OPTIMAL CONTROL OF DAMS USING $\mathrm{P}_{\lambda,\tau}^{M}$ POLICIES AND PENALTY COST WHEN THE INPUT PROCESS IS AN INVERSE GAUSSIAN PROCESS

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ABSTRACT

We consider $P_{\lambda,\tau}^{M}$ policy of a dam in which the water input is an inverse Gaussian process. The release rate of the water is changed from 0 to M and from M to 0 ($M > 0$) at the moments when the water level up crosses levels λ and down crosses level τ ($\tau < \lambda$), respectively. We determine the resolvent of the dam content and compute the total discounted as well as the long-run-average cost. We also find the stationary distribution of the dam content.

I. INTRODUCTION

Adel-Hameed (2000) discuss the optimal control of a dam using $\mathrm{P}_{\lambda,\tau}^{M}$ policies, using total discounted cost as well as the long -run average cost. He assumes that the water input is a compound Poisson process with a positive drift. The release rate is zero until the water reaches level λ, then it is released at rate M until it reaches level ($\tau < \lambda$), once the water reaches level τ the release rate remains zero until level λ is reached again and the cycle is repeated. At any time , the release rate can be increased from zero to M with a starting cost K_1M , or decreased from M to zero with a closing cost K_2M . Moreover, for each unit of output, a reward R is received. Furthermore, there is a penalty cost which accrues at a rate g, where g is a bounded measurable function on the state space. When the release rate is zero, the water content is denoted by $I = (I_t)$. Let W_λ be the first passage time of the process I through the boundary λ. When the release rate is M the content process is denoted by $I^* = (I_t^*)$, the process I^* is a strong Markov process. Let W_τ^* be the first passage of process I^* to the boundary τ. Over time, the content process is obtained by hitching independent copies of the processes I and I^* together. The content process is best described by the bivariate process $B = (Z, R)$, where $Z = (Z_t), R = (R_t)$ describe the dam content, and the release rate respectively. We define the following sequence of stopping times :

$$\begin{aligned}
\hat{T}_0 &= \inf\{t \geq 0 : Z_t \geq \lambda\}, \quad \overset{*}{T}_0 = \inf\{t \geq \hat{T}_0 : Z_t = \tau\} \\
\hat{T}_n &= \inf\{t \geq \overset{*}{T}_{n-1} : Z_t \geq \lambda\}, \; \overset{*}{T}_n = \inf\{t \geq \hat{T}_n : Z_t = \tau\}, \; for \; n \geq 1.
\end{aligned}$$

It follows that the process B is a delayed regenerative process with the regeneration points being the $\overset{*}{T}_n, \ n = 0, 1, ...$. The penalty cost rate function is defined as follows

$$g(z,r) = \begin{cases} g(z) & (z,r) \in (0,\lambda) \times \{0\} \\ g^*(z) & (z,r) \in (\tau,\infty) \times \{M\} \end{cases}$$

where $g : (0, \lambda) :\to R_+, \ g^* : (\tau, \infty) :\to R_+$ are bounded measurable functions.

For any process $Y = (Y_t, t \geq 0\}$, and functional f, $E_y(f)$ denotes the expectation of f conditional on $Y_0 = y$, and $P_y(A)$ denotes the corresponding probability measure. Throughout, we let $R_+ = [0, \infty), \ \ N = \{1, 2, ...\}, N_+ = \{0, 1, ...\}$, and $I_A(\ )$ be the indicator function of any set A. Let $C_g^\alpha(0, x, \lambda)$, be the expected discounted penalty costs during the interval $(0, W_\lambda)$ starting at x, and $C_{g^*}^\alpha(M, y, \tau)$ be the expected discounted penalty costs , starting at y, during the interval $(0, W_\tau^*)$. Furthermore, let $C_g(0, x, \lambda), C_{g^*}(M, y, \tau)$ be the expected non-discounted penalty costs during the same intervals. respectively. It follows that

$$\begin{aligned} C_g^\alpha(0,x,\lambda) &= E_x \int_0^{W_\lambda} e^{-\alpha t} g(I_t)dt, \ C_{g^*}^\alpha(M,y,\tau) \ = E_y \int_0^{W_\tau^*} e^{-\alpha t} g(I_t^*)dt, \\ C_g(0,x,\lambda) &= E_\tau \int_0^{W_\lambda} g(I_t)dt, \ \ C_{g^*}(M,y,\tau) \ = E_y \int_0^{W_\tau^*} g^*(I_t^*)dt \end{aligned}$$

Bae *et al.* (2003) consider the average cost case of the above model, when the dam has a finite capacity. Abdel-Hameed and Nakhi (2006) treat the case where the water input is a diffusion process. The release rate depends on the water content. Dohi *et al.* (1995) consider the case where the water must be released at a fixed time T. They assume that the input is a wiener process. In this paper, we discuss the case where the water input is an inverse Gaussian process. The assumption that the water input is of that type is more realistic than the cases where it is assumed that it is a Wiener process. This is true because the inverse Gaussian process has increasing sample paths. We determine the total discounted as well as the long -run average costs. In Section II, we discuss the resolvent operators of the processes of interest. In Section III we obtain formulas for the cost functional using the total discounted as well as the long-run average cost criteria.

II. THE DAM CONTENT PROCESS AND ITS CHARACTERISTICS

Assume that the water input in the dam $I = \{I_t, t \geq 0\}$ is an inverse Gaussian process, with parameters $\mu, \ \sigma^2 \ \ \geq 0$. We let $p(t, x, y\ )$ be the probability transition function of the process I given $I_0 = x$, and $p(t, y\ )$ be its probability transition function given $I_0 = 0$. Since the process I is additive, we have that for $y \geq 0$ and $x \ \geq 0$

$$\begin{aligned} p(t, x, y\ ) &= p(t, y - x\ ) \\ &= \begin{cases} \frac{t}{\sigma\sqrt{2\pi(y-x)^3}} \exp\{-\frac{[\mu(y-x)-t]^2}{2(y-x)\sigma^2}\}, \ y > x \\ 0 \qquad\qquad\qquad\qquad\qquad\qquad , y \leq x. \end{cases} \end{aligned} \tag{1}$$

It is known that I is a pure-jump process, with state space $(0, \infty)$,and with jump measure ν concentrated on $(0, \infty)$ given by

$$\nu(dy) = \frac{1}{\sigma\sqrt{2\pi}} \frac{\exp(-y\mu^2/2\sigma^2)}{y^{3/2}}$$

It follows that for $\alpha \geq 0$

$$Ee^{-\alpha I_t} = e^{-\frac{t}{\sigma^2}\{\sqrt{2\alpha\sigma^2+\mu^2}-\mu\}}$$

Direct differentiation of the above function gives, $EX_t = \frac{t}{\mu}$, and $Var(X_t) = \frac{t\sigma^2}{\mu^3}$.

To evaluate the cost functional and other parameters of the process during the first part of a cycle, we define the Levy process killed at λ, as follows:

$$X = \{I_t, t < W_\lambda\}$$

Let U_α be the resolvent operator of the process I, defined for every bounded function f by $U_\alpha f(x) = \int\limits_0^\infty f(x + y)U_\alpha(dy)$. Then U_α is the unique solution of the equation

$$\int\limits_0^\infty f(x + y)U_\alpha(dy) = E_x \int\limits_0^\infty e^{-\alpha t} f(X_t)dt.$$

It follows with $f_\beta(x) = exp(-\beta x), \beta \geq 0$, that

$$U_\alpha f_\beta(0) = \frac{\sigma^2}{\alpha\sigma^2 + \{\sqrt{2\beta\sigma^2 + \mu^2} - \mu\}.}$$

Throughout we let $\varphi_z(.)$ as the standard normal density function, erf() and erf $c()$ be the well known error and complimentary error functions, respectively. Inverting the above function w.r.t β we have

$$\begin{aligned} U_\alpha(dy) &= \frac{\sigma}{\sqrt{y}}\varphi_z(\sqrt{y}\mu/\sigma)dy + (\frac{\mu - \alpha\sigma^2}{2})e^{\alpha y(\frac{\alpha\sigma^2}{2}-\mu)} \operatorname{erf} c\{\sqrt{y}\frac{\alpha\sigma^2 - \mu}{\sqrt{2\sigma^2}}\}dy \quad (2). \\ &= u_\alpha(y)dy \end{aligned}$$

where

$$u_\alpha(y) = \frac{\sigma}{\sqrt{y}}\varphi_z(\sqrt{y}\mu/\sigma) + (\frac{\mu - \alpha\sigma^2}{2})e^{\alpha y(\frac{\alpha\sigma^2}{2}-\mu)} \operatorname{erf} c\{\sqrt{y}\frac{\alpha\sigma^2-\mu}{\sqrt{2\sigma^2}}\}.$$

It follows that, for $x \leq \lambda$,

$$\begin{aligned} C_g^\alpha(0,x,\lambda) &= \int_0^{\lambda-x} g(x+y)U_\alpha(dy), \\ C_g(0,x,\lambda) &= \int_0^{\lambda-x} g(x+y)U_0(dy). \end{aligned}$$

From Equation (8) of [7], it follows that for $x \leq \lambda$,

$$\begin{aligned} E_x(\exp(-\alpha W_\lambda)) &= \alpha U_\alpha I_{[\lambda,\infty)}(x). \\ &= \frac{\alpha\sigma^2-\mu}{\alpha\sigma^2-2\mu}e^{\alpha(\lambda-x)(\frac{\alpha\sigma^2}{2}-\mu)} \operatorname{erf} c(\sqrt{\lambda-x}\frac{\alpha\sigma^2-\mu}{\sqrt{2\sigma^2}}) \\ &\quad -\frac{\mu}{\alpha\sigma^2-2\mu} \operatorname{erf} c(\frac{\sqrt{\lambda-x}\mu}{\sqrt{2\sigma^2}}) \qquad (3) \end{aligned}$$

where the last equation follows by integrating $U_\alpha(dy)$ over the interval $[\lambda,\infty)$ (we omit the proof).

It follows that, for $x \leq \lambda$, the distribution function of W_λ (denoted by $F_{W_\lambda}()$) is given by

$$F_{W_\lambda}(t) = \frac{1}{2} \operatorname{erf} c\{\frac{(\lambda-x)\mu - t}{\sqrt{2\sigma^2}}\} - \frac{1}{2}e^{2\mu t/\sigma^2} \operatorname{erf} c\{\frac{(\lambda-x)\mu+t}{\sqrt{2\sigma^2}}\}, \; t \geq 0.$$

Furthermore, for $x \leq \lambda$

$$\begin{aligned} E_x(W_\lambda) &= U_0 I_{[0,\lambda)}(x) \\ &= \sigma\int_0^{\lambda-x} \frac{1}{\sqrt{y}}\varphi_z(\sqrt{y}\frac{\mu}{\sigma})dy + \frac{\mu}{2}\int_0^{\lambda-x} \operatorname{erf} c(-\sqrt{\frac{y}{2}}\frac{\mu}{\sigma})\; dy. \\ &= \frac{(\lambda-x)\mu}{2} + \sigma\sqrt{\lambda-x}\;\varphi_z(\sqrt{\lambda-x}\frac{\mu}{\sigma}) + \frac{(\lambda-x)\mu^2+\sigma^2}{2\mu} \operatorname{erf}(\sqrt{\frac{\lambda-x}{2}}\frac{\mu}{\sigma}) \quad (4) \end{aligned}$$

where the last equation follows from the equation before last upon tedious calculations which we omit.

To derive $C^{\alpha}_{g^*}(M,y,\tau), C_{g^*}(M,y,\tau), E_y(\exp(-\alpha W^*_\tau))$, and $E_y(W^*_\tau)$. We first note that $I^{\,*}$ is a Levy process and using Doob's optional sampling theorem we have the following result

$$E_x[e^{-\alpha W^*_\tau}] = e^{-(x-\tau)\eta(\alpha)} \text{ . (5)}$$

where $\eta(\alpha)$ is the unique increasing solution of the equation

$$M\eta(\alpha) = \alpha + \frac{\sqrt{2\eta(\alpha)\sigma^2 + \mu^2} - \mu}{\sigma^2} \text{.(6)}$$

It can be seen that the permissible solution of this equation is (we omit the proof)

$$\eta(\alpha) = \frac{\alpha}{M} + \frac{(1-M\mu) + \sqrt{2\alpha M\sigma^2 + (1-M\mu)^2}}{M^2\sigma^2}.$$

It follows that, for any $x \geq \tau$,

$$\Pr_x\{W^*_\tau < \infty\} = \begin{cases} 1 & \text{if } \mu M > 1 \\ e^{\frac{-2(x-\tau)(1-\mu M)}{M^2\sigma^2}} & \text{if } \mu M \leq 1. \end{cases}$$

It is also found that the probability density function of W^*_τ $(f_{W^*_\tau}(.))$ is equal to zero for $t < \frac{(x-\tau)}{M}$, and for $t \geq \frac{(x-\tau)}{M}$,

$$f_{W^*_\tau}(t) = \frac{(x-\tau)}{\sigma\sqrt{2\pi(Mt-(x-\tau))^3}} \exp\{-\frac{((M\mu-1)t - \mu(x-\tau))^2}{2(Mt-(x-\tau))\sigma^2}\}.$$

Furthermore,

$$\begin{aligned} E_x W^*_\tau &= \frac{(x-\tau)\mu}{(\mu M - 1)} \quad \text{if } \mu M > 1, \quad (7) \\ &= \infty \qquad\qquad \text{if } \mu M \leq 1, \end{aligned}$$

and

$$\begin{aligned} Var_x(W^*_\tau) &= \frac{(x-\tau)M^2\sigma^2}{(\mu M - 1)^3} \quad \text{if } \mu M > 1, \\ &= \infty \qquad\qquad \text{if } \mu M \leq 1. \end{aligned}$$

We now define, the killed process

$$X^* = \{\overset{*}{I}_t, t \leq W^*_\tau\}$$

It can be shown that the process X^* is a strong Markov process. Furthermore, it has state space (τ,∞). Starting at $x \in (\tau,\infty)$, let $f(x,y,t)$ be the transition probability function of $\overset{*}{I}$. Let U^*_α be the resolvent operator of the process X^*, it follows that for $x \in (\tau, \infty)$,

$$U^*_\alpha(dy - x) = [p^*_\alpha(y - x) - \exp(-(x - \tau)\eta(\alpha))p^*_\alpha(y - \tau)]dy.$$

where for any z ,

$$p^*_\alpha(z) = \int_0^\infty \exp(-\alpha t)p(t, z + Mt)dt,$$

where $p(t, x)$ is the transition probability of the process I, starting at zero, defined before.

Furthermore, for $x, y \geq \tau$,

$$f(x, y, t) = p(t, y - x + Mt) - \int_0^t p(t - s, y - \tau + M(t - s))f_{W^*_\tau}(s)ds.$$

Thus, for $x \geq \tau$,

$$C^\alpha_{g^*}(M, x, \tau) = \int_\tau^\infty g^*(y)U^*_\alpha(dy - x),$$

and

$$C_{g^*}(M, x, \tau) = \int_\tau^\infty g^*(y)U^*_0(dy - x).$$

Now we need to compute the joint distribution function of the pair $(W_\lambda, I_{W_\lambda})$ given $I_0 = 0$, denoted by $f_0(t, x)$. We define $v_\alpha(x) = u_\alpha(x)I\{x > \lambda\}$ and $v^0_\alpha(x) = u_\alpha(x)I\{x = \lambda\}$. For any function g we let $L_\beta(g)$ be its Laplace transform with respect to β. From Equation (8) P. 2067 of [7] we have, for $\alpha \geq 0$, $\beta \geq 0$

$$\begin{aligned}
E[e^{-\alpha W_\lambda - \beta I_{W_\lambda}}] &= [\alpha + \frac{\sqrt{2\beta\sigma^2 + \mu^2} - \mu}{\sigma^2}]L_\beta(v_\alpha). \\
&= \alpha L_\beta(v_\alpha) + \frac{\sqrt{2\beta\sigma^2 + \mu^2} - \mu}{\sigma^2}L_\beta(v_\alpha). \\
&= \alpha L_\beta(v_\alpha) + \frac{2\beta}{\sqrt{2\beta\sigma^2 + \mu^2} - \mu}L_\beta(v_\alpha) - \frac{2\mu}{\sigma^2}L_\beta(v_\alpha) \\
&= \alpha L_\beta(v_\alpha) + 2L_\beta(u_0 * v^o_\alpha) + +2L_\beta(u_0)L_\beta(v'_\alpha) - \frac{2\mu}{\sigma^2}L_\beta(v_\alpha) \\
&= \alpha L_\beta(v_\alpha) + 2L_\beta(u_0 * v^o_\alpha) + +2L_\beta(u_0 * v'_\alpha) - \frac{2\mu}{\sigma^2}L_\beta(v_\alpha). \qquad (8)
\end{aligned}$$

Inverting the above function with respect to α, β, we have

$$f_0(t,x) = \frac{\partial}{\partial t}p(t,x) + 2\{p_t(\lambda)u_0(x-\lambda) + \int_\lambda^x u_0(x-y)\frac{\partial}{\partial y}p(t,y)dy - \frac{\mu}{\sigma^2}p(t,x)\}]I\{t \geq 0, x > \lambda\} \quad (9)$$

To find the marginal pdf of I_{W_λ}, we define

$$L_\beta(\lambda) \stackrel{def}{=} E_0(e^{--\beta I_{W_\lambda}}).$$

Letting $\alpha \downarrow 0$ in Equation (8) we get

$$L_\beta(\lambda) = 2[L_\beta(u_0 * v_0^0) + L_\beta(u_0 * v_0') - \frac{\mu}{\sigma^2}L_\beta(v_0)].$$

It follows that

$$\begin{aligned} E_x(I_{W_\lambda}) &= \frac{U_0 I_{(0,\lambda-x]}(0)}{\mu} \quad (10) \\ &= \frac{E_x(W_\lambda)}{\mu} \end{aligned}$$

Furthermore, the pdf of I_{W_λ}, given $I_0 = 0$ (denoted by $f_{I_{W_\lambda}}()$) is given by

$$f_{I_{W_\lambda}}(x) = 2[u_0(\lambda)u_0(x-\lambda) + \int_\lambda^x u_0(x-y)u_0^{'}(y)dy - \frac{\mu}{\sigma^2}u_0(x)\}]I\{x > \lambda\}.$$

III. THE TOTAL DISCOUNTED AND LONG RUN-AVERAGE COSTS AND THE STATIONARY DISTRIBUTION OF THE DAM CONTENT

We now discuss the computations of the cost functional using the total discounted cost as well as the long-run average cost. Let W be the length of the first cycle, i.e., $W = \overset{*}{T}_1 - \overset{*}{T}_0$, and $C_\alpha(x)$ be the expected cost during the interval $[0, \overset{*}{T}_0)$, when $Z_0 = x$. Then, it follows that the total discounted cost associated with an $P^M_{\lambda,\tau}$ policy is given by

$$C_\alpha(\lambda,\tau) = C_\alpha(x) + \frac{E_x(\exp(-\alpha\overset{*}{T}_0)E_\tau C_\alpha(1)}{1 - E_\tau(\exp(-\alpha W))}$$

where $C_\alpha(1)$ is the total discounted cost during the interval $(0, W)$. For $x \leq \lambda$, we have

$$
\begin{aligned}
C_\alpha(x) &= M\{K_2 + K_1 E_x(e^{-\alpha W_\lambda}) - RE_x \int_{W_\lambda}^{\overset{*}{T}_0} e^{-\alpha t} dt\} \\
&\quad + \int_0^{\overset{*}{T}_0} e^{-\alpha t} g(Z_t, R_t) dt \\
&= M\{K_2 + K_1 E_x(e^{-\alpha W_\lambda}) - RE_x \int_{W_\lambda}^{\overset{*}{T}_0} e^{-\alpha t} dt\} \\
&\quad + E_x \int_0^{W_\lambda} e^{-\alpha t} g(I_t) dt + E_x \int_{W_\lambda}^{\overset{*}{T}_0} e^{-\alpha t} g^*(I_t^*) dt \\
&= M\{K_2 + K_1 E_x(e^{-\alpha W_\lambda}) - RE_x \{e^{-\alpha W_\lambda} E_{I_{W_\lambda}} \int_0^{\overset{*}{W}_\tau} e^{-\alpha t} dt\} \\
&\quad + C_g^\alpha(0,\tau,\lambda) + E_x[e^{-\alpha W_\lambda} E_{I_{W_\lambda}} \int_0^{\overset{*}{W}_\tau} e^{-\alpha t} g^*(I_t^*) dt) \\
&= M\{K_2 + K_1 E_x(e^{-\alpha W_\lambda})\} + C_g^\alpha(0,\tau,\lambda) + E_x[e^{-\alpha W_\lambda} C^\alpha_{g^*-RM}(M, I_{W_\lambda}, \tau)], \quad (11)
\end{aligned}
$$

where the third equation above follows from the second equation upon conditioning on the sigma algebra generated by $(W_\lambda, I_{W_\lambda})$.Furthermore,

$$E_\tau C_\alpha(1) = C_\alpha(\tau),$$

where $C_\alpha(\tau)$ is obtained from Equation (10) upon substituting τ for x.

Throughout the remainder of this section we let $\varphi(\alpha) = \frac{\sqrt{2\alpha\sigma^2+\mu^2}-\mu}{\sigma^2}$. Now, for $x < \lambda$

$$
\begin{aligned}
E_x(e^{-\alpha \overset{*}{T}_0}) &= E_x[e^{-\alpha W_\lambda} E_{I_{W_\lambda}}(e^{-\alpha \overset{*}{W}_\tau})] \\
&= E_x[e^{-\alpha W_\lambda} e^{-\eta(\alpha)(I_{W_\lambda} - \tau)\eta(\alpha)}] \\
&= E_0[e^{-\alpha W_{\lambda-x}} e^{-\eta(\alpha)(I_{W_{\lambda-x}} + x - \tau)}] \\
&= e^{-\eta(\alpha)(x-\tau)} E_0[e^{-\alpha W_{\lambda-x} - \eta(\alpha)(I_{W_{\lambda-x}})}] \\
&= (\alpha + \varphi(\eta(\alpha)) e^{-\eta(\alpha)(x-\tau)} \int_{[\lambda-x,\infty)} e^{-z\eta(\alpha)} U_\alpha(dz) \\
&= M\eta(\alpha) e^{-\eta(\alpha)(x-\tau)} \int_{[\lambda-x,\infty)} e^{-z\eta(\alpha)} U_\alpha(dz), \qquad (12)
\end{aligned}
$$

where the first equation follows since for $x < \lambda$, $\overset{*}{T} = W_\lambda + W_\tau^*$ and upon conditioning on the sigma algebra generated by $(W_\lambda, I_{W_\lambda})$, the second equation follows from Equation (5) above, the fifth equation follows from Equation (8) of reference [7] and the last equation follows from Equation (6) above. However, for $x \geq \lambda$, we have

$$
\begin{aligned}
E_x(e^{-\alpha \overset{*}{T}_0}) &= E(e^{-\alpha W^*_\tau}) \\
&= e^{-\eta(\alpha)(x-\tau)}.
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
E_\tau(e^{-\alpha W}) &= E_\tau(e^{-\alpha \overset{*}{T}_0}) \\
&= M\eta(\alpha)\int_{[\lambda-\tau,\infty)} e^{-x\eta(\alpha)}U_\alpha(dx),
\end{aligned}
$$

where the last equation follows from Equation (12) above.

Now we turn our attention to computing the cost functional for the long- run cost average case. It follows by a Tauberian theorem that the long run average cost per unit of time, denoted by $C(\lambda,\tau)$ is given by

$$
C(\lambda,\tau) = \frac{M[K - RE_\tau(E_{I_{W_\lambda}}(W^*_\tau))] + E_\tau[C_{g^*}(M, I_{W_\lambda},\tau)] + \ C_g(0,\lambda,\tau)}{E_\tau(W)}. \quad (13)
$$

Note that $E_\tau(W) = E_\tau W_\lambda + E_\tau E_{I_{W_\lambda}}(W^*_\tau) = \infty$, if $M\mu \leq 1$. However, if $M\mu > 1$, we have

$$
\begin{aligned}
E_\tau(W) &= E_\tau(W_\lambda) + E_\tau E_{I_{W_\lambda}}(W^*_\tau) \\
&= E_0(W_{\lambda-\tau}) + \frac{\mu E_\tau(I_{W_\lambda} - \tau)\mu}{(\mu M - 1)} \\
&= E_0(W_{\lambda-\tau}) + \frac{\mu E_0(I_{W_{\lambda-\tau}})}{(\mu M - 1)} \\
&= E_0(W_{\lambda-\tau}) + \frac{E_0(W_{\lambda-\tau})}{(\mu M - 1)} \\
&= \frac{\mu M E_0(W_{\lambda-\tau})}{(\mu M - 1)}. \qquad (14)
\end{aligned}
$$

the second equation follows from Equation (7) above, the fourth equation follows from Equation (10) above and $E_0(W_{\lambda-\tau})$ is given in Equation (4) above.

Now we turn our attention to finding the stationary distribution of the content process. It follows that, when $M\mu > 1$, the content process is ergodic. We let $Z = \lim_{t\longrightarrow\infty} Z_t$ and $F(z)$ be the distribution function of the process Z. For simplicity of the notation, we now denote U_0 and U^*_0 by U and U^* ,respectively. It follows from Equations (13) and (14) above that

$$
\begin{aligned}
F(z) &= \frac{(M\mu-1)[C^0_{I_{[0,z]}}(0,\tau,\lambda)+E_0[C^\alpha_{[0,z]}(M,I_{W_{\lambda-\tau}}+\tau,\tau)}{M\mu E_0(W_{\lambda-\tau})} \\
&= \frac{(M\mu-1)U_{I_{[0,\lambda\wedge z-\tau)}}(0)+E_0[U^*_{I_{[0,z-\tau)}}(I_{W_\lambda})]}{M\mu E_0(W_{\lambda-\tau})}. \quad (15)
\end{aligned}
$$

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