Inequality for Variance of Weighted Sum

of Correlated Random Variables

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Abstract: The upper bound inequality for variance of weighted sum of correlated random variables is derived according to Cauchy-Schwarz 's inequality, while the weights are non-negative with sum of 1. We also give a novel proof with positive semidefinite matrix method. And the variance inequality of sum of correlated random variable with general weights is also obtained. Then, the variance inequalities are applied to the Chebyshev's inequality and law of large numbers for sum of correlated random variables.

Key words: Variance; Covariance; Random Variable; Chebyshev's Inequality; Correlated Random Variable; Positive Semidefinite Matrix; Law of Large Numbers.

1. Introduction

Suppose $X_1,\cdots,X_n$ are any random variables (discrete, continuous or else) with finite expectation and variance, for any weights $\alpha_1,\cdots,\alpha_n$,which are real numbers. Let $E(X)$, $Var(X)$, $Cov(X,Y)$ denote the expectation, variance and covariance respectively. Let

$$\rho=\frac{Cov(X,Y)}{\sqrt{Var(X)}\sqrt{Var(Y)}}$$

denote the correlation coefficient between X and Y , and $|\rho|\le 1$. And, the well-known Cauchy -Schwarz inequality is

$$|Cov(X,Y)|\le\sqrt{Var(X)}\sqrt{Var(Y)}\,. \tag{1}$$

For first and second moments of $\xi=\sum_{i=1}^{n}\alpha_i X_i$, the expectation is

$$E(\sum_{i=1}^{n}\alpha_i X_i)=\sum_{i=1}^{n}\alpha_i E(X_i)\,. \tag{2}$$

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And , the variance of $\xi = \sum_{i=1}^{n} \alpha_i X_i$ is

$$Var(\sum_{i=1}^{n} \alpha_i X_i) = \sum_{i=1}^{n} \alpha_i^2 Var(X_i) + \sum_{1 \le i < j \le n} 2\alpha_i \alpha_j Cov(X_i, X_j) . \quad (3)$$

If the variables are uncorrelated (a strong condition is independent),

$$Var(\sum_{i=1}^{n} \alpha_i X_i) = \sum_{i=1}^{n} \alpha_i^2 Var(X_i) \quad (4)$$

All of the above definitions and formulae are referred from [1-6]. Motivated by the relationship of $E(\sum_{i=1}^{n} \alpha_i X_i) = \sum_{i=1}^{n} \alpha_i E(X_i)$, what is the relationship among $Var(\sum_{i=1}^{n} \alpha_i X_i)$, $\sum_{i=1}^{n} \alpha_i^2 Var(X_i)$ and $\sum_{i=1}^{n} \alpha_i Var(X_i)$ for correlated random variables?

If there are no constraints on weights, there are no identical relationship among $Var(\sum_{i=1}^{n} \alpha_i X_i)$, $\sum_{i=1}^{n} \alpha_i^2 Var(X_i)$ and $\sum_{i=1}^{n} \alpha_i Var(X_i)$. For example, if all $\alpha_1, \cdots, \alpha_n$ are negative,

$$Var(\sum_{i=1}^{n} \alpha_i X_i) \ge 0 \ge \sum_{i=1}^{n} \alpha_i Var(X_i) .$$

And, various examples are as follows.

Example 1 Suppose $X_1, \cdots, X_n$ are independent random variables,

If $\alpha_1 = \alpha_2 \cdots = \alpha_n = 1$, $Var(\sum_{i=1}^{n} \alpha_i X_i) = \sum_{i=1}^{n} \alpha_i Var(X_i)$.

If $\alpha_i > 1, i = 1, \cdots, n.$, $Var(\sum_{i=1}^{n} \alpha_i X_i) = \sum_{i=1}^{n} \alpha_i^2 Var(X_i) > \sum_{i=1}^{n} \alpha_i Var(X_i)$.

If $0 \le \alpha_i \le 1, i = 1, \cdots, n.$, $Var(\sum_{i=1}^{n} \alpha_i X_i) = \sum_{i=1}^{n} \alpha_i^2 Var(X_i) < \sum_{i=1}^{n} \alpha_i Var(X_i)$.

Example 2. Let $n = 2$, $(X_1, X_2) \sim N(0,1;0,1;1)$ denote the 2-dimension Normal distribution.

If $\alpha_1 = \alpha_2 = 1$, $Var(\alpha_1 X_1 + \alpha_2 X_2) = 4 > \alpha_1 Var(X_1) + \alpha_2 Var(X_2) = 2$.

If $\alpha_1 = \alpha_2 = \frac{1}{2}$, $Var(\alpha_1 X_1 + \alpha_2 X_2) = 1 = \alpha_1 Var(X_1) + \alpha_2 Var(X_2)$.

If $\alpha_1 = \frac{1}{2}, \alpha_2 = \frac{1}{3}$, $Var(\alpha_1 X_1 + \alpha_2 X_2) = \frac{25}{36} < \alpha_1 Var(X_1) + \alpha_2 Var(X_2) = \frac{5}{6}$.

Example 3. Let $n = 2$, $(X_1, X_2) \sim N(0,1;0,1;-1)$ denote the 2-dimension Normal distribution.

If $\alpha_1 = \alpha_2 = 1$, $Var(\alpha_1 X_1 + \alpha_2 X_2) = 0 < \alpha_1 Var(X_1) + \alpha_2 Var(X_2) = 2$.

Example 4. . Let $n = 2$, $\alpha_1 = \alpha_2 = \frac{1}{2}$, $(X_1, X_2) \sim N(0,1;0,1;1)$,

$$Var(\alpha_1 X_1 + \alpha_2 X_2) = 1 > \sum_{i=1}^{2} \alpha_i^2 Var(X_i) = \frac{1}{2}.$$

Let $n = 2$, $\alpha_1 = \alpha_2 = \frac{1}{2}$, $(X_1, X_2) \sim N(0,1;0,1;-1)$,

$$Var(\alpha_1 X_1 + \alpha_2 X_2) = 0 < \sum_{i=1}^{2} \alpha_i^2 Var(X_i) = \frac{1}{2}.$$

From the above examples, we can conclude that, if all weights $\alpha_1, \cdots, \alpha_n$ are negative, or sum of all absolute values of weights $\alpha_1, \cdots, \alpha_n$ are larger than 1, the relationship among $Var(\sum_{i=1}^{n} \alpha_i X_i)$, $\sum_{i=1}^{n} \alpha_i^2 Var(X_i)$ and $\sum_{i=1}^{n} \alpha_i Var(X_i)$ has no identical conclusion. Example 4 also illustrates that the lower bound of $Var(\sum_{i=1}^{n} \alpha_i X_i) \geq 0$ can be reached. Thus, we start our investigation from a simple case of non-negative weights $\alpha_1, \cdots, \alpha_n$ with sum of 1.

The rest of the paper is organized as follows. The variance inequalities are derived in Section 2. The Chebyshev's inequality of correlated random variables is obtained in Section 3. The law of large numbers of correlated random variables is obtained in Section 4. The conclusion is given in Section 5.

2. Variance inequality of correlated random variables

Consider the formula (3) , we first give the variance inequality with Cauchy-Schwarz inequality.

Theorem 1. Let $\alpha_1, \alpha_2, \cdots, \alpha_n$ be any real numbers with $0 \leq \alpha_i \leq 1, i = 1, \cdots, n$, and $\sum_{i=1}^{n} \alpha_i = 1$.

Suppose $X_1, \cdots, X_n$ are any random variables with $Var(X_i) < +\infty$, $i = 1, \cdots, n$, then

$$Var(\sum_{i=1}^{n}\alpha_i X_i) \le \sum_{i=1}^{n}\alpha_i Var(X_i)\,. \tag{5}$$

Proof. For any X_i and X_j, we zoom the Cauchy-Schwarz inequality,

$$|Cov(X_i, X_j)| \le \sqrt{Var(X_i)}\sqrt{Var(X_j)} \le \frac{Var(X_i)+Var(X_j)}{2}\,.$$

Applying above formula to formula (3) and combine the same items, we have

$$\begin{aligned} Var(\sum_{i=1}^{n}\alpha_i X_i) &\le \sum_{i=1}^{n}\alpha_i^2 Var(X_i) + \sum_{1\le i<j\le n} 2\alpha_i\alpha_j \frac{Var(X_i)+Var(X_j)}{2} \\ &= \sum_{j=1}^{n}\alpha_j \sum_{i=1}^{n}\alpha_i Var(X_i) = (\sum_{i=1}^{n}\alpha_i)[\sum_{i=1}^{n}\alpha_i Var(X_i)] = \sum_{i=1}^{n}\alpha_i Var(X_i) \end{aligned} \tag{6}$$

□

Note that using different Cauchy-Schwarz inequality form could obtain another upper bound of variance.

Theorem 1'. Let $\alpha_1, \alpha_2, \cdots, \alpha_n$ be any real numbers with $0 \le \alpha_i \le 1, i = 1, \cdots, n$, and $\sum_{i=1}^{n}\alpha_i = 1$.

Suppose $X_1, \cdots, X_n$ are any random variables with $Var(X_i) < +\infty$, $i = 1, \cdots, n$, then

$$Var(\sum_{i=1}^{n}\alpha_i X_i) \le (\sum_{i=1}^{n}\alpha_i^2)\sum_{i=1}^{n}Var(X_i)\,. \tag{5'}$$

Proof. Denote $\sigma_i = \sqrt{Var(X_i)}$. According to Cauchy-Schwarz inequality

$$(\sum_{i=1}^{n}a_i b_i)^2 \le (\sum_{i=1}^{n}a_i^2)(\sum_{i=1}^{n}b_i^2)\,,$$

$$\begin{aligned} Var(\sum_{i=1}^{n}\alpha_i X_i) &= \sum_{i=1}^{n}\sum_{j=1}^{n}\rho_{ij}\alpha_i\alpha_j\sqrt{Var(X_i)}\sqrt{Var(Y_i)} \le \sum_{i=1}^{n}\sum_{j=1}^{n}\alpha_i\alpha_j\sigma_i\sigma_j \\ &= (\sum_{i=1}^{n}\alpha_i\sigma_i)^2 \le (\sum_{i=1}^{n}\alpha_i^2)(\sum_{i=1}^{n}\sigma_i^2) = (\sum_{i=1}^{n}\alpha_i^2)(\sum_{i=1}^{n}Var(X_i)) \end{aligned}$$

□

To illustrate the difference between two upper bounds (5)(5'), we examine the values of $\sum_{i=1}^{n}\alpha_i Var(X_i)$ and $(\sum_{i=1}^{n}\alpha_i^2)\sum_{i=1}^{n}Var(X_i)$ in the range of $\alpha_i \in \{0.1, 0.2, \cdots; 1\}$, $\sum_{i=1}^{n}\alpha_i = 1$, $Var(X_i) \in \{0.1, 0.2, \cdots, 2\}$, $i = 1, \cdots, n$. The comparison results are list in Table 1.

As shown in **Table 1**, $\sum_{i=1}^{n}\alpha_i Var(X_i)$ is smaller than $(\sum_{i=1}^{n}\alpha_i^2)\sum_{i=1}^{n}Var(X_i)$ in most cases of simulations.

And, when $\alpha_1 = \alpha_2 = \cdots \alpha_n = \frac{1}{n}$, $\sum_{i=1}^{n}\alpha_i Var(X_i) = (\sum_{i=1}^{n}\alpha_i^2)\sum_{i=1}^{n}Var(X_i) = \frac{1}{n}\sum_{i=1}^{n}Var(X_i)$. For convenience, we discuss the upper bound (5) in the rest of the paper.

Table 1. Upper bound comparison simulation

n	Number of computation cases $\alpha_i \in \{0.1, 0.2, \cdots, 1\}$, $\sum_{i=1}^{n}\alpha_i = 1$ $Var(X_i) \in \{0.1, 0.2, \cdots, 2\}$, $i = 1, \cdots, n$	Number of case $\sum_{i=1}^{n}\alpha_i Var(X_i) > (\sum_{i=1}^{n}\alpha_i^2)\sum_{i=1}^{n}Var(X_i)$	Ratio (%)
2	3600	520	14.44
3	288000	29137	10.11
4	13760000	799763	5.81

In fact, we can obtain the conclusion with matrix method. We give the proof in **Theorem 2.**

Theorem 2. Let $\alpha_1, \alpha_2, \cdots, \alpha_n$ be any real numbers, and $0 \le \alpha_i \le 1, i = 1, \cdots, n$, $\sum_{i=1}^{n}\alpha_i = 1$,

Suppose $X_1, \cdots, X_n$ are any random variables with $Var(X_i) < +\infty$, $i = 1, \cdots, n$, then

$$Var(\sum_{i=1}^{n}\alpha_i X_i) \le \sum_{i=1}^{n}\alpha_i Var(X_i). \qquad (7)$$

Proof.

Denote $\sigma_i = \sqrt{Var(X_i)}$, $\rho_{ij} = \frac{Cov(X_i, X_j)}{\sqrt{Var(X_i)}\sqrt{Var(X_j)}}$, and $\rho_{ii} = 1$.

Then

$$Var(\sum_{i=1}^{n}\alpha_i X_i) - \sum_{i=1}^{n}\alpha_i Var(X_i) = \sum_{i=1}^{n}\sum_{i=1}^{n}Cov(\alpha_i X_i, \alpha_j X_j) - \sum_{i=1}^{n}\alpha_i Var(X_i)$$

$$= \sum_{i=1}^{n}\sum_{i=1}^{n}\rho_{ij}\alpha_i\alpha_j\sigma_i\sigma_j - \sum_{i=1}^{n}\alpha_i\sigma_i^2$$

$$= -(\sum_{i=1}^{n}\alpha_i\sigma_i^2 - \sum_{i=1}^{n}\sum_{i=1}^{n}\rho_{ij}\alpha_i\alpha_j\sigma_i\sigma_j)$$

We will prove that

$$\sum_{i=1}^{n}\alpha_i\sigma_i^2 - \sum_{i=1}^{n}\sum_{i=1}^{n}\rho_{ij}\alpha_i\alpha_j\sigma_i\sigma_j \ge 0. \qquad (8)$$

Since $|\rho_{ij}| \le 1$, we have

$$\sum_{i=1}^{n}\alpha_i\sigma_i^2-\sum_{i=1}^{n}\sum_{i=1}^{n}\rho_{ij}\alpha_i\alpha_j\sigma_i\sigma_j\ge\sum_{i=1}^{n}\alpha_i\sigma_i^2-\sum_{i=1}^{n}\sum_{i=1}^{n}\alpha_i\alpha_j\sigma_i\sigma_j\,.$$

We will prove

$$\sum_{i=1}^{n}\alpha_i\sigma_i^2-\sum_{i=1}^{n}\sum_{i=1}^{n}\alpha_i\alpha_j\sigma_i\sigma_j\ge 0 \qquad (9).$$

In fact,

$$\begin{aligned}
&\sum_{i=1}^{n}\alpha_i\sigma_i^2-\sum_{i=1}^{n}\sum_{i=1}^{n}\alpha_i\alpha_j\sigma_i\sigma_j\\
&=[\sigma_1,\cdots,\sigma_n]\begin{bmatrix}\alpha_1 & 0 & \cdots & 0\\ 0 & \alpha_2 & \cdots & 0\\ \cdots & \cdots & \cdots & \cdots\\ 0 & 0 & \cdots & \alpha_n\end{bmatrix}\begin{bmatrix}\sigma_1\\ \vdots\\ \sigma_n\end{bmatrix}-[\sigma_1,\cdots,\sigma_n]\begin{bmatrix}\alpha_1^2 & \alpha_1\alpha_2 & \cdots & \alpha_1\alpha_n\\ \alpha_1\alpha_2 & \alpha_2^2 & \cdots & \alpha_2\alpha_n\\ \cdots & \cdots & \cdots & \cdots\\ \alpha_1\alpha_n & \alpha_2\alpha_n & \cdots & \alpha_n^2\end{bmatrix}\begin{bmatrix}\sigma_1\\ \vdots\\ \sigma_n\end{bmatrix}\\
&=[\sigma_1,\cdots,\sigma_n]\left(\begin{bmatrix}\alpha_1 & 0 & \cdots & 0\\ 0 & \alpha_2 & \cdots & 0\\ \cdots & \cdots & \cdots & \cdots\\ 0 & 0 & \cdots & \alpha_n\end{bmatrix}-\begin{bmatrix}\alpha_1^2 & \alpha_1\alpha_2 & \cdots & \alpha_1\alpha_n\\ \alpha_1\alpha_2 & \alpha_2^2 & \cdots & \alpha_2\alpha_n\\ \cdots & \cdots & \cdots & \cdots\\ \alpha_1\alpha_n & \alpha_2\alpha_n & \cdots & \alpha_n^2\end{bmatrix}\right)\begin{bmatrix}\sigma_1\\ \vdots\\ \sigma_n\end{bmatrix}\\
&=[\sigma_1,\cdots,\sigma_n]\begin{bmatrix}\alpha_1-\alpha_1^2 & -\alpha_1\alpha_2 & \cdots & -\alpha_1\alpha_n\\ -\alpha_1\alpha_2 & \alpha_2-\alpha_2^2 & \cdots & -\alpha_2\alpha_n\\ \cdots & \cdots & \cdots & \cdots\\ -\alpha_1\alpha_n & -\alpha_2\alpha_n & \cdots & \alpha_n-\alpha_n^2\end{bmatrix}\begin{bmatrix}\sigma_1\\ \vdots\\ \sigma_n\end{bmatrix}
\end{aligned}$$

This is a quadratic form. We only prove that Matrix

$$A=\begin{bmatrix}\alpha_1-\alpha_1^2 & -\alpha_1\alpha_2 & \cdots & -\alpha_1\alpha_n\\ -\alpha_1\alpha_2 & \alpha_2-\alpha_2^2 & \cdots & -\alpha_2\alpha_n\\ \cdots & \cdots & \cdots & \cdots\\ -\alpha_1\alpha_n & -\alpha_2\alpha_n & \cdots & \alpha_n-\alpha_n^2\end{bmatrix}$$

is a positive semidefinite matrix. Consider the k-th ($1\le k\le n$) principal minors.

$$A\begin{pmatrix}i_1 & i_2 & \cdots & i_k\\ i_1 & i_2 & \cdots & i_k\end{pmatrix}=\begin{vmatrix}\alpha_{i_1}-\alpha_{i_1}^2 & -\alpha_{i_1}\alpha_{i_2} & \cdots & -\alpha_{i_1}\alpha_{i_k}\\ -\alpha_{i_1}\alpha_{i_2} & \alpha_{i_2}-\alpha_{i_2}^2 & \cdots & -\alpha_{i_2}\alpha_{i_k}\\ \cdots & \cdots & \cdots & \cdots\\ -\alpha_{i_1}\alpha_{k} & -\alpha_{i_2}\alpha_{i_k} & \cdots & \alpha_{i_k}-\alpha_{i_k}^2\end{vmatrix}=\prod_{m=1}^{k}\alpha_{i_m}\begin{vmatrix}1-\alpha_{i_1} & -\alpha_{i_1} & \cdots & -\alpha_{i_1}\\ -\alpha_{i_2} & 1-\alpha_{i_2} & \cdots & -\alpha_{i_2}\\ \cdots & \cdots & \cdots & \cdots\\ -\alpha_{i_k} & -\alpha_{i_k} & \cdots & 1-\alpha_{i_k}\end{vmatrix}$$

$$=(\prod_{m=1}^{k}\alpha_{i_m})\begin{vmatrix}1-\sum_{m=1}^{k}\alpha_{i_m} & 1-\sum_{m=1}^{k}\alpha_{i_m} & \cdots & 1-\sum_{m=1}^{k}\alpha_{i_m}\\ -\alpha_{i_2} & 1-\alpha_{i_2} & \cdots & -\alpha_{i_2}\\ \cdots & \cdots & \cdots & \cdots\\ -\alpha_{i_k} & -\alpha_{i_k} & \cdots & 1-\alpha_{i_k}\end{vmatrix}=(\prod_{m=1}^{k}\alpha_{i_m})(1-\sum_{m=1}^{k}\alpha_{i_m})\begin{vmatrix}1 & 1 & \cdots & 1\\ -\alpha_{i_2} & 1-\alpha_{i_2} & \cdots & -\alpha_{i_2}\\ \cdots & \cdots & \cdots & \cdots\\ -\alpha_{i_k} & -\alpha_{i_k} & \cdots & 1-\alpha_{i_k}\end{vmatrix}$$

$$=(\prod_{m=1}^{k}\alpha_{i_m})(1-\sum_{m=1}^{k}\alpha_{i_m})\begin{vmatrix}1 & 1 & \cdots & 1\\ 0 & 1 & \cdots & 0\\ \cdots & \cdots & \cdots & \cdots\\ 0 & 0 & \cdots & 1\end{vmatrix}=(\prod_{m=1}^{k}\alpha_{i_m})(1-\sum_{m=1}^{k}\alpha_{i_m})\geq 0 \qquad (10)$$

Then, A is positive semidefinite. Thus, formula (8) holds, and formula (7) holds. This ends of proof . □

If $0\leq\alpha_i\leq 1, i=1,\cdots,n, 0<\sum_{i=1}^{n}\alpha_i<1,$ we have the following theorem.

Theorem 3. Let $\alpha_1,\alpha_2,\cdots,\alpha_n$ be real numbers, $0\leq\alpha_i\leq 1, i=1,\cdots,n, 0\leq\sum_{i=1}^{n}\alpha_i<1$. Suppose $X_1,\cdots,X_n$ are any random variables with $Var(X_i)<+\infty$, $i=1,\cdots,n$ then

$$Var(\sum_{i=1}^{n}\alpha_i X_i)\leq\sum_{i=1}^{n}\alpha_i Var(X_i)\,. \qquad (11)$$

Proof. Examine the formula (10) in the proof of Theorem 1, the condition $0\leq\alpha_i\leq 1$, $0<\sum_{i=1}^{n}\alpha_i<1$, can guarantee that conclusion of formula (8) is still true. Hence, Theorem 3 holds.

□

While $\alpha_i\geq 0, i=1,\cdots,n, \sum_{i=1}^{n}\alpha_i>0$, since the weights can be normalized to sum 1, the upper bound of $Var(\sum_{i=1}^{n}\alpha_i X_i)$ is described as follows.

Theorem 4 Let $\alpha_1,\alpha_2,\cdots,\alpha_n$ be real numbers, $\alpha_i\geq 0, i=1,\cdots,n$. Suppose $X_1,\cdots,X_n$ are any random variables with $Var(X_i)<+\infty$, $i=1,\cdots,n$, then

$$Var(\sum_{i=1}^{n}\alpha_i X_i) \le (\sum_{i=1}^{n}\alpha_{\text{i}})\sum_{i=1}^{n}\alpha_i Var(X_i) \tag{12}$$

Proof. If $\alpha_i = 0, i = 1, \cdots, n$, formula (12) holds.

If $\sum_{i=1}^{n}\alpha_i > 0$, according to Theorem 2, we obtain

$$Var(\sum_{i=1}^{n}\alpha_i X_i) = Var[(\sum_{i=1}^{n}\alpha_i)\sum_{i=1}^{n}\frac{\alpha_i}{\sum_{i=1}^{n}\alpha_i}X_i] = (\sum_{i=1}^{n}\alpha_i)^2 Var[\sum_{i=1}^{n}\frac{\alpha_i}{\sum_{i=1}^{n}\alpha_i}X_i]$$

$$\le (\sum_{i=1}^{n}\alpha_i)^2\sum_{i=1}^{n}\frac{\alpha_i}{\sum_{i=1}^{n}\alpha_i}Var(X_i) \le (\sum_{i=1}^{n}\alpha_i)\sum_{i=1}^{n}\alpha_i Var(X_i)$$

□

In fact, the conclusion of Theorem 4 can also be obtained by checking the formula (6) in Theorem 1. Furthermore, since $Var(\pm X) = Var(X)$ and,

$$\sum_{i=1}^{n}\alpha_i X_i = \sum_{i=1}^{n}|\alpha_i|[\text{sgn}(\alpha_i)X_i], \tag{13}$$

where $\text{sgn}(\bullet)$ is the sign function. We extend the weights to general case without any limitation, and have the following conclusion.

Theorem 5 Let $\alpha_1, \cdots, \alpha_n$ be any n real numbers. Suppose $X_1, \cdots, X_n$ are any random variables with $Var(X_i) < +\infty$, $i = 1, \cdots, n$, then

$$Var(\sum_{i=1}^{n}\alpha_i X_i) \le (\sum_{i=1}^{n}|\alpha_{\text{i}}|)\sum_{i=1}^{n}|\alpha_i| Var(X_i) \tag{14}$$

Proof. If $\alpha_i = 0, i = 1, \cdots, n$, formula (14) holds.

If $\sum_{i=1}^{n}|\alpha_i| > 0$, according to Theorem 4, we obtain

$$Var(\sum_{i=1}^{n}\alpha_i X_i) = \sum_{i=1}^{n}Var(|\alpha_i|[\text{sgn}(\alpha_i)X_i]) \le (\sum_{i=1}^{n}|\alpha_{\text{i}}|)[\sum_{i=1}^{n}|\alpha_i| Var(\text{sgn}(\alpha_i)X_i)]$$

$$= (\sum_{i=1}^{n}|\alpha_{\text{i}}|)[\sum_{i=1}^{n}|\alpha_i| Var(X_i)]$$

□

Obviously, we can obtain the following corollaries.

Corollary 1. Let $X_1, \cdots, X_n$ be any n random variables. Suppose $X_1, \cdots, X_n$ are any random variables with $Var(X_i) < +\infty$, $i = 1, \cdots, n$, then

$$Var(\frac{1}{n}\sum_{i=1}^{n} X_i) \le \frac{1}{n}\sum_{i=1}^{n} Var(X_i) \tag{15}$$

Proof Let $\alpha_i = \frac{1}{n}, i = 1, \cdots, n$, according to Theorem 2, the conclusion holds. □

Corollary 1 shows that variance of mean could not exceed mean of variances for correlated random variables, it is also true for independent random variables through checking formula (4) and Example 1.

Corollary 2 If $0 \le |\alpha_i| \le 1, i = 1, \cdots, n, \sum_{i=1}^{n} |\alpha_i| \le 1,$ for any random variables $X_1, \cdots, X_n$ with $Var(X_i) < +\infty$, $i = 1, \cdots, n$, we have

$$Var(\sum_{i=1}^{n} \alpha_i X_i) \le (\sum_{i=1}^{n} |\alpha_i|)\sum_{i=1}^{n} |\alpha_i| Var(X_i) \le \sum_{i=1}^{n} |\alpha_i| Var(X_i) \le \sum_{i=1}^{n} Var(X_i) \tag{16}$$

Proof. As $0 \le |\alpha_i| \le 1, i = 1, \cdots, n, \sum_{i=1}^{n} |\alpha_i| \le 1,$ according to **Theorem 5,** the inequalities are true from left to right. □

If the variables are uncorrelated (or independent), the relationship among $Var(\sum_{i=1}^{n} \alpha_i X_i)$, $\sum_{i=1}^{n} \alpha_i^2 Var(X_i)$, and $\sum_{i=1}^{n} \alpha_i Var(X_i)$ are as follows.

Corollary 3 If $0 \le |\alpha_i| \le 1, i = 1, \cdots, n, \sum_{i=1}^{n} |\alpha_i| \le 1,$ and $X_1, \cdots, X_n$ be any n uncorrelated (or independent) random variables,

$$Var(\sum_{i=1}^{n} \alpha_i X_i) = \sum_{i=1}^{n} \alpha_i^2 Var(X_i) \le \sum_{i=1}^{n} |\alpha_i| Var(X_i) \le \sum_{i=1}^{n} Var(X_i) \tag{17}$$

Proof. According to formula (3) and $\alpha_i^2 \le |\alpha_i|$, while $0 \le |\alpha_i| \le 1$, $i = 1, \cdots, n$. The conclusion is obviously true. □

In addition, an inequality for the covariance and variance among correlated random variables is achieved as follows.

Corollary 4. Let $X_1,\cdots,X_n$ be any n correlated random variables, with $Var(X_i)<+\infty$, $i=1,\cdots,n$. Then

$$-\frac{1}{n}\sum_{i=1}^{n}Var(X_i)\le\frac{2}{n}\sum_{1\le i<j\le n}Cov(X_i,X_j)\le(1-\frac{1}{n})\sum_{i=1}^{n}Var(X_i) \tag{18}$$

Proof. $Var(\frac{1}{n}\sum_{i=1}^{n}X_i)=\frac{1}{n^2}Var(\sum_{i=1}^{n}X_i)=\frac{1}{n^2}[\sum_{i=1}^{n}Var(X_i)+2\sum_{1\le i<j\le n}Cov(X_i,X_j)]$

$$=\frac{1}{n}[\frac{1}{n}\sum_{i=1}^{n}Var(X_i)+\frac{2}{n}\sum_{1\le i<j\le n}Cov(X_i,X_j)] \tag{19}$$

As $Var(\frac{1}{n}\sum_{i=1}^{n}X_i)\ge 0$, the left inequality is proved.

Using formula (15) $Var(\frac{1}{n}\sum_{i=1}^{n}X_i)\le\frac{1}{n}Var(\sum_{i=1}^{n}X_i)$ and formula (19) , the right inequality is obtained.

□

3. Chebyshev inequality of correlated random variables

Chebyshev's inequality is an important probability formula, when sum of correlated random variables are considered, we have the following inequality.

Theorem 6 (Chebyshev inequality) Let $\alpha_1,\cdots,\alpha_n$ be any n real numbers, and $X_1,\cdots,X_n$ be any n correlated random variables with $Var(X_i)<+\infty$, $i=1,\cdots,n$. For any $\delta>0$, we have

$$P(|\sum_{i=1}^{n}\alpha_i X_i-\sum_{i=1}^{n}\alpha_i EX_i|>\delta)\le\frac{(\sum_{i=1}^{n}|\alpha_i|)}{\delta^2}\sum_{i=1}^{n}|\alpha_i|Var(X_i) \tag{20}$$

Proof.

According to Chebyshev inequality and Theorem 5, we obtain

$$P(|\sum_{i=1}^{n}\alpha_i X_i-\sum_{i=1}^{n}\alpha_i EX_i|>\delta)\le\frac{Var(\sum_{i=1}^{n}\alpha_i X_i)}{\delta^2}\le\frac{(\sum_{i=1}^{n}|\alpha_i|)}{\delta^2}\sum_{i=1}^{n}|\alpha_i|Var(X_i)$$

□

Let $\alpha_i = \frac{1}{n}, i = 1, \cdots, n$, according to formula (20), we have the following conclusion.

Corollary 5. Let $X_1, \cdots, X_n$ be any n correlated random variables with $Var(X_i) < +\infty$, $i = 1, \cdots, n$. For any $\delta > 0$,

$$P(|\frac{1}{n}\sum_{i=1}^{n} X_i - \frac{1}{n}\sum_{i=1}^{n} EX_i| > \delta) \le \frac{1}{\delta^2}[\frac{1}{n}\sum_{i=1}^{n} Var(X_i)] \tag{21}$$

When $X_1, \cdots, X_n$ are uncorrelated (or independent) random variables, the Chebyshev's inequality is

$$P(|\frac{1}{n}\sum_{i=1}^{n} X_i - \frac{1}{n}\sum_{i=1}^{n} EX_i| > \delta) \le \frac{1}{\delta^2}[\frac{1}{n^2}\sum_{i=1}^{n} Var(X_i)] \tag{22}$$

Formula (21) illustrates a relationship among $\frac{1}{n}\sum_{i=1}^{n} X_i$, $\frac{1}{n}\sum_{i=1}^{n} EX_i$ and $\frac{1}{n}\sum_{i=1}^{n} Var(X_i)$, and it could be applied to probability estimation of $\frac{1}{n}\sum_{i=1}^{n} X_i$, regardless of the correlation among $X_1, \cdots, X_n$. When $X_1, \cdots, X_n$ are uncorrelated or independent, the formula (21) sharpens to formula (22). This conclusion also demonstrates that correlation among variables enlarges the departure of $\frac{1}{n}\sum_{i=1}^{n} X_i$ from its expectation.

Corollary 6. Let $X_1, \cdots, X_n$ be any n correlated random variables, with $0 < Var(X_i) < +\infty$, $i = 1, \cdots, n$. For any $\delta > 0$,

$$P(|\frac{1}{n}\sum_{i=1}^{n} \frac{X_i - EX_i}{\sqrt{VarX_i}}| > \delta) \le \frac{1}{\delta^2} \tag{23}$$

$$P(|\sum_{i=1}^{n} \frac{X_i - EX_i}{\sqrt{VarX_i}}| > \delta) \le \frac{n^2}{\delta^2} \tag{24}$$

Proof. According to formula (21)

$$P(|\frac{1}{n}\sum_{i=1}^{n} \frac{X_i - EX_i}{\sqrt{VarX_i}}| > \delta) \le \frac{1}{\delta^2}[\frac{1}{n}\sum_{i=1}^{n} Var(\frac{X_i}{\sqrt{VarX_i}})] = \frac{1}{\delta^2}.$$

Thus, formula (23) holds.

According to formula (20), for variables $\frac{X_1}{\sqrt{VarX_1}}, \cdots, \frac{X_n}{\sqrt{VarX_n}}$ with weights $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 1$, we have

$$P(|\sum_{i=1}^{n} \frac{X_i - EX_i}{\sqrt{VarX_i}}| > \delta) \le \frac{1}{\delta^2}[n\sum_{i=1}^{n} Var(\frac{X_i}{\sqrt{VarX_i}})] = \frac{n^2}{\delta^2}.$$

Then, formula (24) holds.

This ends of the proof. □

Obviously, an alternative proof of formula (24) is easily derived from formula (23) as follows.

$$P(|\sum_{i=1}^{n} \frac{X_i - EX_i}{\sqrt{VarX_i}}| > \delta) = P(|\frac{1}{n}\sum_{i=1}^{n} \frac{X_i - EX_i}{\sqrt{VarX_i}}| > \frac{\delta}{n}) \le \frac{1}{(\frac{\delta}{n})^2} \le \frac{n^2}{\delta^2}.$$

The Chebyshev inequalities in **Theorem 6, Corollary 5** and **Corollary** 6 are different from the Chebyshev inequality in Theorem 1.1 in [7]. In [7], the Chebyshev inequality models the probability of "$(X - EX)^T \Sigma^{-1}(X - EX) > \varepsilon$", where $X = (X_1, \cdots, X_n)^T$, Σ is a positive matrix, its inverse event "$(X - EX)^T \Sigma^{-1}(X - EX) \le \varepsilon$" can be treated as the random event of the vector X from expectation EX with a Mahalanobis distance in high dimension random vector space [8]. And, "$(X - EX)^T \Sigma^{-1}(X - EX) \le \varepsilon$" means an n-dimension ellipsoid neighborhood, while the event "$|\sum_{i=1}^{n} \frac{X_i - EX_i}{\sqrt{VarX_i}}| \le \delta$"represents a more complex neighborhood. Since, the event "$\sum_{i=1}^{n} |\frac{X_i - EX_i}{\sqrt{VarX_i}}| \le \delta$" represents an n-dimension cube neighborhood of scaled random vector $(\frac{X_1}{\sqrt{VarX_1}}, \frac{X_2}{\sqrt{VarX_2}}, \cdots, \frac{X_n}{\sqrt{VarX_n}})$ from expectation $(\frac{EX_1}{\sqrt{VarX_1}}, \frac{EX_2}{\sqrt{VarX_2}}, \cdots, \frac{EX_n}{\sqrt{VarX_n}})$, and "$\sum_{i=1}^{n} |\frac{X_i - EX_i}{\sqrt{VarX_i}}| \le \delta$" $\subset$ "$|\sum_{i=1}^{n} \frac{X_i - EX_i}{\sqrt{VarX_i}}| \le \delta$", the event "$|\sum_{i=1}^{n} \frac{X_i - EX_i}{\sqrt{VarX_i}}| > \delta$" is included in the domain of event "$\sum_{i=1}^{n} |\frac{X_i - EX_i}{\sqrt{VarX_i}}| > \delta$". The intuitionistic domain of "$|\sum_{i=1}^{n} \frac{X_i - EX_i}{\sqrt{VarX_i}}| > \delta$" in this paper is slight complex. Hence, **Theorem 6, Corollary** 5 and **Corollary** 6 address another Chebyshev inequality of high dimensional random variables.

In other words, for the sequence of random variables $X_1, \cdots, X_n, \cdots$, let $X = (X_1, \cdots, X_n)^T$, $S_n = \frac{1}{n}\sum_{i=1}^{n} X_i$. $(X - EX)^T \Sigma^{-1}(X - EX)$ is the quadratic form of $X_1, \cdots, X_n$, it's a n-dimension distance. However, $|S_n - ES_n|$ is the one-dimension distance of sum S_n from its expectation ES_n. The representation difference of mathematical formulae also shows the different between Theorem 6, Corollary 5 and Corollary 6 and Theorem 1.1 in [7].

4. Law of large numbers of correlated random variables

Suppose $X_1, \cdots, X_n, \cdots$ are any random variable sequences, let $S_n = \frac{1}{n}\sum_{i=1}^{n} X_i$, for any $\varepsilon > 0$, if

$$\frac{1}{n^2} Var(\sum_{i=1}^{n} X_i) \to 0 \qquad (n \to \infty), \tag{25}$$

Then,

$$\lim_{n \to \infty} P(|\frac{1}{n}\sum_{i=1}^{n} X_i - \frac{1}{n}\sum_{i=1}^{n} EX_i | < \varepsilon) = 1 \tag{26}$$

(25) is the Markov sufficient condition [6,9] of (26). (26) is denoted as $\frac{1}{n}\sum_{i=1}^{n} X_i \xrightarrow{P} \frac{1}{n}\sum_{i=1}^{n} EX_i$, and called weak law of large numbers (WLLN).

If any of two variables of $X_1, \cdots, X_n, \cdots$ are uncorrelated, and there exists a constant $C > 0$,

$$DX_i \le C, \tag{27}$$

(26) holds. It is called Chebyshev Law of large numbers [6,9].

When $X_1, \cdots, X_n, \cdots$ are correlated, the bound condition (27) could not ensure that (25) is true, especially when the correlations are unknown. According to (15), we have a Markov Law of large numbers.

Theorem 7 (Markov Law of large numbers) Let $X_1, \cdots, X_n, \cdots$ be any correlated random variables sequences with $Var(X_i) < +\infty$, $i = 1, 2, \cdots$. For any $\varepsilon > 0$, if

$$\frac{1}{n}\sum_{i=1}^{n} Var(X_i) \to 0 \qquad (n \to +\infty) \tag{28}$$

Then,

$$\lim_{n\to\infty} P(|\frac{1}{n}\sum_{i=1}^{n} X_i - \frac{1}{n}\sum_{i=1}^{n} EX_i| < \varepsilon) = 1$$

Proof. According to (15)

$$\frac{1}{n^2}Var(\sum_{i=1}^{n} X_i) = Var(\frac{1}{n}\sum_{i=1}^{n} X_i) \le \frac{1}{n}Var(\sum_{i=1}^{n} X_i)$$

Using (28) and (21), we have $\lim_{n\to\infty} P(|\frac{1}{n}\sum_{i=1}^{n} X_i - \frac{1}{n}\sum_{i=1}^{n} EX_i| < \varepsilon) = 1$.

□

(28) is a sufficient condition for law of large numbers without known the correlation between variables. Theorem 7 provides a convenient condition to verify the law of large numbers or probability convergence of correlated random variable sequence.

Furthermore, we can provide another sufficient condition with the following Lemma.

Lemma 1. Let $X_1, \cdots, X_n, \cdots$ be any correlated random variables sequence with $Var(X_i) < +\infty$, $i = 1, 2, \cdots$. For any $0 < r \le s$, we have,

$$[\frac{1}{n}\sum_{i=1}^{n}(Var(X_i))^r]^{\frac{1}{r}} \le [\frac{1}{n}\sum_{i=1}^{n}(Var(X_i))^s]^{\frac{1}{s}} \tag{29}$$

Proof. For nonnegative values $Var(X_1), \cdots, Var(X_n)$, we define a random variable ξ with probability distribution

$$P(\xi = Var(X_i)) = \frac{1}{n}, \quad i = 1, \cdots, n.$$

For any $0 < r \le s$, using the Lyapunov's inequality [3], we have

$$(E|\xi|^r)^{\frac{1}{r}} \le (E|\xi|^s)^{\frac{1}{s}},$$

which is

$$[\sum_{i=1}^{n}\frac{1}{n}(Var(X_i))^r]^{\frac{1}{r}} \le [\sum_{i=1}^{n}\frac{1}{n}(Var(X_i))^s]^{\frac{1}{s}}.$$

Hence, (29) holds. □

Utilizing (29), we can have the following theorem.

Theorem 8 (Markov Law of large numbers) Let $X_1, \cdots, X_n, \cdots$ be any correlated random variables sequences with $Var(X_i) < +\infty$, $i = 1, 2, \cdots$. For any $\varepsilon > 0$, $s \geq 1$, if

$$[\frac{1}{n}\sum_{i=1}^{n}(Var(X_i))^s]^{\frac{1}{s}} \to 0 \qquad (n \to +\infty) \tag{30}$$

Then,

$$\lim_{n\to\infty} P(|\frac{1}{n}\sum_{i=1}^{n} X_i - \frac{1}{n}\sum_{i=1}^{n} EX_i| < \varepsilon) = 1$$

Proof. Let $r = 1$, $s \geq 1$ in (27), we have

$$\frac{1}{n}\sum_{i=1}^{n} Var(X_i) \leq [\frac{1}{n}\sum_{i=1}^{n}(Var(X_i))^s]^{\frac{1}{s}} \tag{31}$$

According to (30)(31), the condition (25) holds. Applying Theorem 7, we have

$$\lim_{n\to\infty} P(|\frac{1}{n}\sum_{i=1}^{n} X_i - \frac{1}{n}\sum_{i=1}^{n} EX_i| < \varepsilon) = 1.$$

□

Now, we give an example of correlated random variables sequence.

Example 5. Let $X_1, \cdots, X_n, \cdots$ be independent and identical distribution $N(\mu, \sigma^2)$ ($\sigma > 0$).

Let $S_n = \frac{1}{n}\sum_{i=1}^{n} X_i$, then $E(S_n) = \mu$, $Var(S_n) = \frac{\sigma^2}{n}$, $n = 1, 2, \cdots$.

$$\begin{aligned}
&Cov(S_i, S_{i+k}) = E(S_i - ES_i)(S_{i+k} - ES_{i+k}) = E(S_i S_{i+k}) - ES_i ES_{i+k} \\
&= E[\frac{1}{i}\sum_{j=1}^{i} X_j (\frac{1}{i+k}\sum_{j=1}^{i} X_j + \frac{1}{i+k}\sum_{j=i+1}^{i+k} X_j)] - ES_i ES_{i+k} \\
&= \frac{1}{i}\frac{1}{i+k}E(\sum_{j=1}^{i} X_j)^2 + \frac{1}{i}\frac{1}{i+k}E(\sum_{j=1}^{i} X_j \sum_{j=i+1}^{i+k} X_j) - ES_i ES_{i+k} \\
&= \frac{\sigma^2 + i\mu^2}{(i+k)} + \frac{1}{i}\frac{1}{i+k} ik\mu^2 - \mu^2 = \frac{\sigma^2}{(i+k)}
\end{aligned}$$

For the correlated random variable sequence $S_1, S_2, \cdots, S_n, \cdots$, let $Y_n = \frac{1}{n}\sum_{i=1}^{n} S_i$. We have,

$$EY_n = E(\frac{1}{n}\sum_{i=1}^{n} S_i) = \mu,$$

The Markov sufficient condition (25),

$$Var(Y_n)=Var(\frac{1}{n}\sum_{i=1}^{n}S_i)=\frac{1}{n^2}Var(\sum_{i=1}^{n}S_i)=\frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}Cov(S_i,S_j)=\frac{1}{n^2}[\sum_{i=1}^{n}\frac{\sigma^2}{i}+2\sum_{1\le i<j\le n}\frac{\sigma^2}{j}]$$
$$=\frac{1}{n^2}[\sum_{i=1}^{n}\frac{\sigma^2}{i}+2\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\frac{\sigma^2}{j}]=\frac{1}{n^2}[\sum_{i=1}^{n}\frac{\sigma^2}{i}+2\sum_{j=2}^{n}(j-1)\frac{\sigma^2}{j}]$$
$$=\frac{1}{n^2}[\sum_{i=1}^{n}\frac{\sigma^2}{i}+2(n-1)\sigma^2-2\sum_{j=2}^{n}\frac{\sigma^2}{j}]=\frac{1}{n^2}[2n\sigma^2-\sum_{i=1}^{n}\frac{\sigma^2}{i}]$$
$$=\frac{\sigma^2}{n}[2-\frac{1}{n}(C+\ln n+\varepsilon_n)]\to 0,\quad n\to+\infty$$

The Markov sufficient condition (28) in Theorem 7,

$$Var(Y_n)\le\frac{1}{n}\sum_{i=1}^{n}Var(S_i)=\frac{1}{n}\sum_{i=1}^{n}\frac{\sigma^2}{i}=\frac{\sigma^2}{n}(C+\ln n+\varepsilon_n)\to 0,\quad n\to+\infty$$

where $C=0.577216\cdots$ is Euler constant, $\varepsilon_n\to 0,\quad n\to+\infty$.

Then，for any $\varepsilon>0$， $\lim_{n\to\infty}P(|\frac{1}{n}\sum_{i=1}^{n}S_i-\frac{1}{n}\sum_{i=1}^{n}ES_i|<\varepsilon)=1$.

Example 5 shows that the law of large numbers of the correlated random variable sequence $S_1,S_2,\cdots,S_n,\cdots$ holds, and the sufficient condition (28) is simpler to be calculated than the sufficient condition (25). Note that Theorem 7 works in the case of non-stationary time series [11,12]. For the stationary time series, Theorem 7 fails.

As for the correlated random variables satisfying (27), we have the following theorem.

Theorem 9 (Chebyshev Law of large numbers) Let $X_1,\cdots,X_n,\cdots$ be any correlated random variables sequence. If there exists a constant $C>0$, $Var(X_i)<C$, $i=1,2,\cdots$, for any $\varepsilon>0$, $s>0$, then,

$$\lim_{n\to\infty}P(|\frac{1}{n^{1+s}}\sum_{i=1}^{n}X_i-\frac{1}{n^{1+s}}\sum_{i=1}^{n}EX_i|<\varepsilon)=1$$

Proof:

$$Var(\frac{1}{n^{1+s}}\sum_{i=1}^{n}X_i)=\frac{1}{n^{2+2s}}Var(\sum_{i=1}^{n}X_i)\le\frac{1}{n^{2+2s}}n\sum_{i=1}^{n}Var(X_i)\le\frac{1}{n^{1+2s}}\sum_{i=1}^{n}Var(X_i)$$
$$\le\frac{1}{n^{1+2s}}nC=\frac{1}{n^{2s}}C\to 0,\qquad n\to\infty \quad . \qquad (32)$$

According to (21)(32), we have

$$\lim_{n\to\infty} P(|\frac{1}{n^{1+s}}\sum_{i=1}^{n} X_i - \frac{1}{n^{1+s}}\sum_{i=1}^{n} EX_i| < \varepsilon) = 1.$$

□

As a special case of Theorem 9, we can obtain the theorem for the random variables with same expectation and variance.

Theorem 10 Let $X_1,\cdots,X_n,\cdots$ be any correlated random variable sequence with same expectation $E(X_i)=\mu$ and variance $Var(X_i)=\sigma^2$, $i=1,2,\cdots$, for any $\varepsilon>0$, $s>0$, then

$$\lim_{n\to\infty} P(|\frac{1}{n^{1+s}}\sum_{i=1}^{n} X_i - \frac{1}{n^{1+s}}\sum_{i=1}^{n} EX_i| < \varepsilon) = 1.$$

Proof. The proof is similar to Theorem 9. We omitted it. □

Corresponding to the famous Bernoulli Law of large numbers of independent and identical distribution (iid) random varables , we have

Theorem 11(Bernoulli Law of large numbers) Let $X_1,\cdots,X_n,\cdots$ be any correlated random variables sequence with identically distribution:

$P(X_i=1)=p$, $P(X_i=0)=1-p$, $i=1,2,\cdots$, $(0<p<1)$

For any $\varepsilon>0$, $s>0$, then

$$\lim_{n\to\infty} P(|\frac{1}{n^{1+s}}\sum_{i=1}^{n} X_i - \frac{p}{n^s}| < \varepsilon) = 1.$$

Theorem 9,10,11 discuss the stable of $\{\frac{1}{n^{1+s}}\sum_{i=1}^{n} X_i\}$, motivated by Bernoulli Law of large numbers in 1713, we are still interested in the stable of $\{\frac{1}{n}\sum_{i=1}^{n} X_i\}$, as its significance lies in not only the extension of Bernoulli WLLN, but also the moment method of estimation of point estimation [13], we continue to discuss the WLLN of correlated random variables.

To illustrate our motivation, we first give an example of correlated random variables of 0-1 distribution $b(1,p)$.

Example 6.（Random telegraph signals ([14],p24)） Let $\{X(t), t\in(-\infty,+\infty)\}$ be a stochastic process with state space $S=\{0,1\}$, which satisfies

(1) For any $t \in (-\infty, +\infty)$, $P(X(t)=0) = P(X(t)=1) = \frac{1}{2}$.

(2) For any t, the number of zero crossings in the interval $(\mathrm{t},\ \mathrm{t}+\Delta\mathrm{t})$ is described by a Poisson process $N(t+\Delta t) - N(t) \sim \Pi(\lambda \Delta t)$, $\lambda > 0$.

(3) $X(t)$ is independent to $N(t+\Delta t) - N(t)$.

$\{X(t), t \in (-\infty, +\infty)\}$ are called random telegraph signals. The special case $\{X(n), n = 0,1,2,\cdots\cdots\}$ can be treated as the correlated case of Bernoulli sequences. We discuss the WLLN of $\{X(n), n = 0,1,2,\cdots\cdots\}$.

The expected value is

$$E[X(t)] = 0 \times P\{X(t)=0\} + 1 \times P\{X(t)=1\} = \frac{1}{2}$$

For any $\tau \ge 0$,

$$\{X(t)X(t+\tau)=1\} = \{X(t)=1, X(t+\tau)=1\} = \{X(t)=1, N(t+\tau)-N(t)=\text{even number}\}$$

$$= \{X(t)=1, \bigcup_{k=0}^{\infty}\{N(\tau)=2k\}\}$$

$$P\{X(t)X(t+\tau)=1\} = P\{X(t)=1\}P\left\{\bigcup_{k=0}^{\infty} N(\tau)=2k\right\} = \frac{1}{2}\sum_{k=0}^{\infty} P\{N(\tau)=2k\} = \frac{1}{4}[1+e^{-2\lambda\tau}]$$

The autocorrelation function is

$$R_X(t,t+\tau) = E[X(t)X(t+\tau)] = 1 \times P\{X(t)X(t+\tau)=1\} + 0 \times P\{X(t)X(t+\tau)=0\} = \frac{1}{4}[1+e^{-2\lambda\tau}]$$

$$Cov(X(t), X(t+\tau)) = \frac{1}{4}e^{-2\lambda\tau}$$

Therefore, {X(n)} is correlated sequences.

$$D(\frac{1}{n}\sum_{k=1}^{n} X_k) = \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n} Cov(X_i, X_j) = \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{1}{4}e^{-2\lambda|j-i|} = \frac{1}{4n^2}\sum_{i=1}^{n}\sum_{j=1}^{n} e^{-2\lambda|j-i|}$$

$$= \frac{1}{4n^2}[n + 2\sum_{k=1}^{n-1}(n-k)e^{-2\lambda k}] = \frac{1}{4n^2}[n + 2\sum_{k=1}^{n-1}(n-k)e^{-2\lambda k}]$$

$$= \frac{1}{4n^2}[n + 2\sum_{k=1}^{n-1} ne^{-2\lambda k} - 2\sum_{k=1}^{n-1} ke^{-2\lambda k}]$$

$$= \frac{1}{4n}[1 + 2\frac{e^{-2\lambda} - e^{-2\lambda n}}{1 - e^{-2\lambda}} - \frac{2}{n}\sum_{k=1}^{n-1} ke^{-2\lambda k}] \to 0,\ \ n \to \infty \qquad (33)$$

$$\frac{1}{n}\sum_{k=1}^{n} D(X_k) = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{4} = \frac{1}{4} \not\to 0,\ \ n \to \infty \qquad (34)$$

If we extend the assumption of (1) of *Example* 6 to $P(X(t)=1) = p$, $P(X(t)=0) = 1-p$ $(0 < p < 1)$, $\{X(n), n = 0,1,2,\cdots\cdots\}$ can be treated is the correlated case of Bernoulli sequence, it is a stationary sequence. Formula (33) shows that the WLLN still holds, while (34) shows Theorem 7 fails to verify this problem. Hence, we modify Theorem 7 as follows,

Theorem 12 (Markov Law of large numbers) Let $X_1,\cdots,X_n,\cdots$ be any correlated random variables sequences with $Var(X_i)<+\infty$, $i=1,2,\cdots$. For any $\varepsilon>0$, if

$$(1)\frac{1}{n}\sum_{i=1}^{n}Var(X_i)\to 0 \qquad (n\to+\infty) \tag{35}$$

OR

(2) $\exists C>0$, $0<s<1$, $|\frac{1}{n}\sum_{i=1}^{n}Var(X_i)|<C$, or $|\frac{1}{n}\sum_{i=1}^{n}Var(X_i)|=O(n^s)$, and

$$|\frac{1}{n}\sum_{1\le i<j\le n}Cov(X_i,X_j)|<C \text{, or } |\frac{1}{n}\sum_{1\le i<j\le n}Cov(X_i,X_j)|=O(n^s) . \tag{36}$$

Then,

$$\lim_{n\to\infty}P(|\frac{1}{n}\sum_{i=1}^{n}X_i-\frac{1}{n}\sum_{i=1}^{n}EX_i|<\varepsilon)=1,$$

where $|\frac{1}{n}\sum_{i=1}^{n}Var(X_i)|=O(n^s)$ denotes that $|\frac{1}{n}\sum_{i=1}^{n}Var(X_i)|$ is the same order infinite of n^s.

Proof:

(1) is the conclusion of Theorem 7.

(2) Since

$$Var(\frac{1}{n}\sum_{i=1}^{n}X_i)=\frac{1}{n}[\frac{1}{n}\sum_{i=1}^{n}Var(X_i)+\frac{2}{n}\sum_{1\le i<j\le n}Cov(X_i,X_j)]$$

When $\exists C>0$, $0<s<1$, $|\frac{1}{n}\sum_{i=1}^{n}Var(X_i)|<C$, or $|\frac{1}{n}\sum_{i=1}^{n}Var(X_i)|=O(n^s)$, and

$|\frac{1}{n}\sum_{1\le i<j\le n}Cov(X_i,X_j)|<C$, or $|\frac{1}{n}\sum_{1\le i<j\le n}Cov(X_i,X_j)|=O(n^s)$ hold, we have

$$Var(\frac{1}{n}\sum_{i=1}^{n}X_i)=\frac{1}{n}[\frac{1}{n}\sum_{i=1}^{n}Var(X_i)+\frac{2}{n}\sum_{1\le i<j\le n}Cov(X_i,X_j)]\to 0,\ n\to\infty.$$

Using Chebyshev's inequality,

$$P(|\frac{1}{n}\sum_{i=1}^{n}X_i-\frac{1}{n}\sum_{i=1}^{n}EX_i|>\varepsilon)\le\frac{1}{\varepsilon^2}Var(\frac{1}{n}\sum_{i=1}^{n}X_i)\to 0,\ n\to\infty.$$

Which ends the proof of the theorem. □

Then, the discussion of WLLN of famous Chebyshev and Bernoulli Law of large numbers [15,16] could be verified by the second condition of Theorem 12 when we discuss the correlated case.

In fact, Theorem 7,8,9,10,11,12 discuss the stable [9, P61-62, 66-67] in the case of correlated random variables (no independence assumption). If any two random variables are uncorrelated (or independent), Theorem 7,8,9,10,11,12 still hold, where $s = 0$. Theorem 12 could be applied to the first moment estimation in stationary and non-stationary (while first moment is constant) time series. And, we can also construct the high order moment estimation method by generalize Theorem 12 to discuss $\lim_{n\to\infty} P(|\frac{1}{n}\sum_{i=1}^{n} X_i^r - \frac{1}{n}\sum_{i=1}^{n} EX_i^r | < \varepsilon) = 1$,where $r > 0$, we omit the discussion for the sake of paper length.

In addition, the upper bound in Theorem 1' can be similarly applied to the discussions of Chebyshev inequality and law of large numbers, we omit them here. And, the theoretical discussion will be given in the future work.

5. Conclusion

The variance inequalities for weighted sum of correlated random variables are established by two methods Cauchy-Schwarz's inequality and positive semidefinite matrix. And the Chebyshev inequality is also extended to sum of correlated random variables. We also discuss the Markov WLLN, Chebyshev WLLN and Bernoulli WLLN in correlated random variables. In our discussion, independence and identical distribution are not necessary, and we only discuss the random variables with second moments. The WLLN discuss of correlated random variables could be utilized in the stationary or non-stationary time series, when their first moments are the same constant, the point estimation holds, otherwise the WLLN describes the stable convergence problem. Future work will focus on applying the conclusions to modifying probability inequalities and analyzing statistical data in various applied fields.

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