Consistent Approximations of Some Geometric Differential Operators and Their Convergences

Guoliang Xu *
The Institute of Computational Mathematics,
Chinese Academy of Sciences, Beijing, China
Email: xuguo@lsec.cc.ac.cn

Abstract

The numerical integration of many geometric partial differential equations involves discrete approximations of several first- and second-order geometric differential operators. In this paper, we consider consistent discretized approximations of these operators based on a quadratic fitting scheme. An asymptotic error analysis is conducted which shows that the discrete approximations of the first- and second-order geometric differential operators have the quadratic and linear convergence rates, respectively.

Key words: Geometric Differential Operators; Consistent discretized approximations; Triangular Surface Mesh; Error Analysis.

1 Introduction

In many application areas, such as image processing, surface processing, computer graphics and computer aided geometric design, discrete approximations of various geometric differential operators are often required. For instance, to solve geometric partial differential equations (PDE) using a divided-difference-like method, discrete approximations of surface normal, gradient, mean curvature, Gaussian curvature, Laplace-Beltrami and Giaquinta-Hildebrandt operators are prerequisite (see [33], [36]). Discrete schemes have been proposed individually for each of these differential operators from different point of views (see [17], [21], [24], [22], [31] for references). For Laplace-Beltrami operators, many discretization scheme have been proposed including Taubin's discretization (see [27], 1995; [28], 2000), Fujiwara's discretization (see [11], 1995), Desbrun et al's discretization (see [6], 1999), Mayer's discretization (see [20], 2001) and Meyer et al's discretization (see [22],2002). All these discretizations are in the linear form:

$$\Delta f(\mathbf{x}_i) = \sum_{j\in N(i)} w_{ij}\left[f(\mathbf{x}_j) - f(\mathbf{x}_i)\right], \tag{1.1}$$

where $\mathbf{x}_i$ and $\mathbf{x}_j$ are the vertices of the surface triangulation M, $N(i)$ is the index set of one-ring neighbors of vertex $\mathbf{x}_i$, w_{ij} are constants depending on the mesh vertices, independent of f. On

*Project support in part by NSFC grant 60773165 and National Key Basic Research Project of China (2004CB318000).

the approximation of the mean curvature, there exist also several approaches, such as the ones proposed by Chen, Hamann and Taubin to name a few [4, 14, 26]. But these approaches do not yield the linear form (1.1).

For Gaussian curvature, there are also many disctretized approaches. One class of these approaches is based on the local fitting techniques, such as paraboloid fitting (see [14, 16, 25]), quadratic fitting (see [21, 31]) or higher order fitting (see [3]), circular fitting (see [4, 19]) and implicit fitting by 3D a function (see [9]). The second class of methods is based on a theorem for the Gauss map (see [21, 23]). For a given neighborhood of a surface point, the theorem says that the Gaussian curvature at the surface point can be approximated by the ratio of the area of the spherical image of the neighborhood and the area of the neighborhood itself. The third class of approaches is based on the Gauss-Bonnet theorem (see [1, 10, 15, 21, 25]). The derived scheme for triangulated surfaces is usually called *angle deficit scheme* or *Gauss-Bonnet scheme*. Apart from these classes, there are some other approaches, e.g, Taubin's approach [26] based on the eigen-analysis, Watanabe and Belyaev's approach (see [29]) based on the integral formulas of the normal curvature and its square, Wollmann's approach (see [30]) based on Euler theorem (see [2], page 145) and Meusnier theorem (see [2], page 142). Using the theory of normal cycles, Cohen-Steiner and Morvan derive an efficient and reliable curvature estimation algorithm (see [5]). Error bound of the estimated curvature is given in the case restricted Delaunay triangulations.

Since all these differential operators are individually dicretized based on various theorems in differential geometry, they are not consistent in general, meaning they do not come from a single surface. Furthermore, except for the schemes based on the interpolation or fitting, non of these schemes converges without any restriction on the regularity of the meshes considered. Finally, the semi-implicit discretization of the geometric PDEs require the discrete differential operators to have a linear form, while the widely used discrete schemes, for instance the discrete approximation of Gaussian curvature based on Gauss-Bonnet theorem (see [1]), are not in this form. Therefore, it is desirable to use a fitting scheme so that resulted discrete differential operators are consistent, convergent and have the required forms as well.

A quadratic fitting scheme is proposed in [31] based on local parametrization technique, and it is frequently used afterward in solving geometric partial differential equations and yields very desirable results (see [35, 36, 38]). Basing on an extensive numerical experiment, the author of [31] claims that the approximate mean curvature computed from the parametric quadratic fitting surface converges to the exact mean curvature. However, this fact has never been formally proved. In this paper, several commonly used differential operators are consistently approximated based on the parametric quadratic fitting algorithm. A asymptotic error analysis on the quadratic fitting is conducted, which firmly support the claim made in [31]. We prove that the obtained discrete approximations of the first- and second-order geometric differential operators from the quadratic fitting algorithm have the quadratic and linear convergence rates, respectively.

The rest of the paper is organized as follows: Section 2 introduces some used notations and a set of geometric differential operators. Discretization schemes for these differential operators are considered in Section 3. Convergence analysis of these discrete differential operators are conducted in section 4. Section 5 concludes the paper.

2 Geometric Differential Operators

In this section, we introduce the used notations and a set of geometric differential operators, including gradient, mean curvature, Gaussian curvature, Laplace-Beltrami operator and Giaquinta-Hildebrandt operator.

Curvatures. Let $\mathcal{S} = \{\mathbf{x}(u,v),\ (u,v) \in \Omega \subset \mathbb{R}^2\}$ be a regular smooth parametric surface in $\mathbb{R}^3$. For simplicity, we sometimes write (u,v) as (u^1,u^2). Let $g_{\alpha\beta} = \langle \mathbf{x}_{u^\alpha}, \mathbf{x}_{u^\beta}\rangle$ $(\alpha,\beta = 1,2)$ be the coefficients of the first fundamental form of $\mathcal{S}$ with $\mathbf{x}_{u^\alpha} = \frac{\partial \mathbf{x}}{\partial u^\alpha}$. Set $g = \det([g_{\alpha\beta}])$, $[g^{\alpha\beta}] = [g_{\alpha\beta}]^{-1}$. Let $b_{\alpha\beta} = \langle \mathbf{n}, \mathbf{x}_{\alpha\beta}\rangle$ be the coefficients of the second fundamental form of $\mathcal{S}$ with $\mathbf{x}_{u^\alpha u^\beta} = \frac{\partial^2 \mathbf{x}}{\partial u^\alpha \partial u^\beta}$ and $\mathbf{n} = (\mathbf{x}_u \times \mathbf{x}_v)/\|\mathbf{x}_u \times \mathbf{x}_v\|$. Then the *mean curvature* H and the *Gaussian curvature* K are given by

$$H = \frac{b_{11}g_{22} - 2b_{12}g_{12} + b_{22}g_{11}}{2g}, \quad K = \frac{b_{11}b_{22} - b_{12}^2}{g}.$$

The mean curvature vector and Gaussian curvature vector are $\mathbf{H} = H\mathbf{n}$ and $\mathbf{K} = K\mathbf{n}$, respectively.

Tangential gradient operator. Let $f \in C^1(\mathcal{S})$. Then the *tangential gradient operator* ∇ acting on f is given by (see [7], page 102)

$$\begin{aligned} \nabla f &= [\,\mathbf{x}_u,\ \mathbf{x}_v\,][\,g^{\alpha\beta}\,][\,f_u, f_v\,]^T \in \mathbb{R}^3 \\ &= g_u^\nabla f_u + g_v^\nabla f_v, \end{aligned} \tag{2.1}$$

where $g_u^\nabla = \frac{1}{g}(g_{22}\mathbf{x}_u - g_{12}\mathbf{x}_v)$ and $g_v^\nabla = \frac{1}{g}(g_{11}\mathbf{x}_v - g_{12}\mathbf{x}_u)$. Obviously, ∇ is a first-order differential operator.

Second tangential operator. Let $f \in C^1(\mathcal{S})$. Then the *second tangential operator* $\diamond$ acting on f is defined as (see [36])

$$\begin{aligned} \diamond f &= [\,\mathbf{x}_u,\ \mathbf{x}_v\,][\,K\,b^{\alpha\beta}\,][\,f_u, f_v\,]^T \in \mathbb{R}^3 \\ &= g_u^\diamond f_u + g_v^\diamond f_v, \end{aligned} \tag{2.2}$$

where $g_u^\diamond = \frac{1}{g}(b_{22}\mathbf{x}_u - b_{12}\mathbf{x}_v)$ and $g_v^\diamond = \frac{1}{g}(b_{11}\mathbf{x}_v - b_{12}\mathbf{x}_u)$. It is easy to see that $\diamond f$ is of first order with respect to the function f, but second-order with respect to the surface. Hence, we classify it as a second-order operator.

Divergence operator. Let $\mathbf{v}$ be a C^1 smooth vector field on $\mathcal{S}$. Then the *divergence* of $\mathbf{v}$ is defined by

$$\operatorname{div}(\mathbf{v}) = \frac{1}{\sqrt{g}}\left[\frac{\partial}{\partial u}, \frac{\partial}{\partial v}\right]\left[\ \sqrt{g}\,[\,g^{\alpha\beta}\,]\,[\mathbf{x}_u, \mathbf{x}_v]^T\mathbf{v}\,\right].$$

Laplace-Beltrami operator. Let $f \in C^2(\mathcal{S})$. Then ∇f is a smooth vector field on $\mathcal{S}$. The Laplace-Beltrami operator (LBO) Δ applying to f is defined by (see [8])

$$\Delta f = \operatorname{div}(\nabla f).$$

From the definition of ∇ and div, it is easy to derive that

$$\begin{aligned}
\Delta f &= \frac{1}{\sqrt{g}}\left[\frac{\partial}{\partial u},\frac{\partial}{\partial v}\right]\left[\ \sqrt{g}\,[\,g^{\alpha\beta}\,]\,[\,f_u, f_v\,]^T\,\right] \\
&= g_u^{\Delta} f_u + g_v^{\Delta} f_v + g_{uu}^{\Delta} f_{uu} + g_{uv}^{\Delta} f_{uv} + g_{vv}^{\Delta} f_{vv}, \qquad (2.3)
\end{aligned}$$

where

$$\begin{aligned}
g_u^{\triangle} &= -[g_{11}(g_{22}g_{122}-g_{12}g_{222})+2g_{12}(g_{12}g_{212} - g_{22}g_{112})+g_{22}(g_{22}g_{111}-g_{12}g_{211})]/g^2, \\
g_v^{\triangle} &= -[g_{11}(g_{11}g_{222}-g_{12}g_{122})+2g_{12}(g_{12}g_{112} - g_{11}g_{212})+g_{22}(g_{11}g_{211}-g_{12}g_{111})]/g^2, \\
g_{uu}^{\triangle} &= g_{22}/g, \quad g_{uv}^{\triangle} = -2g_{12}/g, \quad g_{vv}^{\triangle} = g_{11}/g,
\end{aligned}$$

and $g_{\alpha\beta\delta} = \langle \mathbf{x}_{u^\alpha}, \mathbf{x}_{u^\beta u^\delta}\rangle$. Obviously, Δ is a second-order differential operator. It is well know that $\Delta \mathbf{x} = 2H\mathbf{n}$.

Giaquinta-Hildebrandt Operator. Let $f \in C^2(\mathcal{S})$. Then the Giaquinta-Hildebrandt operator (GHO) acting on f is given by

$$\Box f = \mathrm{div}(\Diamond f).$$

From the definition of $\Diamond$ and div, it is easy to derive that (see [12], page 84)

$$\begin{aligned}
\Box f &= \frac{1}{\sqrt{g}}\left[\frac{\partial}{\partial u},\frac{\partial}{\partial v}\right]\left[\ \sqrt{g}\,[\,Kb^{\alpha\beta}\,]\,[\,f_u, f_v\,]^T\,\right] \qquad (2.4) \\
&= g_u^{\Box} f_u + g_v^{\Box} f_v + g_{uu}^{\Box} f_{uu} + g_{uv}^{\Box} f_{uv} + g_{vv}^{\Box} f_{vv}, \qquad (2.5)
\end{aligned}$$

where

$$\begin{aligned}
g_u^{\Box} &= -[b_{11}(g_{22}g_{122}-g_{12}g_{222})+2b_{12}(g_{12}g_{212} - g_{22}g_{112})+b_{22}(g_{22}g_{111}-g_{12}g_{211})]/g^2, \\
g_v^{\Box} &= -[b_{11}(g_{11}g_{222}-g_{12}g_{122})+2b_{12}(g_{12}g_{112} - g_{11}g_{212})+b_{22}(g_{11}g_{211}-g_{12}g_{111})]/g^2, \\
g_{uu}^{\Box} &= b_{22}/g, \quad g_{uv}^{\Box} = -2b_{12}/g, \quad g_{vv}^{\Box} = b_{11}/g.
\end{aligned}$$

Differential operator $\Box$ is introduced by Giaquinta and Hildebrandt (see [12], pages 82–85), we therefore call it as Giaquinta-Hildebrandt operator. Since b_{ij} involves the second order derivatives of the surface considered, equation (2.4) implies that $\Box f$ is a third order differential operator at first glance. However, (2.5) shows that it is of second order, since the terms involving the third order derivatives are canceled fortunately. Similar to the relation $\Delta\mathbf{x} = 2H\mathbf{n}$, we have $\Box\mathbf{x} = 2K\mathbf{n}$ (see [37]).

The differential operators introduced above are frequently used in defining various geometric partial differential equations. Using them, a plenty of geometric differential partial equations have been constructed (see [34]). For instance, to minimize a third order energy functional $\int_{\mathcal{S}} \|\nabla f(H,K)\|^2 \mathrm{d}A$, we obtain the following Euler-Lagrange equation (see [18])

$$\begin{aligned}
\Delta_s(f_H\Delta f) + 2\Box(f_K\Delta f) + 4KHf_K\Delta f + 4H^2 f_H\Delta f - 2Kf_H\Delta f \\
- 2H\|\nabla f\|^2 + 2\,\langle \nabla_s f, \Diamond f\rangle = 0.
\end{aligned}$$

It is easy yo see almost all the differential operators introduced above are involved in this equation. However, it is still unclear that if these operators are sufficient to describe all the geometric partial equations up to six order.

3 Discretizations of Geometric Differential Operators

Discrete Surface, and M a triangulation of surface $\mathcal{S}$. Let $\{\mathbf{x}_i\}_{i=1}^N$ be the vertex set of $\mathcal{S}$. For vertex $\mathbf{x}_i$ with valence n, denote by $N(i) = \{i_1, i_2, \cdots, i_n\}$ the set of the vertex indices of one-ring neighbors of $\mathbf{x}_i$, and $N_1(i) = \{i_0, i_1, i_2, \cdots, i_n\}$ with $i_0 = i$. We assume in the following that these $i_1, \cdots, i_n$ are arranged such that the triangles $[\mathbf{x}_i \mathbf{x}_{i_k} \mathbf{x}_{i_{k-1}}]$ and $[\mathbf{x}_i \mathbf{x}_{i_k} \mathbf{x}_{i_{k+1}}]$ are in $\mathcal{S}$, and $\mathbf{x}_{i_{k-1}}$, $\mathbf{x}_{i_{k+1}}$ opposite to the edge $[\mathbf{x}_i \mathbf{x}_{i_k}]$.

We devote our attention to the discretization of geometric differential operators because of the needs in solving various geometric partial differential equations. To solve the geometric PDEs using a divided-difference-like method, discrete approximations of the gradient, the mean curvature, Gaussian curvature, Laplace-Beltrami operator and Giaquinta-Hildebrandt operator are required (see [36]). In order to use a semi-implicit scheme, the approximations of the above mentioned differential operators require to have linear forms (see (3.3)–(3.7)). There are several discretization schemes of Laplace-Beltrami operator and Gaussian curvature (see [31, 32] for references). However, some of them are not in the required linear form and may not consistent in the following sense.

Definition 3.1 *A set of approximate geometric differential operators is said consistent if there exists a C^2 smooth surface S, such that the approximate operators coincide with the exact counterparts of S.*

If a set approximate differential operators are not consistent, some unexpected results may obtained in the applications. For instance, if we compute the principal curvatures $k_{1,2} = H \pm \sqrt{H^2 - K}$ from the approximated H and K, there is the danger that $H^2 - K < 0$. One may brutely take $H^2 - K = 0$ if $H^2 - K < 0$. But this will yield discontinuous principal curvatures.

Here we use a quadratic fitting of the surface data and function data to calculate the approximate differential operators. The algorithm we adopted is from [31]. Let $\mathbf{x}_i$ be a vertex of M with valence n, $\mathbf{x}_j$ be its neighbor vertices for $j \in N(i)$.

Algorithm 3.1. *Quadratic Fit*

1. **Local Parametrization**. Compute angles

$$\alpha_k = cos^{-1} \frac{\langle \mathbf{x}_{i_k} - \mathbf{x}_i, \mathbf{x}_{i_{k+1}} - \mathbf{x}_i \rangle}{\|\mathbf{x}_{i_k} - \mathbf{x}_i\| \|\mathbf{x}_{i_{k+1}} - \mathbf{x}_i\|}, \quad k = 1, \cdots, n,$$

   and then compute the angles

$$\beta_k = 2\pi\alpha_k / \sum_{j=1}^{n} \alpha_j, \quad k = 1, \cdots, n.$$

   Set $\mathbf{q}_0 = [0, 0]^T$, $\theta_1 = 0$ and $\mathbf{q}_k = \|\mathbf{x}_{i_k} - \mathbf{x}_i\| [cos\theta_k, sin\theta_k]^T$, $\theta_k = \beta_1 + \cdots + \beta_{k-1}$, $k = 1, \cdots, n$.

2. **Local Fitting**. Take the basis functions

$$\{B_l(u, v)\}_{l=0}^5 = \{1, u, v, \frac{1}{2}u^2, uv, \frac{1}{2}v^2\},$$

and determine the coefficient $\mathbf{c}_l \in \mathbb{R}^3$ of $\mathbf{x}(u,v) = \sum_{l=0}^{5} \mathbf{c}_l B_l(u,v)$ such that

$$\sum_{l=0}^{5} \mathbf{c}_l B_l(\mathbf{q}_k) = \mathbf{x}_{i_k}, \quad k = 0, \cdots, n$$

in the least square sense. This system is solved by solving the normal equation. Let $A = (B_l(\mathbf{q}_k))_{k=0,l=0}^{n,5} \in \mathbb{R}^{(n+1)\times 6}$, and let

$$C = (A^T A)^{-1} A^T \in \mathbb{R}^{6\times(n+1)}, \tag{3.1}$$

then $[\mathbf{c}_0, \cdots, \mathbf{c}_5] = [\mathbf{x}_{i_0}, \cdots, \mathbf{x}_{i_n}] C^T$.

Remark 3.1 *The construction algorithm above may fail if the coefficient matrix of the normal equation is singular or nearly singular. In this case, we look for a least square solution with minimal normal. Let $A^T AX = b$ be the linear system in the matrix form. We find a least square solution x such that $\|X\|_2 = min$. That is, we replace $(A^T A)^{-1}$ in (3.1) with $(A^T A)^+$, the Moore-Penrose inverse. It is well known that $(A^T A)^+$ could be computed by the SVD decomposition of $A^T A$ (see [13], Chapter 5). Let $V = \mathrm{diag}[\sigma_1, \cdots, \sigma_6]$, where $\sigma_1 \geq \cdots \geq \sigma_6 \geq 0$ are the singular value of $A^T A$. If the computed singular value $\sigma_i < 10^{-8}$, we regard this singular value as zero (we use double precision arithmetic operations). In the practice, $\|\mathbf{x}_{i_k} - \mathbf{x}_i\|$ may be very small and the singular values are also small. Then the truncation of the singular values mentioned above may be misleading. To overcome this difficulty, the matrix A is normalized by multiplying a diagonal matrix $\mathrm{diag}[1, h^{-1}, h^{-1}, h^{-2}, h^{-2}, h^{-2}]$ on the left, where $h = \max_k \|\mathbf{x}_{i_k} - \mathbf{x}_i\|$.*

Partial derivatives. Given discrete function values $f(\mathbf{x}_{i_k})$, the fitting function is $f(u,v) = \sum_{l=0}^{5} d_l B_l(u,v)$ with $[d_0, \cdots, d_5]^T = C[f(\mathbf{x}_{i_0}), \cdots, f(\mathbf{x}_{i_n})]^T$. We compute partial derivatives of $\mathbf{x}(u,v)$ and $f(u,v)$ at the origin up to the second order. Denote the second, third, fourth, fifth and sixth rows of C as C_1, C_2, C_{11}, C_{12} and C_{22}, respectively, then we can see that

$$\begin{array}{rll} \mathbf{x}_{u^\alpha} & = [\mathbf{x}_{i_0}, \cdots, \mathbf{x}_{i_n}] C_\alpha^T, & \alpha = 1,2, \\ \frac{\partial f}{\partial u^\alpha} & = [f(\mathbf{x}_{i_0}), \cdots, f(\mathbf{x}_{i_n})] C_\alpha^T, & \alpha = 1,2, \\ \mathbf{x}_{u^\alpha u^\beta} & = [\mathbf{x}_{i_0}, \cdots, \mathbf{x}_{i_n}] C_{\alpha\beta}^T, & 1 \leq \alpha \leq \beta \leq 2, \\ \frac{\partial^2 f}{\partial u^\alpha \partial u^\beta} & = [f(\mathbf{x}_{i_0}), \cdots, f(\mathbf{x}_{i_n})] C_{\alpha\beta}^T, & 1 \leq \alpha \leq \beta \leq 2. \end{array} \tag{3.2}$$

Tangential gradient operator. Substituting (3.2) into (2.1), we get an approximation of tangential gradient operator as follows:

$$\nabla f(\mathbf{x}_i) \approx \sum_{j\in N_1(i)} w_{ij}^{\nabla} f(\mathbf{x}_j), \quad w_{i,i_j}^{\nabla} = g_u^{\nabla} c_1^{(j)} + g_v^{\nabla} c_2^{(j)} \in \mathbb{R}^3. \tag{3.3}$$

Here $c_\alpha^{(j)}$ are the j-th component of C_α .

Second tangential operator. Substituting (3.2) into (2.2), we get an approximation of second tangential operator as follows:

$$\diamond f(\mathbf{x}_i) \approx \sum_{j\in N_1(i)} w_{ij}^{\diamond} f(\mathbf{x}_j), \quad w_{i,i_j}^{\diamond} = g_u^{\diamond} c_1^{(j)} + g_v^{\diamond} c_2^{(j)} \in \mathbb{R}^3. \tag{3.4}$$

Laplace-Beltrami operator. Substituting (3.2) into (2.3), we get an approximation of LBO as follows:

$$\Delta f(\mathbf{x}_i) \approx \sum_{j\in N_1(i)} w_{ij}^{\triangle} f(\mathbf{x}_j), \tag{3.5}$$

where $w_{i,i_j}^{\triangle} = g_u^{\triangle} c_1^{(j)} + g_v^{\triangle} c_2^{(j)} + g_{uu}^{\triangle} c_{11}^{(j)} + g_{uv}^{\triangle} c_{12}^{(j)} + g_{vv}^{\triangle} c_{22}^{(j)}$ and $c_{\alpha\beta}^{(j)}$ are the j-th component of $C_{\alpha\beta}$.

Giaquinta-Hildebrandt operator. Substituting (3.2) into (2.5), we get an approximation of $\Box$ as follows:

$$\Box f(\mathbf{x}_i) \approx \sum_{j\in N_0(i)} w_{ij}^{\Box} f(\mathbf{x}_j), \tag{3.6}$$

where $w_{i,i_j}^{\Box} = g_u^{\Box} c_1^{(j)} + g_v^{\Box} c_2^{(j)} + g_{uu}^{\Box} c_{11}^{(j)} + g_{uv}^{\Box} c_{12}^{(j)} + g_{vv}^{\Box} c_{22}^{(j)}$.

Mean curvature normal and mean curvature. Using the relation $\Delta\mathbf{x} = 2\mathbf{H}$, we have

$$\mathbf{H}(\mathbf{x}_i) \approx \frac{1}{2}\sum_{j\in N_1(i)} w_{ij}^{\triangle}\mathbf{x}_j, \quad H(\mathbf{x}_i) \approx \frac{1}{2}\sum_{j\in N_1(i)} w_{ij}^{\triangle}\mathbf{n}(\mathbf{x}_i)^T\mathbf{x}_j, \tag{3.7}$$

where $\mathbf{n}(\mathbf{x}_i)$ is the surface normal at $\mathbf{x}_i$, which is computed as $(\mathbf{x}_u \times \mathbf{x}_v)/\|\mathbf{x}_u \times \mathbf{x}_v\|$.

Gaussian curvature normal and Gaussian curvature. Using the relation $\Box\mathbf{x} = 2K\mathbf{n}$, we have

$$K(\mathbf{x}_i)\mathbf{n}(\mathbf{x}_i) \approx \frac{1}{2}\sum_{j\in N_1(i)} w_{ij}^{\Box}\mathbf{x}_j, \quad K(\mathbf{x}_i) \approx \frac{1}{2}\sum_{j\in N_1(i)} w_{ij}^{\Box}\mathbf{n}(\mathbf{x}_i)^T\mathbf{x}_j.$$

4 Convergence of Discrete Differential Operators

It is well known that for a given function $f(x,y)$, the errors of the coefficients c_{ij} of the Lagrange interpolation polynomial $\sum_{i+j\le n} c_{ij}x^iy^j$ of degree n versus the Taylor expansion of $f(x,y)$ around the origin are bounded by (see [3])

$$|c_{ij} - f_{ij}(0)/i!j!| \le Ch^{n+1-(i+j)},$$

where $f_{ij} = \frac{\partial^{i+j} f}{\partial x^i \partial y^j}$, C is a constant and h is the maximal distance of the interpolation nodes to the origin. For approximation (the least square fitting), a similar result holds (see [3]). However, these results do not imply explicitly the convergence for our quadratic fitting algorithm because of the following two reasons. First, our fitting surface is in the parametric form. Second, the nodes of the fitting are determined in a way that is different from the function case (see Step 1 of Algorithm 3.1). In the following, we analyze the asymptotic properties of the fitting surface generated by Algorithm 3.1.

Definition 4.1 *Let* $K := \left\{\mathbf{q}_i = [x_i, y_i]^T \in \mathbb{R}^2\right\}_{i=0}^n$, $n \ge 5$. *If the matrix*

$$A(K) := \begin{bmatrix} 1 & x_0 & y_0 & x_0^2/2 & x_0y_0 & y_0^2/2 \\ 1 & x_1 & y_1 & x_1^2/2 & x_1y_1 & y_1^2/2 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & x_n & y_n & x_n^2/2 & x_ny_n & y_n^2/2 \end{bmatrix}$$

is full rank in column, then we say that the node set K is well-posed for the problem of the quadratic fitting: Determining a bivariate polynomial $G(x,y) = \sum_{i+j\le 2} a_{ij}x^i y^j/i!j!$ such that

$$G(\mathbf{q}_i) = f(\mathbf{q}_i), \quad i = 0, 1, \cdots, n, \tag{4.1}$$

in the least square sense, for a given function $f(x,y)$.

If the node set K is well-posed, then the quadratic fitting problem has unique solution. In the following, we assume that $\mathbf{q}_0, \mathbf{q}_1, \cdots, \mathbf{q}_n$ are mutual distinct and further assume that $\mathbf{q}_0 = [0,0]^T$ for simplicity.

Lemma 4.1 *Let $K = \{\mathbf{q}_i\}_{i=0}^n$ be a well-posed node set for the quadratic fitting problem. Then the node set $K^{(h)} := hK := \{h\mathbf{q}_i\}_{i=0}^n$ is also well-posed for any $h > 0$. More general, let $L \in \mathbb{R}^{2\times 2}$ be a nonsingular matrix. Then the node set $K^{(L)} := \{L\mathbf{q}_i\}_{i=0}^n$ is well-posed.*

Proof. Since $h > 0$ and

$$A(K^{(h)}) = A(K)\Lambda, \quad \Lambda = \mathrm{diag}[1, h, h, h^2, h^2, h^2], \tag{4.2}$$

the first conclusion of the Lemma follows. To prove the second conclusion, let $L = (a_{ij})_{i,j=1}^2$. Then under the transform L,

$$A(K^{(L)}) = A(K)\mathrm{diag}\left\{1, L, \begin{bmatrix} a_{11}^2 & a_{11}a_{21} & a_{21}^2 \\ 2a_{11}a_{12} & a_{11}a_{22} + a_{12}a_{21} & 2a_{21}a_{22} \\ a_{12}^2 & a_{12}a_{22} & a_{22}^2 \end{bmatrix}\right\}.$$

It is not difficult to calculate that the determinant of the last 3×3 block matrix above is $(a_{11}a_{22} - a_{12}a_{21})^3 \ne 0$, since $\det(L) = a_{11}a_{22} - a_{12}a_{21} \ne 0$. Hence the lemma is proved. $\diamond$

Lemma 4.2 *Suppose $f(x,y)$ is a sufficiently smooth bivariate function in the neighborhood of the origin $\mathbf{q}_0$. Let $K = \{\mathbf{q}_i\}_{i=0}^n$ be a well-posed node set, and*

$$G^{(h)}(x,y) = \sum_{i+j\le 2} a_{ij}^{(h)} x^i y^j/i!j!$$

the quadratic fitting function of f on the node set $K^{(h)} = hK$, $h > 0$. Then

$$\left|a_{ij}^{(h)} - f_{ij}(\mathbf{q}_0)\right| \le c_{ij}\|[A(K)^T A(K)]^{-1}A(K)\|h^{3-i-j}, \tag{4.3}$$

where c_{ij} are constants depending on f, independent of $K^{(h)}$.

Proof. Let

$$X = [a_{00}^{(h)}, a_{10}^{(h)}, a_{01}^{(h)}, a_{20}^{(h)}, a_{11}^{(h)}, a_{02}^{(h)}]^T, \quad F = [f(\mathbf{q}_0), f(h\mathbf{q}_1), \cdots, f(h\mathbf{q}_n)]^T.$$

Then from the fitting problem (4.1), we have

$$A(K^{(h)})^T A(K^{(h)})X = A(K^{(h)})^T F,$$

and, by (4.2),

$$A(K)^T A(K)\Lambda X = A(K)^T F. \tag{4.4}$$

Since

$$f(h\mathbf{q}_i) = f(\mathbf{q}_0) + h\mathbf{q}_i^T \nabla f(\mathbf{q}_0) + \frac{1}{2}h^2\mathbf{q}_i^T \nabla^2 f(\mathbf{q}_0)\mathbf{q}_i + O(h^3),$$

we have

$$F = A(K)\Lambda F_0 + O(h^3),$$

where $F_0 = [f(\mathbf{q}_0), f_{10}(\mathbf{q}_0), f_{01}(\mathbf{q}_0), f_{20}(\mathbf{q}_0), f_{11}(\mathbf{q}_0), f_{02}(\mathbf{q}_0)]^T$. Substituting this into (4.4), we have

$$A(K)^T A(K)\Lambda X = A(K)^T A(K)\Lambda F_0 + A(K)^T O(h^3),$$

and

$$X = F_0 + \Lambda^{-1}[A(K)^T A(K)]^{-1}A(K)^T O(h^3).$$

Then (4.3) follows. $\diamond$

Corollary 4.1 *Suppose f is a sufficiently smooth bivariate function in the neighborhood of the origin. Let $K = \{\mathbf{q}_i\}_{i=0}^n$ be a well-posed node set. Let $G^{(h)}$ be the quadratic fitting function of f on the node set $K^{(h)}$. Then we have*

$$\|\mathbf{n}(G^{(h)}) - \mathbf{n}(f)\| \le C_0 h^2, \quad |k_i(G^{(h)}) - k_i(f)| \le C_i h, \quad i = 1, 2,$$

where $\mathbf{n}(f)$ and $k_i(f)$ denote the normal and the principal curvatures of f at $\mathbf{q}_0$, respectively.

Remark 4.1 *The corollary says that the normal $\mathbf{n}(G^{(h)})$ has quadratic convergence rate, curvatures have linear convergence rate. These conclusions match Meek and Walton's results (see [21], Lemma 4.1).*

Lemma 4.3 *Let $K = \{\mathbf{q}_i\}_{i=0}^n$ be a well-posed node set. Then for any $B > \|[A(K)^T A(K)]^{-1} A(K)\|$, there exists an $\epsilon > 0$, such that*

(i). $\mathbf{q}_0 \notin \Omega_i$, $\Omega_i \cap \Omega_j = \emptyset$, $i \ne j$, $i, j \ge 1$, where $\Omega_i = \{\mathbf{q} \in \mathbb{R}^2 : \|\mathbf{q} - \mathbf{q}_i\| < \epsilon\}$.

(ii). For any node set $R := \{\mathbf{r}_i \in \Omega_i\}_{i=0}^n$, we have $\|[A(R)^T A(R)]^{-1}A(R)\| < B$.

(iii). For this ϵ, let $\Omega_i^{(h)} = \{\mathbf{q} \in \mathbb{R}^2 : \|\mathbf{q} - h\mathbf{q}_i\| < \epsilon h\}$ $(h > 0)$. Then the quadratic fitting problem on the node set $R^{(h)} := \{\mathbf{r}_i \in \Omega_i^{(h)}\}_{i=0}^n$ has unique solution

$$G^{(h)}(x, y) = \sum_{i+j\le 2} a_{ij}^{(h)} x^i y^j / i! j!$$

and

$$\left|a_{ij}^{(h)} - f_{ij}(\mathbf{q}_0)\right| \le c_{ij} B h^{3-i-j}, \quad i + j \le 2, \tag{4.5}$$

where c_{ij} are constants depending on f, independent of $R^{(h)}$.

Proof. Since K is a well-posed node set, $\mathbf{q}_i$ are distinct. It is obvious that there exists an $\epsilon_1 > 0$ such that (i) holds. Notice that the elements of the matrix $A(R)$ are continuous functions of $\mathbf{r}_0, \cdots, \mathbf{r}_n$. Hence the inverse of $A(R)^T A(R)$ exists in the neighborhood of $\mathbf{q}_0, \cdots, \mathbf{q}_n$. Then there exists an $\epsilon \le \epsilon_1$ such that (ii) holds. Similar to the proof of Lemma 4.2, (4.5) can be derived. $\diamond$

Lemma 4.4 *Suppose $f(x, y)$ is a smooth function around the origin $\mathbf{q}_0$, satisfying $f(\mathbf{q}_0) = 0$ and $\nabla f(\mathbf{q}_0) = \mathbf{0}$. Let $\mathbf{q} \in \mathbb{R}^2$ be a point in the neighborhood of the origin. Then there exists a constant C, such that*

$$\sqrt{||\mathbf{q} - \mathbf{q}_0||^2 + f(\mathbf{q})^2} - ||\mathbf{q} - \mathbf{q}_0|| \le C||\mathbf{q} - \mathbf{q}_0||^3.$$

Proof. Since

$$f(\mathbf{q}) = f(\mathbf{q}_0) + (\mathbf{q} - \mathbf{q}_0)^T \nabla f(\mathbf{q}_0) + O(||\mathbf{q} - \mathbf{q}_0||^2) = O(||\mathbf{q} - \mathbf{q}_0||^2),$$

we have

$$\sqrt{||\mathbf{q} - \mathbf{q}_0||^2 + f(\mathbf{q})^2} = \sqrt{||\mathbf{q} - \mathbf{q}_0||^2 + O(||\mathbf{q} - \mathbf{q}_0||^4)} = ||\mathbf{q} - \mathbf{q}_0|| + O(||\mathbf{q} - \mathbf{q}_0||^3).$$

$\diamond$

Lemma 4.5 *Let $f(x, y)$ be a smooth function around the origin $\mathbf{q}_0$, satisfying $f(\mathbf{q}_0) = 0$ and $\nabla f(\mathbf{q}_0) = \mathbf{0}$. Let $\mathbf{q}_1, \mathbf{q}_2 \in \mathbb{R}^2$ be two points in the neighborhood of the origin, and $\mathbf{q}_i \neq \mathbf{0}$. Set*

$$\theta = cos^{-1} \frac{\langle \mathbf{q}_1, \mathbf{q}_2 \rangle}{||\mathbf{q}_1|| ||\mathbf{q}_2||}, \quad \theta_f^{(h)} = cos^{-1} \frac{\langle \mathbf{x}_1, \mathbf{x}_2 \rangle}{||\mathbf{x}_1|| ||\mathbf{x}_2||},$$

where $\mathbf{x}_i = [h\mathbf{q}_i^T, f(h\mathbf{q}_i)]^T$, $i = 1, 2$. Then there exists a constant C, such that

$$|\theta_f^{(h)} - \theta| \le Ch^2.$$

Proof. Since

$$\begin{aligned} f(h\mathbf{q}_i) &= f(\mathbf{q}_0) + h\mathbf{q}_i^T \nabla f(\mathbf{q}_0) + \frac{1}{2} h^2 \mathbf{q}_i^T \nabla^2 f(\mathbf{q}_0) \mathbf{q}_i + O(h^3) \\ &= \frac{1}{2} h^2 \mathbf{q}_i^T \nabla^2 f(\mathbf{q}_0) \mathbf{q}_i + O(h^3), \\ ||\mathbf{x}_i|| &= ||\mathbf{q}_i|| h + O(h^3), \quad \langle \mathbf{x}_1, \mathbf{x}_2 \rangle = \langle \mathbf{q}_1, \mathbf{q}_2 \rangle h^2 + O(h^4), \end{aligned}$$

we have

$$\begin{aligned} \theta_f^{(h)} &= cos^{-1} \frac{\langle \mathbf{x}_1, \mathbf{x}_2 \rangle}{||\mathbf{x}_1|| ||\mathbf{x}_2||} = cos^{-1} \frac{\langle \mathbf{q}_1, \mathbf{q}_2 \rangle + O(h^2)}{||\mathbf{q}_1|| ||\mathbf{q}_2|| + O(h^2)} \\ &= cos^{-1} \frac{\langle \mathbf{q}_1, \mathbf{q}_2 \rangle}{||\mathbf{q}_1|| ||\mathbf{q}_2||} + O(h^2) \\ &= \theta + O(h^2). \end{aligned}$$

$\diamond$

Lemma 4.6 *Let f be a smooth function around the origin $\mathbf{q}_0$, satisfying $\nabla f(\mathbf{q}_0) = \mathbf{0}$. Let $K = \{\mathbf{q}_i\}_{i=0}^n$ be a well-posed node set, and*

$$G^{(h)}(x,y) = \sum_{i+j\le 2} \mathbf{a}_{ij}^{(h)} x^i y^j / i! j!$$

the quadratic fitting function generated by Algorithm 3.1 using the sampling data $[h\mathbf{q}_i^T, f(h\mathbf{q}_i)]^T \in \mathbb{R}^3$. Then for any given $B > \|[A(K)^T A(K)]^{-1} A(K)\|$, there exists a $h_0 > 0$, such that

$$\left\|\mathbf{a}_{ij}^{(h)} - F_{ij}(\mathbf{q}_0)\right\| \le C_{ij} B h^{3-i-j}, \quad i+j \le 2, \ \ as \ \ 0 \le h < h_0, \tag{4.6}$$

where C_{ij} are constant depending on f, $F(x,y) = [x,y,f(x,y)]^T$ and $F_{ij} = \frac{\partial^{i+j} F}{\partial x^i \partial y^j}$.

Proof. Without loss of generality (WLG), we may assume $f(\mathbf{q}_0) = 0$. Let $\mathbf{q}_i'^{(h)}$ be the fitting nodes defined by Algorithm 3.1, using the sampling data $[h\mathbf{q}_i^T, f(h\mathbf{q}_i)]^T$. Let

$$\mathbf{q}_i^{(h)} = \mathbf{q}_i h = r_i^{(h)} [\cos\theta_i^{(h)}, \sin\theta_i^{(h)}]^T, \quad \mathbf{q}_i'^{(h)} = r_i'^{(h)} [\cos\theta_i'^{(h)}, \sin\theta_i'^{(h)}]^T.$$

Then by Lemma 4.4 and Lemma 4.5, we have

$$|r_i^{(h)} - r_i'^{(h)}| < Ch^3, \quad |\theta_i^{(h)} - \theta_i'^{(h)}| < Ch^2.$$

Notes that, $r_i^{(h)} = O(h)$, $r_i'^{(h)} = O(h)$. Hence

$$\|\mathbf{q}_i'^{(h)} - \mathbf{q}_i^{(h)}\| \le Ch^3.$$

Therefore, when h is small enough, $\mathbf{q}_i'^{(h)} \in \Omega_i^{(h)}$. Then by Lemma 4.3, (4.5) holds for the third component of the vector-valued function $F(x,y) = [x,y,f(x,y)]^T$. Similarly, (4.5) holds for the first and second components of $F(x,y)$. Therefore, (4.6) holds. $\diamond$

Lemma 4.6 is for the sampling data of a function. Next we consider the sampling data of a parametric surface. Let $\mathbf{x}(u,v)$ be a smooth parametric surface in the xyz-space $\mathbb{R}^3$. Let $\mathbf{q}_0$, $\mathbf{q}_1 \in \mathbb{R}^2$, and $\mathbf{q}_0 = 0$, $\mathbf{q}_1 \ne \mathbf{0}$. Suppose $\mathbf{x}(\mathbf{q}_0) = \mathbf{0}$. First we define a linear map $\sigma_{\mathbf{q}_1}(u,v)$ from uv-plane to the tangent plane of $\mathbf{x}(u,v)$ at $\mathbf{q}_0$, which is defined as follows. In the xyz-space, take another Descartes coordinate system XYZ, such that the origin of the system is $\mathbf{x}(\mathbf{q}_0)$, the unit Z, X and Y-direction are

$$\mathbf{e}_3 = \frac{\mathbf{x}_u(\mathbf{q}_0) \times \mathbf{x}_v(\mathbf{q}_0)}{\|\mathbf{x}_u(\mathbf{q}_0) \times \mathbf{x}_v(\mathbf{q}_0)\|}, \quad \mathbf{e}_1 = \frac{[\mathbf{x}_u(\mathbf{q}_0), \mathbf{x}_v(\mathbf{q}_0)]\mathbf{q}_1}{\|[\mathbf{x}_u(\mathbf{q}_0), \mathbf{x}_v(\mathbf{q}_0)]\mathbf{q}_1\|} \quad \text{and} \ \ \mathbf{e}_2 = \mathbf{e}_1 \times \mathbf{e}_3,$$

respectively. Obviously, XY-plane is the tangent plane of $\mathbf{x}(u,v)$ at $\mathbf{q}_0$. Define

$$\sigma_{\mathbf{q}_1}(u,v) : (u,v) \to (X,Y) \ \text{ such that } \ [\mathbf{e}_1, \mathbf{e}_2][X,Y]^T = [\mathbf{x}_u(\mathbf{q}_0), \mathbf{x}_v(\mathbf{q}_0)][u,v]^T.$$

That is

$$\sigma_{\mathbf{q}_1}(u,v) = [\mathbf{e}_1, \mathbf{e}_2]^T [\mathbf{x}_u(\mathbf{q}_0), \mathbf{x}_v(\mathbf{q}_0)][u,v]^T.$$

Now let us define another map $\sigma(u, v)$ from uv-place to XY-plane as follows

$$\sigma(u, v) : (u, v) \to (X, Y) \text{ such that } [X, Y]^T = [\mathbf{e}_1, \mathbf{e}_2]^T \mathbf{x}(u, v).$$

That is, $\sigma(u, v) = [\mathbf{e}_1, \mathbf{e}_2]^T \mathbf{x}(u, v)$, which is of nonlinear in general. Since the Jacobian matrix $[\mathbf{x}_u, \mathbf{x}_v]^T [\mathbf{e}_1, \mathbf{e}_2]$ of $\sigma(u, v)$ is nonsingular around $\mathbf{q}_0$, $\sigma(u, v)$ is an invertible map in a neighborhood of $\mathbf{q}_0$. It is not difficult to see that, $\sigma(u, v)$ is the coordinate of the projection point of $\mathbf{x}(u, v)$ on XY-plane. Using maps $\sigma_{\mathbf{q}_1}(u, v)$ and $\sigma(u, v)$, we can prove the following result.

Theorem 4.1 *Let $\mathbf{x}(u, v) \in \mathbb{R}^3$ be a smooth parametric surface around the origin $\mathbf{q}_0$, $K = \{\mathbf{q}_i\}_{i=0}^n$ a well-posed node set on the uv-plane. Let*

$$G^{(h)}(X, Y) = \sum_{i+j \le 2} \mathbf{s}_{ij}^{(h)} X^i Y^j / i! j!$$

be the quadratic fitting surface generated by Algorithm 3.1 using the sampling data $\mathbf{x}(h\mathbf{q}_i)$. Then we have

$$\left| \mathbf{s}_{ij}^{(h)} - \mathbf{x}_{ij}(\mathbf{q}_0) \right| \le C_{ij} B h^{3-i-j}, \quad i + j \le 2, \tag{4.7}$$

where C_{ij} are constants depending merely on $\mathbf{x}(u, v)$, B is a given constant as in Lemma 4.6.

$$\mathbf{x}_{ij}(\mathbf{q}_0) = \left. \frac{\partial^{i+j} \mathbf{x}(\sigma^{-1}(X, Y))}{\partial X^i \partial Y^j} \right|_{(X,Y)=[0,0]^T}.$$

Proof. WLG, we may assume $\mathbf{x}(\mathbf{q}_0) = \mathbf{0}$. Using the linear map $\sigma_{\mathbf{q}_1}(u, v)$, the well-posed node set $K = \{\mathbf{q}_i\}_{i=0}^n$ on the uv-plane is mapped to another well-posed node set $K' = \{\mathbf{q}'_i\}_{i=0}^n$ on the XY-plane. Using map $\sigma(u, v)$, we can identify the parametric surface $\mathbf{x}(\mathbf{q})$ as the graph $[X, Y, f(X, Y)]^T$ of a function $f(X, Y)$ on the XY-plane, where $f(X, Y)$ is defined as follows.

For any point $\mathbf{x}(\mathbf{q})$ on the parametric surface, its independent variable on the XY-plane is the projection of the point. Denoting the projection point as $P(\mathbf{x}(\mathbf{q}))$, we have ,

$$P(\mathbf{x}(\mathbf{q})) = [\mathbf{e}_1, \mathbf{e}_2][X, Y]^T, \quad [X, Y]^T = \sigma(u, v).$$

The function value $f(X, Y)$ at $P(\mathbf{x}(\mathbf{q}))$ is $\mathbf{e}_3^T[\mathbf{x}(\mathbf{q}) - P(\mathbf{x}(\mathbf{q}))] = \mathbf{e}_3^T \mathbf{x}(\mathbf{q})$. Hence

$$\mathbf{x}(\mathbf{q}) = [\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3] \left[X, Y, f(X, Y)\right]^T. \tag{4.8}$$

Since

$$\begin{aligned} P(\mathbf{x}(h\mathbf{q}_i)) &= [\mathbf{x}_u(\mathbf{q}_0), \mathbf{x}_u(\mathbf{q}_0)][g_{\alpha\beta}]^{-1}[\mathbf{x}_u(\mathbf{q}_0), \mathbf{x}_u(\mathbf{q}_0)]^T \mathbf{x}(h\mathbf{q}_i) \\ &= [\mathbf{x}_u(\mathbf{q}_0), \mathbf{x}_u(\mathbf{q}_0)][g_{\alpha\beta}]^{-1}[\mathbf{x}_u(\mathbf{q}_0), \mathbf{x}_u(\mathbf{q}_0)]^T \left[[\mathbf{x}_u(\mathbf{q}_0), \mathbf{x}_u(\mathbf{q}_0)]h\mathbf{q}_i + O(h^2)\right] \\ &= [\mathbf{x}_u(\mathbf{q}_0), \mathbf{x}_u(\mathbf{q}_0)]h\mathbf{q}_i + O(h^2), \end{aligned}$$

and

$$\mathbf{q}'_i = [\mathbf{x}_u(\mathbf{q}_0), \mathbf{x}_u(\mathbf{q}_0)]\mathbf{q}_i,$$

we have

$$\|\sigma(h\mathbf{q}_i) - \sigma_{\mathbf{q}_1}(h\mathbf{q}_i)\| = \|h\mathbf{q}_i' - P(\mathbf{x}(h\mathbf{q}_i))\| = O(h^2).$$

Therefore, $\mathbf{x}(h\mathbf{q}_i)$ can be regarded as the sampling of $f(X, Y)$ on the XY-plane. The difference of the sampling points $\{P(\mathbf{x}(h\mathbf{q}_i))\}_{i=0}^n$ and the well-posed node set $\{h\mathbf{q}_i'\}_{i=0}^n$ on the XY-plane are bounded by $O(h^2)$. Hence, by Lemma 4.3 we know that, if h is small enough, $\{P(\mathbf{x}(h\mathbf{q}_i))\}_{i=0}^n$ is also a well-posed node set on the XY-plane. In Lemma 4.6, take the function $f(x, y)$ as $f(X, Y)$, and let $\sum_{i+j\le 2} \mathbf{a}_{ij}^{(h)} X^i Y^j / i! j!$ be the fitting surface, then (4.6) holds. Let

$$\mathbf{s}_{ij}^{(h)} = [\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3]\mathbf{a}_{ij}^{(h)}.$$

From (4.8), we have

$$\begin{aligned}
[\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3] F_{ij}(X, Y) &= \frac{\partial^{i+j} [\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3] F(X, Y)}{\partial X^i \partial Y^j} \\
&= \frac{\partial^{i+j} \mathbf{x}(u, v)}{\partial X^i \partial Y^j} = \frac{\partial^{i+j} \mathbf{x}(\sigma^{-1}(X, Y))}{\partial X^i \partial Y^j},
\end{aligned}$$

Therefore, (4.7) holds. $\diamond$

Theorem 4.1 says that the first-order partial derivatives of the fitting surface have quadratic convergence rates, the second-order partial derivatives have linear convergence rates. Therefore, we have the following corollary.

Corollary 4.2 *Under the conditions of Theorem 4.1, the discretized first- and second-order geometric differential operators have the quadratic and linear convergence rates, respectively.*

It should be pointed out that except for the tangential gradient operator, which is a first-order geometric differential operator, all the other geometric differential operators introduced in section 2 are the second-order.

Remark 4.2 *The numerical experiments conducted in [31] show that for certain type domain triangulations, the discrete approximation of the mean curvature, derived from the quadratic fitting algorithm, have quadratic convergence rate. This convergence rate is one order higher (super-convergence) than what we obtained from the theoretical analysis. Therefore, an open problem left is what is the necessary and sufficient condition for domain triangulation, under which the super-convergence happens for any smooth parametric surface to be sampled. We believe that if $n = 2m$, $m > 2$, and the node set $K = \{\mathbf{q}_i\}_{i=0}^n$ satisfies*

$$\mathbf{q}_{i+m} + \mathbf{q}_i = \mathbf{0}, \quad i = 1, \cdots, m,$$

then the quadratic fitting surface, obtained from the sampling data of a smooth parametric surface on the node set $K^h = hK$, will have super-convergence property.

Remark 4.3 *In Algorithm 3.1, the local parametrization step is crucial to the quality of the fitting surface. Here an interesting problem worth further study is: what is any other local parametrization technique which yields even better fitting surface?*

5 Conclusions

In the literature, many approximate schemes on geometric differential operators have been proposed. Except for the schemes based on the interpolation or fitting, non of these schemes converges without any restriction on the regularity of the meshes considered. It has been claimed in [31] that the mean curvature computed from a parametric quadratic fitting surface converges. However, this fact has never been formally proved. In this paper, several commonly used differential operators are approximated based on a parametric quadratic fitting. We have proved that under very mild conditions, these approximated differential operators are convergent in the linear (for the second-order operators) or quadratic (for the first-order operators) rates with respect to the mesh size.

References

[1] L. Alboul and R. van Damme. *Polyhedral metrics in surface reconstruction: tight triangulation*. Technical Report, University of Twente, Dep. of Applied Mathematics, 1995.

[2] M. P. Do Carmo. *Differential Geometry of Curves and Surfaces*. Englewood Cliffs, New Jersey, 1976.

[3] F. Cazals and M. Pouget. Estimating Differential Quantities using Polynomial Fitting of Osculating Jets. *Computer Aided Geometric Design*, 22(2):767–784, 2005.

[4] X. Chen and F. Schmitt. Intrinsic surface properties from surface triangulation. In Proceedings, European Conference on Computer Vision, pages 739–743, 1992.

[5] D. Cohen-Steiner and J.-M. Morvan. Restricted delaunay triangulations and normal cycle. In In Proc. 19th Annu. ACM Sympos. Comput. Geom., 2003.

[6] M. Desbrun, M. Meyer, P. Schröder, and A. H. Barr. Implicit fairing of irregular meshes using diffusion and curvature flow. In *SIGGRAPH99*, pages 317–324, Los Angeles, USA, 1999.

[7] M. P. do Carmo. *Differential Geometry of Curves and Surfaces*. Prentice-Hall, Englewood Cliffs, NJ, 1976.

[8] M. P. do Carmo. *Riemannian Geometry*. Boston, 1992.

[9] I. Douros and B. F. Buxton. Three-dimensional surface curvature estimation using quadric surface patches. *Numerisation 3D - Scanning*, 2002.

[10] N. Dyn, K. Hormann, S. J. Kim, and D. Levin. Optimizing 3d triangulation using discrete curvature analysis. In T. Lyche and L. L. Schmaker, editors, *Mathematical Methods in CAGD*, pages 135–146. Vanderbilt University Press, 2001.

[11] K. Fujiwara. Eigenvalues of laplacians on a closed riemannian manifold and its nets. In Proceedings of the AMS, pages 2585–2594, 1995.

[12] M. Giaquinta and S. Hildebrandt. *Calculus of Variations I*. Springer-Verlag, Berlion Heidelberg, 1996.

[13] G. Golub and C. Van Loan. Matrix Computations. The Johns Hopkins University Press, 1996.

[14] B. Hamann. Curvature approximation for triangulated surfaces. In G. Farin et al, editor, *Geometric Modelling*, pages 139–153. Springer-Verlag, 1993.

[15] S. J. Kim, C. H. Kim, and D. Levin. Surface simplification using a discrete curvature norm. *Computers and Graphics*, 26(5):657–663, 2002.

[16] P. Krsek, T. Pajdla, and V. Hlavac. Estimation of differential parameters on triangulated surfaces. In *21st Workshop of the Austrian Association for Pattern Recognition*, May 1997.

[17] T. Langer, A. G. Belyaev, and H. Seidel. Analysis and design of discrete normals and curvatures. MPI-I-2005-4-003, Max-Planck-Institute for Informatik, March 2005.

[18] D. Liu and G. Xu. A general sixth order geometric partial differential equation and its application in surface modeling. *Journal of Information and Computational Science*, 4(1):129–140, 2007.

[19] R. Martin. Estimation of principal curvatures from range data. *International Journal of Shape Modeling*, 4:99–111, 1998.

[20] U. F. Mayer. Numerical Solutions for the Surface Diffusion Flow in Three Space Dimensions. *Computational and Applied Mathematics (to appear)*, 2001.

[21] D. Meek and D. Walton. On surface normal and Gaussian curvature approximations given data sampled from a smooth surface. *Computer Aided Geometric Design*, 17:521–543, 2000.

[22] M. Meyer, M. Desbrun, P. Schröder, and A. Barr. Discrete differential-geometry operators for triangulated 2-manifolds. In *Proceedings of Visual Mathematics'02*, Berlin, Germany, 2002.

[23] J. Peng, Q. Li, C. J. Kuo, and M. Zhou. Estimating Gaussian Curvature from 3D Meshes. *(to appear) SPIE Electronic Image*, 2003.

[24] A. Razdan and M. Bae. Curvature estimation scheme for triangle meshes using biquadratic Bézier patch. *Computer-Aided Design*, 22?(1):1–11?, 2005.

[25] E. Stokely and S. Y. Wu. Surface parameterization and curvature measurement of arbitrary 3d-objects: Five practical methods. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 14(8):833–840, 1992.

[26] G. Taubin. Estimating the tensor of curvatures of a surface from a polyhedral approximation. In Proceedings 5th Intl. Conf. on Computer Vision (ICCV'95), pages 902–907, 1995.

[27] G. Taubin. A signal processing approach to fair surface design. In *SIGGRAPH '95 Proceedings*, pages 351–358, 1995.

[28] G. Taubin. Signal processing on polygonal meshes. In *EUROGRAPHICS*. 2000.

[29] K. Watanabe and A. G. Belyaev. Meshes: Detection of salient curvature features on polygonal surfaces. *Computer Graphics Forum*, 20(3), 2001.

[30] C. Wollmann. Estimation of principle curvatures of approximated surfaces. *Computer Aided Geometric Design*, 17:621–630, 2000.

[31] G. Xu. Discrete Laplace-Beltrami operators and their convergence. *Computer Aided Geometric Design*, 21(8):767–784, 2004.

[32] G. Xu. Convergence analysis of a discretization scheme for Gaussian curvature over triangular surfaces. *Computer Aided Geometric Design*, 23(2):193–207, 2006.

[33] G. Xu, Q. Pan, and C. Bajaj. Discrete surface modelling using partial differential equations. *Computer Aided Geometric Design*, 23(2):125–145, 2006.

[34] G. Xu and Q. Zhang. Construction of Geometric Partial Differential Equations in Computational Geometry. *Mathematica Numerica Sinica*, 28(4):337–356, 2006.

[35] G. Xu and Q. Zhang. G^2 surface modeling using minimal mean–curvature–variation flow. *Computer-Aided Design*, 39(5):342–351, 2007.

[36] G. Xu and Q. Zhang. A General Framework for Surface Modeling Using Geometric Partial Differential Equations. *Computer Aided Geometric Design*, 25(3):181–202, 2008.

[37] Q. Zhang and G. Xu. Geometric partial differential equations for minimal curvature variation surfaces. Research Report No. ICM-06-03, Institute of Computational Mathematics, Chinese Academy of Sciences, 2006.

[38] Q. Zhang, G. Xu, and J. Sun. A general sixth order geometric flow and its applications in surface processing. In *Cyberworlds Conference Proceedings 2007*, Hannover, Oct. 24-27 2007. IEEE.