta_{15} + 1$. As before, we conclude that $0 \leq a_3\Delta_{14} + a_4 \leq \Delta_{12}$. However, these two constraints can only be satisfied by $a_3 = a_4 = 0$, because $\Delta_{14} = \Delta_{12} + \Delta_{24}$ and $\Delta_{15} > 1$. Finally, we consider a facet \mathcal{F}_{134i_4} . By computing $\sigma_{134i_4}(w) = a_2\Delta_{12}\Delta_{24}\Delta_{2i_4}$ and $\sigma_{134i_4}(p) = \Delta_{12}\Delta_{24}\Delta_{2i_4} - 1$, we conclude that $a_2 = 0$. But this means $w = 0$, a contradiction to $w \neq 0$. \square

By using this proposition, we also obtain

Corollary 3.3. *Let $\mathcal{P} = C_d(\tau_1, \dots, \tau_n)$, where $d \geq 5$. If there is some i with $2 \leq i \leq n - 2$ such that $\Delta_{i,i+1} = 1$, then \mathcal{P} is not very ample.*

Proof. We prove this by induction on d .

When $d = 5$, let $\mathcal{F}_i = \text{conv}(\{v_1, v_i, v_{i+1}, v_{i+2}, v_{i+3}\})$ for $2 \leq i \leq n - 3$ and $\mathcal{F}_{n-2} = \text{conv}(\{v_{n-4}, v_{n-3}, v_{n-2}, v_{n-1}, v_n\})$. By Gale's evenness condition, each \mathcal{F}_i is a facet of \mathcal{P}^* . When $\Delta_{i,i+1} = 1$ for some i with $2 \leq i \leq n - 2$, it then follows from Proposition 3.2 that \mathcal{F}_i is not very ample. Thus, \mathcal{P} itself is non-very ample, either. (See [8, Lemma 1].)

Now, let $d \geq 6$. For $2 \leq i \leq n - d + 2$, we set

$$\mathcal{F}_i = \begin{cases} \text{conv}(\{v_1, v_i, \dots, v_{i+d-2}\}), & \text{when } d \text{ is odd,} \\ \text{conv}(\{v_{i-1}, v_i, \dots, v_{i+d-2}\}), & \text{when } d \text{ is even.} \end{cases}$$

Again, Gale's evenness condition guarantees that each \mathcal{F}_i is a facet of \mathcal{P}^* . When $\Delta_{i,i+1} = 1$ for some i with $2 \leq i \leq n - 2$, since each facet is also an integral cyclic polytope of dimension $d - 1$, either \mathcal{F}_i or \mathcal{F}_{d-n+2} is not very ample by the hypothesis of induction. Therefore, \mathcal{P} is non-very ample. \square

On the case where $d = 2$, it is well known that there exists a unimodular triangulation for every integral convex polytope of dimension 2. Therefore, integral convex polytopes of dimension 2 are always normal.

On the case where $d = 3$, exhaustive computational experiences lead us to give the following

Conjecture 3.4. *All cyclic polytopes of dimension 3 are normal.*

Moreover, by computational experiences together with Proposition 3.2, we also conjecture a complete characterization of normal cyclic polytopes of dimension 4.

Conjecture 3.5. *A cyclic polytope of dimension 4 is normal if and only if we have*

$$\Delta_{23} \geq 2 \quad \text{and} \quad \Delta_{n-2, n-1} \geq 2.$$

By considering the foregoing two conjectures and Theorem 2.1, the following statement seems to be natural for us.

Conjecture 3.6. *If $\mathcal{P} = C_d(\tau_1, \dots, \tau_n)$ is normal and $\mathcal{P}' = C_d(\tau'_1, \dots, \tau'_n)$ satisfies $\tau'_j - \tau'_i \geq \Delta_{ij}$ for all $1 \leq i < j \leq n$, then \mathcal{P}' is also normal.*

Finally, we also state

Conjecture 3.7. *If an integral cyclic polytope is very ample, then it is also normal.*

Actually, it often happens that a very ample integral convex polytope is also normal, that is to say, the normality of an integral convex polytope is equivalent to what it is very ample. Hence, the above conjecture occurs in the natural way. On the other hand, it is also known that there exists an integral convex polytope which is not normal but very ample. See [1, Exercise 2.24].

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