AN ADAPTIVE UNIAXIAL PERFECTLY MATCHED LAYER METHOD FOR TIME-HARMONIC SCATTERING PROBLEMS

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Abstract. The uniaxial perfectly matched layer (PML) method uses rectangular domain to define the PML problem and thus provides greater flexibility and efficiency in dealing with problems involving anisotropic scatterers. In this paper an adaptive uniaxial PML technique for solving the time harmonic Helmholtz scattering problem is developed. The PML parameters such as the thickness of the layer and the fictitious medium property are determined through sharp a posteriori error estimates. The adaptive finite element method based on a posteriori error estimate is proposed to solve the PML equation which produces automatically a coarse mesh size away from the fixed domain and thus makes the total computational costs insensitive to the thickness of the PML absorbing layer. Numerical experiments are included to illustrate the competitive behavior of the proposed adaptive method. In particular, it is demonstrated that the PML layer can be chosen as close to one wave-length from the scatterer and still yields good accuracy and efficiency in approximating the far fields.

Key words. Adaptivity, uniaxial perfectly matched layer, a posteriori error analysis, acoustic scattering problems.

1. Introduction. We propose and study a uniaxial perfectly matched layer (PML) technique for solving Helmholtz-type scattering problems with perfectly conducting boundary:

$$\Delta u + k^2 u = 0 \quad \text{in } \mathbb{R}^2\backslash\bar{D}, \tag{1.1a}$$

$$\frac{\partial u}{\partial \mathbf{n}_D} = -g \quad \text{on } \Gamma_D, \tag{1.1b}$$

$$\sqrt{r}\left(\frac{\partial u}{\partial r} - \mathbf{i}ku\right) \to 0 \quad \text{as } r = |x| \to \infty. \tag{1.1c}$$

Here $D \subset \mathbb{R}^2$ is a bounded domain with Lipschitz boundary Γ_D, $g \in H^{-1/2}(\Gamma_D)$ is determined by the incoming wave, and $\mathbf{n}_D$ is the unit outer normal to Γ_D. We assume the wave number $k \in \mathbb{R}$ is a constant. We remark that the results in this paper can be extended to the case when $k^2(x)$ is a variable wave number inside some bounded domain, or to solve the scattering problems with other boundary conditions, such as Dirichlet or the impedance boundary condition on Γ_D.

Since the work of Bérénger [3] which proposed a PML technique for solving the time dependent Maxwell equations, various constructions of PML absorbing layers have been proposed and studied in the literature (cf. e.g. Turkel and Yefet [19], Teixeira and Chew [18] for the reviews). The basic idea of the PML technique is to surround the computational domain by a layer of finite thickness with specially designed model medium that would either slow down or attenuate all the waves that propagate from inside the computational domain.

The convergence of the PML method is studied in Lassas and Somersalo [13], Hohage et al [12] for the acoustic scattering problems for circular PML layers and in Lassas and Somersalo [14] for general smooth convex geometry. It is proved in [13, 12, 14] that the PML solution converges exponentially to the solution of the original scattering problem as the thickness of the PML layer tends to infinite. We remark that in practical applications involving PML techniques, one cannot afford to use a very thick PML layer if uniform meshes are used because it requires excessive grid points and hence more computer time and more storage. On the other hand, a thin PML layer requires a rapid variation of the artificial material property which deteriorates the accuracy if too coarse mesh is used in the PML layer.

The adaptive PML technique was proposed in Chen and Wu [8] for a scattering problem by periodic structures (the grating problem), in Chen and Liu [6] for the acoustic scattering problem, and in Chen and Chen [5] for Maxwell scattering problems. The main idea of the adaptive PML technique is to use the a posteriori error estimate to determine the PML parameters and to use the adaptive finite element method to solve the PML equations. The adaptive PML technique provides a complete numerical strategy to solve the scattering problems in the framework of finite element which produces automatically a coarse mesh size away from the fixed domain and thus makes the total computational costs insensitive to the thickness of the PML absorbing layer.

The purpose of this paper is to extend the adaptive PML technique developed for circular PML layer in [8, 6, 5] to deal with the uniaxial PML methods which are widely used in the engineering literature. The main advantage of the uniaxial PML method as opposing to the circular PML method is that it provides greater flexibility and efficiency to solve problems involving anisotropic scatterers. Our technique to prove the PML convergence is different from the techniques developed in [8, 6, 5] for circular PML layers. It is based on the integral representation of the exterior Dirichlet problem for the Helmholtz equation and the idea of the complex coordinate stretching. To the authors' best knowledge, this is the first convergence proof of the uniaxial PML method in the literature. We remark that the boundary of the uniaxial PML layer is only Lipschitz and so the results in [14] cannot be applied.

The layout of the paper is as follows. In section 2 we recall the uniaxial PML formulation for (1.1a)-(1.1c) by following the method of complex coordinate stretching in Chew and Weedon [4]. In section 3 we prove the convergence of the uniaxial PML method. In section 4 we introduce the finite element approximation. In section 5 we derive the a posteriori error estimate which includes both the PML error and the finite element discretization error. Finally in section 6 we describe our adaptive algorithm and present two examples to show the competitive behavior of the adaptive method.

2. The PML equation. Let D be contained in the interior of the rectangle $B_1 = \{x \in \mathbb{R}^2 : |x_1| < L_1/2, |x_2| < L_2/2\}$. Let $\Gamma_1 = \partial B_1$ and $\mathbf{n}_1$ the unit outer normal to Γ_1. We start by introducing the Dirichlet-to-Neumann operator $T : H^{1/2}(\Gamma_1) \to H^{-1/2}(\Gamma_1)$. Given $f \in H^{1/2}(\Gamma_1)$, we define $Tf = \dfrac{\partial \xi}{\partial \mathbf{n}_1}$ on Γ_1, where ξ is the solution of the following exterior Dirichlet problem of the Helmholtz equation

$$\Delta\xi + k^2\xi = 0 \quad \text{in } \mathbb{R}^2\backslash\bar{B}_1, \tag{2.1a}$$

$$\xi = f \quad \text{on } \Gamma_1, \tag{2.1b}$$

$$\sqrt{r}\left(\frac{\partial\xi}{\partial r} - \mathbf{i}k\xi\right) \to 0 \quad \text{as } r = |x| \to \infty. \tag{2.1c}$$

It is well-known that (2.1a)-(2.1c) has a unique solution $\xi \in H^1_{loc}(\mathbb{R}^2\backslash\bar{B}_1)$ (cf. e.g. Colten-Kress [11]). Thus $T : H^{1/2}(\Gamma_1) \to H^{-1/2}(\Gamma_1)$ is well-defined and is a continuous linear operator.

Let $a : H^1(\Omega_1) \times H^1(\Omega_1) \to \mathbb{C}$, where $\Omega_1 = B_1\backslash\bar{D}$, be the sesquilinear form

$$a(\varphi, \psi) = \int_{\Omega_1} \left(\nabla\varphi \cdot \nabla\bar{\psi} - k^2\varphi\bar{\psi}\right) dx - \langle T\varphi, \psi\rangle_{\Gamma_1}, \tag{2.2}$$

where $\langle \cdot, \cdot \rangle_{\Gamma_1}$ stands for the inner product on $L^2(\Gamma_1)$ or the duality pairing between $H^{-1/2}(\Gamma_1)$ and $H^{1/2}(\Gamma_1)$. The scattering problem (1.1a)-(1.1c) is equivalent to the following weak formulation (cf. e.g. [11]): Given $g \in H^{-1/2}(\Gamma_D)$, find $u \in H^1(\Omega_1)$ such that

$$a(u, \psi) = \langle g, \psi \rangle_{\Gamma_D}, \qquad \forall\, \psi \in H^1(\Omega_1). \tag{2.3}$$

The existence of a unique solution of the scattering problem (2.3) is known (cf. e.g. [11], McLean [15]). Then the general theory in Babuška and Aziz [1, Chap. 5] implies that there exists a constant $\mu > 0$ such that the following inf-sup condition is satisfied

$$\sup_{0 \neq \psi \in H^1(\Omega_1)} \frac{|a(\varphi, \psi)|}{\|\,\psi\,\|_{H^1(\Omega_1)}} \geq \ \mu \|\,\varphi\,\|_{H^1(\Omega_1)}, \qquad \forall\, \varphi \in H^1(\Omega_1). \tag{2.4}$$

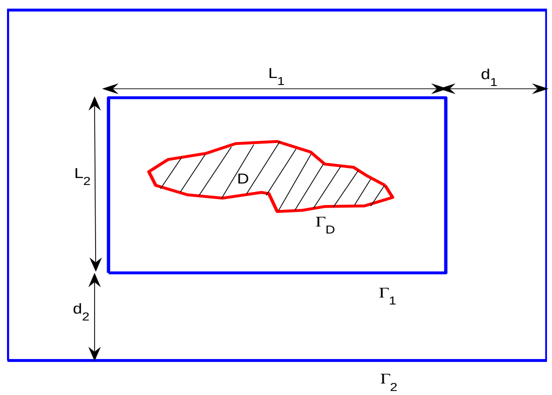

Fig. 2.1. *Setting of the scattering problem with the PML layer.*

Now we turn to the introduction of the absorbing PML layer. Let $B_2 = \{x \in \mathbb{R}^2 : |x_1| < L_1/2 + d_1, |x_2| < L_2/2 + d_2\}$ be the rectangle which contains B_1. Let $\alpha_1(x_1) = 1 + \mathbf{i}\sigma_1(x_1), \alpha_2(x_2) = 1 + \mathbf{i}\sigma_2(x_2)$ be the model medium property which satisfy

$$\sigma_j \in C(\mathbb{R}), \quad \sigma_j \geq 0, \quad \sigma_j(t) = \sigma_j(-t), \quad \text{and} \quad \sigma_j = 0 \ \text{ for } \ |t| \leq L_j/2, \quad j = 1, 2.$$

Denote by $\tilde{x}_j$ the complex coordinate defined by

$$\tilde{x}_j = \begin{cases} x_j & \text{if } |x_j| < L_j/2, \\ \int_0^{x_j} \alpha_j(t)dt & \text{if } |x_j| \geq L_j/2. \end{cases} \tag{2.5}$$

To derive the PML equation, we first notice that by the third Green formula, the solution ξ of the exterior Dirichlet problem (2.1a)-(2.1c) satisfies

$$\xi = -\Psi^k_{\mathrm{SL}}(\lambda) + \Psi^k_{\mathrm{DL}}(f) \qquad \text{in } \mathbb{R}^2 \backslash \bar{B}_1, \tag{2.6}$$

where $\lambda = Tf \in H^{-1/2}(\Gamma_1)$ is the Neumann trace of ξ on Γ_1, and $\Psi_{\rm SL}^k, \Psi_{\rm DL}^k$ are respectively the single and double layer potentials

$$\Psi_{\rm SL}^k(\lambda)(x) = \int_{\Gamma_1} G_k(x,y)\lambda(y)\,\mathrm{d}s(y), \qquad \forall\, \lambda \in H^{-1/2}(\Gamma_1), \tag{2.7}$$

$$\Psi_{\rm DL}^k(f)(x) = \int_{\Gamma_1} \frac{\partial G_k(x,y)}{\partial \mathbf{n}_1(y)} f(y)\,\mathrm{d}s(y), \qquad \forall\, f \in H^{1/2}(\Gamma_1). \tag{2.8}$$

Here G_k is the fundamental solution of the Helmholtz equation satisfying the Sommerfeld radiation condition

$$G_k(x,y) = \frac{\mathbf{i}}{4} H_0^{(1)}(k|x-y|),$$

where $H_0^{(1)}(z)$ is the first Hankel function of order zero.

We follow the method of complex coordinate stretching [4] to introduce the PML equation. For any $z \in \mathbb{C}$, denote by $z^{1/2}$ the analytic branch of $\sqrt{z}$ such that $\mathrm{Im}\,(z^{1/2}) > 0$ for any $z \in \mathbb{C}\backslash[0,+\infty)$. Let $\rho(\tilde{x},y) = [(\tilde{x}_1 - y_1)^2 + (\tilde{x}_2 - y_2)^2]^{1/2}$ be the complex distance and define

$$\tilde{G}_k(x,y) = \frac{\mathbf{i}}{4} H_0^{(1)}(k\rho(\tilde{x},y)).$$

It is easy to see that $\tilde{G}_k$ is smooth for $x \in \mathbb{R}^2\backslash\bar{B}_1$ and $y \in \bar{B}_1$. Now we can define the modified single and double layer potentials [14]

$$\tilde{\Psi}_{\rm SL}^k(\lambda)(x) = \int_{\Gamma_1} \tilde{G}_k(x,y)\lambda(y)\,\mathrm{d}s(y), \qquad \forall\, \lambda \in H^{-1/2}(\Gamma_1), \tag{2.9}$$

$$\tilde{\Psi}_{\rm DL}^k(f)(x) = \int_{\Gamma_1} \frac{\partial \tilde{G}_k(x,y)}{\partial \mathbf{n}_1(y)} f(y)\,\mathrm{d}s(y), \qquad \forall\, f \in H^{1/2}(\Gamma_1). \tag{2.10}$$

It is clear that $\tilde{\Psi}_{\rm SL}^k(\lambda), \tilde{\Psi}_{\rm DL}^k(f)$ are smooth in $\mathbb{R}^2\backslash\bar{B}_1$, and

$$\begin{aligned} \gamma_D^+\tilde{\Psi}_{\rm SL}^k(\lambda) &= \gamma_D^+\Psi_{\rm SL}^k(\lambda), \qquad \forall\, \lambda \in H^{-1/2}(\Gamma_1), \\ \gamma_D^+\tilde{\Psi}_{\rm DL}^k(f) &= \gamma_D^+\Psi_{\rm DL}^k(f), \qquad \forall\, f \in H^{1/2}(\Gamma_1), \end{aligned} \tag{2.11}$$

where $\gamma_D^+ : H^1_{loc}(\mathbb{R}^2\backslash\bar{B}_1) \to H^{1/2}(\Gamma_1)$ is the trace operator.

For any $f \in H^{1/2}(\Gamma_1)$, let $\mathbb{E}(f)(x)$ be the PML extension given by

$$\mathbb{E}(f)(x) = -\tilde{\Psi}_{\rm SL}^k(Tf) + \tilde{\Psi}_{\rm DL}^k(f) \qquad \text{for } x \in \mathbb{R}^2\backslash\bar{B}_1. \tag{2.12}$$

By (2.11) and (2.6) we know that $\gamma_D^+\mathbb{E}(f) = -\gamma_D^+\Psi_{\rm SL}^k(Tf) + \gamma_D^+\Psi_{\rm DL}^k(f) = \gamma_D^+\xi = f$ on Γ_1 for any $f \in H^{1/2}(\Gamma_1)$. For the solution u of the scattering problem (2.3), let $\tilde{u} = \mathbb{E}(u|_{\Gamma_1})$ be the PML extension of $u|_{\Gamma_1}$ which satisfies $\gamma_D^+\tilde{u} = u|_{\Gamma_1}$ on Γ_1. Since $H_0^{(1)}(z)$ decays exponentially on the upper half complex plane [6], heuristically $\tilde{u}(x)$ will decay exponentially when x is away from Γ_1. It is obvious that $\tilde{u}$ satisfies

$$\frac{\partial^2\tilde{u}}{\partial\tilde{x}_1^2} + \frac{\partial^2\tilde{u}}{\partial\tilde{x}_2^2} + k^2\tilde{u} = 0 \qquad \text{in } \mathbb{R}^2\backslash\bar{B}_1,$$

which yields the desired PML equation by the chain rule

$$\nabla\cdot(A\nabla\tilde{u}) + \alpha_1\alpha_2 k^2\tilde{u} = 0 \qquad \text{in } \mathbb{R}^2\backslash\bar{B}_1,$$

where $A = \mathrm{diag}(\alpha_2(x_2)/\alpha_1(x_1), \alpha_1(x_1)/\alpha_2(x_2))$ is a diagonal matrix.

The PML solution $\hat{u}$ in $\Omega_2 = B_2\backslash\bar{D}$ is defined as the solution of the following system

$$\nabla\cdot(A\nabla\hat{u}) + \alpha_1\alpha_2 k^2\hat{u} = 0 \quad \text{in } \Omega_2, \tag{2.13a}$$

$$\frac{\partial\hat{u}}{\partial\mathbf{n}_D} = -g \ \text{ on } \Gamma_D, \quad \hat{u} = 0 \ \text{ on } \Gamma_2. \tag{2.13b}$$

The well-posedness of the PML problem (2.13a)-(2.13b) and the convergence of its solution to the solution of the original scattering problem will be studied in the next section.

3. Convergence analysis. We start by considering the Dirichlet problem of the PML equation in the layer

$$\nabla\cdot(A\nabla w) + \alpha_1\alpha_2 k^2 w = 0 \quad \text{in } \Omega^{\mathrm{PML}} = B_2\backslash\bar{B}_1, \tag{3.1a}$$

$$w = 0 \ \text{ on } \Gamma_1, \quad w = q \ \text{ on } \Gamma_2, \tag{3.1b}$$

where $q \in H^{1/2}(\Gamma_2)$. Introduce the sesquilinear form $c : H^1(\Omega^{\mathrm{PML}}) \times H^1(\Omega^{\mathrm{PML}}) \to \mathbb{C}$ as

$$c(\varphi,\psi) = \int_{\Omega^{\mathrm{PML}}} (A\nabla\varphi\cdot\nabla\bar{\psi} - \alpha_1\alpha_2 k^2\varphi\bar{\psi})\,\mathrm{d}x.$$

Then the weak formulation for (3.1a)-(3.1b) is: Given $q \in H^{1/2}(\Gamma_2)$, find $w \in H^1(\Omega^{\mathrm{PML}})$ such that $w = 0$ on Γ_1, $w = q$ on Γ_2, and

$$c(w,\psi) = 0, \quad \forall\,\psi \in H_0^1(\Omega^{\mathrm{PML}}). \tag{3.2}$$

Notice that, for any $\varphi \in H^1(\Omega^{\mathrm{PML}})$,

$$\mathrm{Re}\,[c(\varphi,\varphi)] = \int_{\Omega^{\mathrm{PML}}} \left(\frac{1+\sigma_1\sigma_2}{1+\sigma_1^2}\left|\frac{\partial\varphi}{\partial x_1}\right|^2 + \frac{1+\sigma_1\sigma_2}{1+\sigma_2^2}\left|\frac{\partial\varphi}{\partial x_2}\right|^2 + (\sigma_1\sigma_2 - 1)k^2|\varphi|^2\right)\mathrm{d}x.$$

Since

$$\frac{1+\sigma_1\sigma_2}{1+\sigma_1^2} \geq \frac{1}{1+\sigma_m^2}, \qquad \frac{1+\sigma_1\sigma_2}{1+\sigma_2^2} \geq \frac{1}{1+\sigma_m^2},$$

where $\sigma_m = \max_{x\in\bar{B}_2}(\sigma_1(x_1), \sigma_2(x_2)) > 0$, we know by using the spectral theory of compact operators that (3.2) has a unique solution for every real k except possibly for a discrete set of values of k (see Collino and Monk [10, Theorem 2] for a similar discussion on the PML equation in the polar coordinates). In this paper we will not elaborate on this issue and simply make the following assumption

(H1) There exists a unique solution to the Dirichlet PML problem (3.2) in the layer.

We remark that for the circular PML method, the unique existence of the PML equation in the layer can be proved under certain conditions on the PML medium property in [13], [6]. The proof of the unique existence of the Dirichlet problem for the uniaxial PML equation is an interesting open problem.

Throughout the paper we will use the weighted H^1-norm

$$\| \varphi \|_{H^1(\Omega)} = \left(\| \nabla\varphi \|^2_{L^2(\Omega)} + |\Omega|^{-1} \| \varphi \|^2_{L^2(\Omega)} \right)^{1/2},$$

for any bounded domain $\Omega \subset \mathbb{R}^2$, where $|\Omega|$ is the Lebesgue measure of Ω. For any $\varphi \in H^1(\Omega^{\rm PML})$, we define

$$\| \varphi \|_{*,\Omega^{\rm PML}} = \left[\int_{\Omega^{\rm PML}} \left(\frac{1}{1+\sigma_1^2} \left| \frac{\partial\varphi}{\partial x_1} \right|^2 + \frac{1}{1+\sigma_2^2} \left| \frac{\partial\varphi}{\partial x_2} \right|^2 + (1+\sigma_1\sigma_2) k^2 |\varphi|^2 \right) \mathrm{d}x \right]^{1/2}.$$

It is easy to see that $\| \cdot \|_{*,\Omega^{\rm PML}}$ is an equivalent norm on $H^1(\Omega^{\rm PML})$. Again by using the general theory in [1, Chap. 5] we know that there exists a constant $\hat{C} > 0$ such that

$$\sup_{0 \neq \psi \in H_0^1(\Omega^{\rm PML})} \frac{|c(\varphi,\psi)|}{\| \psi \|_{*,\Omega^{\rm PML}}} \geq \hat{C} \| \varphi \|_{*,\Omega^{\rm PML}}. \tag{3.3}$$

The constant $\hat{C}$ depends in general on the domain $\Omega^{\rm PML}$ and the wave number k.

Before we state the main result of this section, we make the following assumptions on the fictitious medium property, which is rather mild in the practical applications of the uniaxial PML method

(H2) $\displaystyle\int_0^{\frac{L_1}{2}+d_1} \sigma_1(t)\,\mathrm{d}t = \int_0^{\frac{L_2}{2}+d_2} \sigma_2(t)\,\mathrm{d}t = \sigma, \quad \sigma > 0$ is a constant;

(H3) $\displaystyle\sigma_j(t) = \tilde{\sigma}_j \left(\frac{|t| - L_j/2}{d_j} \right)^m, \quad m \geq 1$ integer, $\tilde{\sigma}_j > 0$ is a constant, $j = 1, 2$.

The following theorem is the main result of this section.

THEOREM 3.1. *Let (H1)-(H3) be satisfied. Then for sufficiently large $\sigma > 0$, the PML problem* (2.13a)-(2.13b) *has a unique solution $\hat{u} \in H^1(\Omega_2)$. Moreover, we have the following error estimate*

$$\| u - \hat{u} \|_{H^1(\Omega_1)} \leq C\hat{C}^{-1} |\alpha_m|^3 (1+kL)^4 \, e^{-(\gamma k\sigma - 1)} \| \hat{u} \|_{H^{1/2}(\Gamma_1)}, \tag{3.4}$$

where $\gamma = \frac{\min(d_1,d_2)}{\sqrt{(L_1+d_1)^2+(L_2+d_2)^2}}$, $L = \max(L_1, L_2)$.

The proof of this theorem will be given in Section 3.3 which depends on the exponential decay estimates of the PML extension in Section 3.1 and the stability estimates of the Dirichelt problem of the PML equation in the layer in Section 3.2.

3.1. Estimates for the PML extension. We start with the following elementary lemma.

LEMMA 3.2. *For any $z_1 = a_1 + \mathbf{i}b_1, z_2 = a_2 + \mathbf{i}b_2$ with $a_1, b_1, a_2, b_2 \in \mathbb{R}$ such that $a_1b_1 + a_2b_2 \geq 0$ and $a_1^2 + a_2^2 > 0$, we have*

$$\mathrm{Im}\,(z_1^2 + z_2^2)^{1/2} \geq \frac{a_1b_1 + a_2b_2}{\sqrt{a_1^2 + a_2^2}}.$$

Proof. For any $a, b \in \mathbb{R}$ we know that

$$\operatorname{Im}(a+\mathbf{i}b)^{1/2} = \sqrt{\frac{-a+\sqrt{a^2+b^2}}{2}}.$$

Here we used the convention that $z^{1/2}$ is the analytic branch of $\sqrt{z}$ such that $\operatorname{Im}(z^{1/2}) > 0$ for any $z \in \mathbb{C}\backslash[0,+\infty)$. It is easy to check that $\operatorname{Im}(a+\mathbf{i}b)^{1/2}$ is a decreasing function in $a \in \mathbb{R}$. Let $z_1^2 + z_2^2 = a + \mathbf{i}b$, then

$$a+\mathbf{i}b = \left(\sqrt{a_1^2+a_2^2} + \mathbf{i}\frac{a_1b_1+a_2b_2}{\sqrt{a_1^2+a_2^2}}\right)^2 - \frac{(a_2b_1-a_1b_2)^2}{a_1^2+a_2^2}.$$

Let $a' = a + \frac{(a_2b_1-a_1b_2)^2}{a_1^2+a_2^2}$, since $a_1b_1+a_2b_2 \geq 0$, we have

$$\operatorname{Im}(a'+\mathbf{i}b)^{1/2} = \frac{a_1b_1+a_2b_2}{\sqrt{a_1^2+a_2^2}}.$$

On the other hand, since $a' \geq a$, we know that $\operatorname{Im}(a+\mathbf{i}b)^{1/2} \geq \operatorname{Im}(a'+\mathbf{i}b)^{1/2}$. This completes the proof. $\square$

Now let $z_j = \tilde{x}_j - y_j = (x_j - y_j) + \mathbf{i}\int_0^{x_j}\sigma_j(t)\,\mathrm{d}t$. For any $x \in \Gamma_2, y \in \bar{\Omega}_1$, it is easy to see that $(x_j - y_j)\int_0^{x_j}\sigma_j(t)\,\mathrm{d}t \geq 0$. Thus, by Lemma 3.2, $\rho(\tilde{x}, y) = (z_1^2+z_2^2)^{1/2}$ satisfies

$$\operatorname{Im}\rho(\tilde{x},y) \geq \frac{|x_1-y_1|\,|\int_0^{x_1}\sigma_1(t)\,\mathrm{d}t| + |x_2-y_2|\,|\int_0^{x_2}\sigma_2(t)\,\mathrm{d}t|}{|x-y|}.$$

Now by (H2) we have, for any $x \in \Gamma_2, y \in \bar{\Omega}_1$,

$$\operatorname{Im}\rho(\tilde{x},y) \geq \frac{\min(d_1,d_2)}{\sqrt{(L_1+d_1)^2+(L_2+d_2)^2}}\,\sigma = \gamma\,\sigma, \tag{3.5}$$

where $\gamma = \frac{\min(d_1,d_2)}{\sqrt{(L_1+d_1)^2+(L_2+d_2)^2}}$.

We need the modified Bessel function $K_\nu(z)$ of order ν, $\nu \in \mathbb{C}$, which is the solution of the differential equation

$$z^2\frac{\mathrm{d}^2w}{\mathrm{d}z^2} + z\frac{\mathrm{d}w}{\mathrm{d}z} - (z^2+\nu^2)w = 0$$

satisfying the asymptotic behavior $K_\nu(z) \sim \mathbf{i}(\frac{\pi}{2z})^{1/2}e^{-z-\frac{\pi}{4}\mathbf{i}}$ as $|z| \to \infty$. $K_\nu(z)$ is connected with $H_\nu^{(1)}(z)$ through the relation $K_\nu(z) = \frac{1}{2}\pi\mathbf{i}e^{\frac{1}{2}\nu\pi\mathbf{i}}H_\nu^{(1)}(\mathbf{i}z)$. Thus we know that

$$\tilde{G}_k(x,y) = \frac{\mathbf{i}}{4}H_0^{(1)}(k\rho(\tilde{x},y)) = \frac{1}{2\pi}K_0(-\mathbf{i}k\rho(\tilde{x},y)). \tag{3.6}$$

We refer to the treatise Watson [20] for extensive studies on the special function $K_\nu(z)$.

LEMMA 3.3. *For any $\nu \in \mathbb{R}$, $\theta_2 \geq \theta_1 > 0$, we have*

$$K_\nu(\theta_2) \leq e^{-(\theta_2-\theta_1)}K_\nu(\theta_1).$$

Proof. We first recall the Schläfli integral representation formula [20, P. 181], for $z \in \mathbb{C}$ such that $|\arg z| < \pi/2$,

$$K_\nu(z) = \int_0^\infty e^{-z\cosh t}\cosh \nu t\,\mathrm{d}t. \tag{3.7}$$

The lemma is a direct consequence of (3.7). $\square$

The following lemma on the estimates of the fundamental solution $\tilde{G}_k$ of the PML equation will play an important role in the analysis in this paper.

LEMMA 3.4. *Let (H1)-(H3) be satisfied. Let γ, σ be so chosen that*

$$\gamma k\sigma \geq 1. \tag{3.8}$$

Then there exists a constant $C > 0$ depending only on γ but independent of k, σ, L_j, d_j, $j = 1, 2$, such that for any $x \in \Gamma_2, y \in \bar{\Omega}_1$,

(i) $|\tilde{G}_k(x, y)| \leq C\, e^{-(\gamma k\sigma - 1)}$;

(ii) $\left|\dfrac{\partial \tilde{G}_k}{\partial x_j}\right| \leq Ck|\alpha_m|\, e^{-(\gamma k\sigma-1)}, \quad j = 1, 2$;

(iii) $\left|\dfrac{\partial \tilde{G}_k}{\partial y_j}\right| \leq Ck\, e^{-(\gamma k\sigma-1)}, \qquad j = 1, 2$;

(iv) $\left|\dfrac{\partial^2 \tilde{G}_k}{\partial x_i \partial y_j}\right| \leq Ck^2|\alpha_m|\, e^{-(\gamma k\sigma-1)}, \quad i, j = 1, 2.$

Here $\alpha_m = \max\limits_{x\in\Gamma_2}(|\alpha_1(x_1)|, |\alpha_2(x_2)|)$.

Proof. By the Schläfli integral representation formula (3.7), it is easy to see that $|K_0(z)| < K_0(\mathrm{Re}\,(z))$ for any $z \in \mathbb{C}$ such that $\mathrm{Re}\,(z) > 0$. Thus, by (3.6) and Lemma 3.3, if $k\,\mathrm{Im}\,\rho(\tilde{x}, y) \geq 1$,

$$|\tilde{G}_k| \leq \frac{1}{2\pi}|K_0(-\mathbf{i}k\rho(\tilde{x}, y))| \leq \frac{1}{2\pi}K_0(k\mathrm{Im}\,\rho(\tilde{x}, y)) \leq \frac{1}{2\pi}K_0(1)e^{-(k\mathrm{Im}\,\rho(\tilde{x},y)-1)}.$$

This proves (i) by (3.5) and (3.8).

To show (ii) we first notice that

$$\frac{\partial \tilde{G}_k}{\partial x_j} = -\frac{\mathbf{i}k}{2\pi}K_0'(-\mathbf{i}k\rho(\tilde{x}, y))\frac{\partial \rho(\tilde{x}, y)}{\partial x_j} = \frac{\mathbf{i}k}{2\pi}K_1(-\mathbf{i}k\rho(\tilde{x}, y))\frac{(\tilde{x}_j - y_j)\alpha_j(x_j)}{\rho(\tilde{x}, y)},$$

where we have used the identity $K_0'(z) = -K_1(z)$. Note that when $|x - y| \geq 2\sigma$, we have

$$|\rho(\tilde{x}, y)| \geq |\mathrm{Re}\,(z_1^2 + z_2^2)|^{1/2} \geq (|x - y|^2 - 2\sigma^2)^{1/2} \geq \frac{1}{\sqrt{2}}|x - y|,$$

where $z_j = (\tilde{x}_j - y_j)$. Thus for any $x \in \Gamma_2, y \in \bar{\Omega}_1$,

$$\frac{|\tilde{x}_j - y_j|}{|\rho(\tilde{x}, y)|} \leq \sqrt{2}\frac{(|x - y|^2 + \sigma^2)^{1/2}}{|x - y|} \leq \sqrt{2}\left(1 + \frac{\sigma^2}{|x - y|^2}\right)^{1/2} \leq \frac{\sqrt{10}}{2}. \tag{3.9}$$

On the other hand, when $|x - y| \leq 2\sigma$, by (3.5) we know that $\mathrm{Im}\,\rho(\tilde{x}, y) \geq \gamma\sigma$. Thus for any $x \in \Gamma_2, y \in \bar{\Omega}_1$,

$$\frac{|\tilde{x}_j - y_j|}{|\rho(\tilde{x}, y)|} \leq \frac{(|x - y|^2 + \sigma^2)^{1/2}}{\mathrm{Im}\,\rho(\tilde{x}, y)} \leq \gamma^{-1}\frac{(|x - y|^2 + \sigma^2)^{1/2}}{\sigma} \leq \sqrt{5}\gamma^{-1}. \tag{3.10}$$

This proves (ii) again by using (3.8) and Lemma 3.3. Similarly, we can prove (iii).

To prove (iv) we note that

$$\frac{\partial^2 \tilde{G}_k}{\partial x_i \partial y_j} = \frac{k^2}{2\pi} K_1'(-\mathbf{i}k\rho(\tilde{x},y)) \frac{-(\tilde{x}_i - y_i)(\tilde{x}_j - y_j)\alpha_i(x_i)}{\rho(\tilde{x},y)^2}$$
$$+ \frac{\mathbf{i}k}{2\pi} K_1(-\mathbf{i}k\rho(\tilde{x},y)) \frac{-\rho^2\delta_{ij}\alpha_i(x_i) + (\tilde{x}_i - y_i)(\tilde{x}_j - y_j)\alpha_i(x_i)}{\rho^3}.$$

By using the identity $K_1'(z) = -\frac{1}{2}(K_0(z) + K_2(z))$, we have

$$|K_1'(-\mathbf{i}k\rho(\tilde{x},y))| \leq \frac{1}{2}(\,|K_0(-\mathbf{i}k\rho(\tilde{x},y))| + |K_2(-\mathbf{i}k\rho(\tilde{x},y))|\,)$$
$$\leq e^{-(\gamma k\sigma - 1)} \frac{1}{2}(K_0(1) + K_2(1)) = e^{-(\gamma k\sigma - 1)} |K_1'(1)|.$$

Therefore, by using (3.9)-(3.10) we obtain

$$\left| \frac{\partial^2 \tilde{G}_k}{\partial x_i \partial y_j} \right| \leq Ck^2 |\alpha_m|\, e^{-(\gamma k\sigma - 1)} + Ck|\alpha_m|\, \sigma^{-1}\, e^{-(\gamma k\sigma - 1)}$$
$$\leq Ck^2 |\alpha_m|\, e^{-(\gamma k\sigma - 1)},$$

where in the last inequality we have used the assumption (3.8) to conclude that $\sigma^{-1} \leq k\gamma$. This completes the proof. ∎

Now we are in the position to estimate the modified single and double layer potentials $\tilde{\Psi}^k_{\mathrm{SL}}, \tilde{\Psi}^k_{\mathrm{DL}}$. Throughout the paper we shall use the weighted $H^{1/2}(\Gamma_j)$ norm, $j = 1, 2$,

$$\|\, v\,\|_{H^{1/2}(\Gamma_j)} = \Big(|\Gamma_j|^{-1} \|\, v\,\|^2_{L^2(\Gamma_j)} + |v|^2_{\frac{1}{2},\Gamma_j} \Big)^{1/2},$$

where

$$|v|^2_{\frac{1}{2},\Gamma_j} = \int_{\Gamma_j} \int_{\Gamma_j} \frac{|v(x) - v(x')|^2}{|x - x'|^2}\, \mathrm{d}s(x)\, \mathrm{d}s(x').$$

Lemma 3.5. *For any $f \in H^{1/2}(\Gamma_1)$, let*

$$v(x) = \tilde{\Psi}^k_{\mathrm{DL}}(f) = \int_{\Gamma_1} \frac{\partial \tilde{G}_k(x,y)}{\partial \mathbf{n}_1(y)} f(y)\, \mathrm{d}s(y)$$

be the double layer potential. Then

$$\|\, v\,\|_{H^{1/2}(\Gamma_2)} \leq C|\alpha_m|(1 + kL)^2\, e^{-(\gamma k\sigma - 1)} \|\, f\,\|_{H^{1/2}(\Gamma_1)},$$

where $L = \max(L_1, L_2)$.

Proof. For any $x \in \Gamma_2$, by Lemma 3.4, we know that

$$|v(x)| \leq \|\partial_{\mathbf{n}_1(y)} \tilde{G}_k(x,\cdot)\|_{L^\infty(\Gamma_1)} \|f\|_{L^1(\Gamma_1)} \leq Ck\, e^{-(\gamma k\sigma - 1)} \|f\|_{L^1(\Gamma_1)}.$$

Hence

$$|\Gamma_2|^{-1/2} \|\, v\,\|_{L^2(\Gamma_2)} \leq Ck\, e^{-(\gamma k\sigma - 1)} \|\, f\,\|_{L^1(\Gamma_1)}.$$

It is easy to see that, for any $x, x' \in \Gamma_2$,

$$|v(x) - v(x')| \le C\|\nabla_x v\|_{L^\infty(\Gamma_2)}|x - x'|.$$

Thus

$$\begin{aligned}|v|_{\frac12,\Gamma_2} \le C|\Gamma_2|\,\|\nabla_x v\|_{L^\infty(\Gamma_2)} &\le C|\Gamma_2| \max_{x\in\Gamma_2, y\in\Gamma_1} |\nabla_y \nabla_x \tilde{G}_k(x,y)|\,\|f\|_{L^1(\Gamma_1)} \\ &\le Ck^2|\alpha_m|\,|\Gamma_2|\,e^{-(\gamma k\sigma-1)}\|f\|_{L^1(\Gamma_1)}.\end{aligned}$$

Since $\|f\|_{L^1(\Gamma_1)} \le C|\Gamma_1|\,\| f \|_{H^{1/2}(\Gamma_1)} \le CL\| f \|_{H^{1/2}(\Gamma_1)}$, we conclude that

$$\| v \|_{H^{1/2}(\Gamma_2)} \le C|\alpha_m|(1+kL)^2\, e^{-(\gamma k\sigma-1)}\| f \|_{H^{1/2}(\Gamma_1)}.$$

This completes the proof. □

Lemma 3.6. *For any $\lambda \in H^{-1/2}(\Gamma_1)$, let*

$$v(x) = \tilde{\Psi}^k_{\mathrm{SL}}(\lambda) = \int_{\Gamma_1} \tilde{G}_k(x,y)\lambda(y)\,\mathrm{d}s(y)$$

be the single layer potential. Then

$$\| v \|_{H^{1/2}(\Gamma_2)} \le C|\alpha_m|(1+kL)^2\, e^{-(\gamma k\sigma-1)}\|\lambda\|_{H^{-1/2}(\Gamma_1)},$$

where $L = \max(L_1, L_2)$.

Proof. For any $x \in \Gamma_2$, we know that

$$\begin{aligned}|v(x)| &\le C\|\lambda\|_{H^{-1/2}(\Gamma_1)}\| \tilde{G}_k(x,\cdot) \|_{H^{1/2}(\Gamma_1)} \\ &\le C\|\lambda\|_{H^{-1/2}(\Gamma_1)}\| \tilde{G}_k(x,\cdot) \|_{H^{1}(\Omega_1)} \\ &\le C\|\lambda\|_{H^{-1/2}(\Gamma_1)}(\| \tilde{G}_k(x,\cdot) \|_{L^\infty(\Omega_1)} + |\Omega_1|^{1/2}\| \nabla_y\tilde{G}_k(x,\cdot) \|_{L^\infty(\Omega_1)}) \\ &\le C\|\lambda\|_{H^{-1/2}(\Gamma_1)}(1+kL)\, e^{-(\gamma k\sigma-1)},\end{aligned}$$

where we have used Lemma 3.4. Hence

$$|\Gamma_2|^{-1/2}\| v \|_{L^2(\Gamma_2)} \le C(1+kL)\, e^{-(\gamma k\sigma-1)}\|\lambda\|_{H^{-1/2}(\Gamma_1)}.$$

On the other hand, similar argument as in Lemma 3.5 yields

$$\begin{aligned}|v|_{\frac12,\Gamma_2} &\le C|\Gamma_2|\,\|\nabla_x v\|_{L^\infty(\Gamma_2)} \\ &\le CL\|\lambda\|_{H^{-1/2}(\Gamma_1)} \max_{x\in\Gamma_2} \| \nabla_x\tilde{G}_k(x,\cdot) \|_{H^{1/2}(\Gamma_1)} \\ &\le CL\|\lambda\|_{H^{-1/2}(\Gamma_1)} \max_{x\in\Gamma_2} \| \nabla_x\tilde{G}_k(x,\cdot) \|_{H^{1}(\Omega_1)} \\ &\le CL\|\lambda\|_{H^{-1/2}(\Gamma_1)} \max_{x\in\Gamma_2}(\, \|\nabla_x\tilde{G}_k(x,\cdot)\|_{L^\infty(\Omega_1)} + |\Omega_1|^{1/2}\|\nabla_y\nabla_x\tilde{G}_k(x,\cdot)\|_{L^\infty(\Omega_1)}\,).\end{aligned}$$

Again by using Lemma 3.4 we obtain

$$|v|_{\frac12,\Gamma_2} \le C|\alpha_m|(1+kL)^2\, e^{-(\gamma k\sigma-1)}\|\lambda\|_{H^{-1/2}(\Gamma_1)}.$$

This completes the proof. □

The following theorem is the main result of this subsection.

THEOREM 3.7. *Let (H1)-(H3) and (3.8) be satisfied. For any $f \in H^{1/2}(\Gamma_1)$, let $\mathbb{E}(f)$ be the PML extension defined in (2.12). Then there exists a constant C depending only on γ but independent of $k, \sigma, L_j, d_j, j = 1, 2$, such that*

$$\| \mathbb{E}(f) \|_{H^{1/2}(\Gamma_2)} \leq C|\alpha_m|(1 + kL)^2\, e^{-(\gamma k\sigma - 1)} \| f \|_{H^{1/2}(\Gamma_1)},$$

where $L = \max(L_1, L_2)$.

Proof. This theorem is a direct consequence of Lemmas 3.5-3.6 and the continuity of the Dirichlet-to-Neumann operator $T : H^{1/2}(\Gamma_1) \to H^{-1/2}(\Gamma_1)$. ∎

3.2. The PML equation in the layer. In this subsection we derive the stability estimates for the Dirichlet problem of the PML equation in the layer.

THEOREM 3.8. *Let (H1) be satisfied. For $q \in H^{1/2}(\Gamma_2)$, let w be the solution of the PML equation in the layer* (3.1a)-(3.1b). *Then there exists a constant $C > 0$ independent of k, σ such that*

$$\| \nabla w \|_{L^2(\Omega^{\rm PML})} \leq C\hat{C}^{-1}|\alpha_m|^2(1 + kL)\| q \|_{H^{1/2}(\Gamma_2)},$$
$$\left\| \frac{\partial w}{\partial \mathbf{n}_1} \right\|_{H^{-1/2}(\Gamma_1)} \leq C\hat{C}^{-1}|\alpha_m|^2(1 + kL)^2\| q \|_{H^{1/2}(\Gamma_2)}.$$

Proof. For any $\zeta \in H^1(\Omega^{\rm PML})$ such that $\zeta = q$ on Γ_2 and $\zeta = 0$ on Γ_1. By the inf-sup condition in (3.3) and using (3.2), we know that

$$\hat{C}\| w - \zeta \|_{*,\Omega^{\rm PML}} \leq \sup_{0\neq\psi\in H_0^1(\Omega^{\rm PML})} \frac{|c(w - \zeta, \psi)|}{\| \psi \|_{*,\Omega^{\rm PML}}} = \sup_{0\neq\psi\in H_0^1(\Omega^{\rm PML})} \frac{|c(\zeta, \psi)|}{\| \psi \|_{*,\Omega^{\rm PML}}}.$$

By Cauchy-Schwarz inequality

$$\begin{aligned} |c(\zeta, \psi)| &\leq C \max_{x\in\Omega^{\rm PML}} \left(|\alpha_2|, |\alpha_1|, |\Omega^{\rm PML}|^{1/2} \frac{k\,|\alpha_1|\,|\alpha_2|}{(1 + \sigma_1\sigma_2)^{1/2}}\right) \| \zeta \|_{H^1(\Omega^{\rm PML})}\| \psi \|_{*,\Omega^{\rm PML}} \\ &\leq C(1 + kL)|\alpha_m|\, \| \zeta \|_{H^1(\Omega^{\rm PML})}\| \psi \|_{*,\Omega^{\rm PML}}. \end{aligned} \tag{3.11}$$

Notice that

$$\| \zeta \|_{*,\Omega^{\rm PML}} \leq C|\alpha_m|(1 + kL)\| \zeta \|_{H^1(\Omega^{\rm PML})},$$

by the triangle inequality and the trace inequality, we conclude that

$$\| w \|_{*,\Omega^{\rm PML}} \leq C\hat{C}^{-1}|\alpha_m|(1 + kL)\| q \|_{H^{1/2}(\Gamma_2)}. \tag{3.12}$$

This shows the first estimate in the theorem by using the definition of $\| \cdot \|_{*,\Omega^{\rm PML}}$.

Next, since $A(x)$ reduces to the identity matrix on Γ_1, we know that, for any $\psi \in H^1(\Omega^{\rm PML})$ such that $\psi = 0$ on Γ_2,

$$\begin{aligned} -\int_{\Gamma_1} \frac{\partial w}{\partial \mathbf{n}_1}\bar{\psi} = \int_{\partial\Omega^{\rm PML}} A\nabla w \cdot \mathbf{n}\,\bar{\psi} &= \int_{\Omega^{\rm PML}} (A\nabla w \cdot \nabla\bar{\psi} + \nabla \cdot (A\nabla w))\,\mathrm{d}x \\ &= \int_{\Omega^{\rm PML}} (A\nabla w \cdot \nabla\bar{\psi} - k^2\alpha_1\alpha_2 w\bar{\psi})\,\mathrm{d}x, \end{aligned}$$

where we have used (3.1a) and the formula of integration by parts. Again by Cauchy-Schwarz inequality and the argument in (3.11) we get

$$\left|\int_{\Gamma_1} \frac{\partial w}{\partial \mathbf{n}_1}\bar{\psi}\right| \leq C(1 + kL)|\alpha_m|\, \| w \|_{*,\Omega^{\rm PML}}\| \psi \|_{H^1(\Omega^{\rm PML})}.$$

This completes the proof by using (3.12) and the trace inequality. ∎

3.3. Convergence of the PML problem. The purpose of this section is to prove Theorem 3.1. We start by introducing the approximate Dirichlet-to-Neumann operator $\hat{T} : H^{1/2}(\Gamma_1) \to H^{-1/2}(\Gamma_1)$ associated with the PML problem. Given $f \in H^{1/2}(\Gamma_1)$, let $\hat{T}f = \frac{\partial \zeta}{\partial \mathbf{n}_1}\Big|_{\Gamma_1}$, where $\zeta \in H^1(\Omega^{\rm PML})$ be the solution of the PML problem in the layer

$$\nabla \cdot (A\nabla\zeta) + \alpha_1\alpha_2 k^2 \zeta = 0 \quad \text{in } \Omega^{\rm PML}, \tag{3.13a}$$

$$\zeta = f \quad \text{on } \Gamma_1, \quad \zeta = 0 \quad \text{on } \Gamma_2. \tag{3.13b}$$

By assumption (H1), $\hat{T}$ is well-defined.

Lemma 3.9. *Let (H1)-(H3) and* (3.8) *be satisfied. We have, for any $f \in H^{1/2}(\Gamma_1)$,*

$$\|Tf - \hat{T}f\|_{H^{-1/2}(\Gamma_1)} \le C\hat{C}^{-1}|\alpha_m|^3(1+kL)^4\, e^{-(\gamma k\sigma - 1)}\| f \|_{H^{1/2}(\Gamma_1)}.$$

Proof. For any $f \in H^{1/2}(\Gamma_1)$, let $\mathbb{E}(f)$ be the PML extension defined in (2.12). Denote by $\gamma_N^+ v = \frac{\partial v}{\partial \mathbf{n}_1}\Big|_{\Gamma_1}$ be the Neumann trace on Γ_1 for any function v defined on $\mathbb{R}^2\backslash\bar{B}_1$. It is easy to see that $\gamma_N^+\tilde{\Psi}^k_{\rm SL}(\lambda) = \gamma_N^+\Psi^k_{\rm SL}(\lambda)$, $\gamma_N^+\tilde{\Psi}^k_{\rm DL}(f) = \gamma_N^+\Psi^k_{\rm DL}(f)$ for any $\lambda \in H^{-1/2}(\Gamma_1)$ and $f \in H^{1/2}(\Gamma_1)$. Thus, by (2.6),

$$\gamma_N^+\mathbb{E}(f) = -\gamma_N^+\Psi^k_{\rm SL}(Tf) + \gamma_N^+\Psi^k_{\rm DL}(f) = \gamma_N^+\xi = Tf.$$

By (3.13a)-(3.13b), we know that $Tf - \hat{T}f = \frac{\partial w}{\partial \mathbf{n}_1}\Big|_{\Gamma_1}$, where $w = \mathbb{E}(f) - \zeta \in H^1(\Omega^{\rm PML})$ satisfies

$$\begin{aligned} &\nabla \cdot (A\nabla w) + \alpha_1\alpha_2 k^2 w = 0 \quad \text{in } \Omega^{\rm PML}, \\ &w = 0 \quad \text{on } \Gamma_1, \quad w = \mathbb{E}(f) \quad \text{on } \Gamma_2. \end{aligned}$$

By Theorem 3.8 and Theorem 3.7,

$$\begin{aligned} \left\| \frac{\partial w}{\partial \mathbf{n}_1} \right\|_{H^{-1/2}(\Gamma_1)} &\le C\hat{C}^{-1}|\alpha_m|^2(1+kL)^2\| \mathbb{E}(f) \|_{H^{1/2}(\Gamma_2)} \\ &\le C\hat{C}^{-1}|\alpha_m|^3(1+kL)^4\, e^{-(\gamma k\sigma - 1)}\| f \|_{H^{1/2}(\Gamma_1)}. \end{aligned}$$

This completes the proof. ☐

Proof of Theorem 3.1: We first prove the estimate (3.4). Let $\hat{u}$ be the solution of the PML problem (2.13a)-(2.13b). Simple integration by parts implies that

$$a(\hat{u}, \psi) + \langle T\hat{u} - \hat{T}\hat{u}, \psi\rangle_{\Gamma_1} = \langle g, \psi\rangle_{\Gamma_D}, \qquad \forall\, \psi \in H^1(\Omega_1).$$

Subtracting with (2.3) we get

$$a(u - \hat{u}, \psi) = \langle T\hat{u} - \hat{T}\hat{u}, \psi\rangle_{\Gamma_1}, \qquad \forall\, \psi \in H^1(\Omega_1).$$

Thus, by the inf-sup condition (2.4) and Lemma 3.9,

$$\| u - \hat{u} \|_{H^1(\Omega_1)} \le C\hat{C}^{-1}|\alpha_m|^3(1+kL)^4\, e^{-(\gamma k\sigma - 1)}\| \hat{u} \|_{H^{1/2}(\Gamma_1)}. \tag{3.14}$$

This is the desired estimate (3.4).

Now we turn to the well-posedness of the PML problem. By the Fredholm alternative theorem we only need to show the uniqueness of the PML problem (2.13a)-(2.13b). For that purpose we let $g = 0$ in (2.13a)-(2.13b). By the uniqueness of the scattering problem we know that the corresponding scattering solution $u = 0$ in Ω_1. Thus (3.14) implies

$$\begin{aligned}\| \hat{u} \|_{H^1(\Omega_1)} &\le C\hat{C}^{-1}|\alpha_m|^3(1+kL)^4\, e^{-(\gamma k\sigma-1)}\| \hat{u} \|_{H^{1/2}(\Gamma_1)}\\ &\le C\hat{C}^{-1}|\alpha_m|^3(1+kL)^4\, e^{-(\gamma k\sigma-1)}\| \hat{u} \|_{H^1(\Omega_1)}.\end{aligned}$$

Thus for sufficiently large σ we conclude that $\hat{u} = 0$ on Ω_1. That $\hat{u}$ also vanishes in Ω_2 is a direct consequence of the unique continuation theorem (cf. e.g. [16, P. 92]). This completes the proof. □

4. Finite element approximation. In this section we introduce the finite element approximations of the PML problems (2.13a)-(2.13b). From now on we assume $g \in L^2(\Gamma_D)$. Let $b : H^1(\Omega_2) \times H^1(\Omega_2) \to \mathbb{C}$ be the sesquilinear form given by

$$b(\varphi,\psi) = \int_{\Omega_2} \left(A\nabla\varphi \cdot \nabla\bar{\psi} - \alpha_1\alpha_2 k^2 \varphi\bar{\psi}\right) dx. \tag{4.1}$$

Denote by $H^1_{(0)}(\Omega_2) = \{v \in H^1(\Omega_2) : v = 0 \text{ on } \Gamma_2\}$. Then the weak formulation of (2.13a)-(2.13b) is: Given $g \in L^2(\Gamma_D)$, find $\hat{u} \in H^1_{(0)}(\Omega_2)$ such that

$$b(\hat{u},\psi) = \int_{\Gamma_D} g\bar{\psi}\,\mathrm{d}s, \qquad \forall\, \psi \in H^1_{(0)}(\Omega_2). \tag{4.2}$$

Let $\mathcal{M}_h$ be a regular triangulation of the domain Ω_2. We assume the elements $K \in \mathcal{M}_h$ may have one curved edge align with Γ_D so that $\Omega_2 = \cup_{K\in\mathcal{M}_h} K$. Let $V_h \subset H^1(\Omega_2)$ be the conforming linear finite element space over Ω_2, and $\overset{\circ}{V}_h = \{v_h \in V_h : v_h = 0 \text{ on } \Gamma_2\}$. The finite element approximation to the PML problem (2.13a)-(2.13b) reads as follows: Find $u_h \in \overset{\circ}{V}_h$ such that

$$b(u_h,\psi_h) = \int_{\Gamma_D} g\bar{\psi}_h\,\mathrm{d}s, \qquad \forall\, \psi_h \in \overset{\circ}{V}_h. \tag{4.3}$$

Following the general theory in [1, Chap. 5], the existence of unique solution of the discrete problem (4.3) and the finite element convergence analysis depend on the following discrete inf-sup condition

$$\sup_{0\ne\psi_h\in\overset{\circ}{V}_h} \frac{|b(\varphi_h,\psi_h)|}{\| \psi_h \|_{H^1(\Omega_2)}} \ge \hat{\mu}\, \| \varphi_h \|_{H^1(\Omega_2)}, \quad \forall\, \varphi_h \in \overset{\circ}{V}_h, \tag{4.4}$$

where the constant $\hat{\mu} > 0$ is independent of the finite element mesh size. Since the continuous problem (4.2) has a unique solution by Theorem 3.1, the sesquilinear form $b : H^1_{(0)}(\Omega_2)\times H^1_{(0)}(\Omega_2) \to \mathbb{C}$ satisfies the continuous inf-sup condition. Then a general argument of Schatz [17] implies (4.4) is valid for sufficiently small mesh size $h < h^*$. In this paper we are interested in a posterioir error estimates and the associated adaptive method. Thus in the following, we simply assume the discrete problem (4.3) has a unique solution.

For any $K \in \mathcal{M}_h$, we denote by h_K its diameter. Let $\mathcal{B}_h$ denote the set of all sides that do not lie on Γ_D and Γ_2. For any $e \in \mathcal{B}_h$, h_e stands for its length. For any $K \in \mathcal{M}_h$, we introduce the residual:

$$R_h := \nabla \cdot (A \nabla u_h|_K) + \alpha_1 \alpha_2 k^2 u_h|_K. \tag{4.5}$$

For any interior side $e \in \mathcal{B}_h$ which is the common side of K_1 and $K_2 \in \mathcal{M}_h$, we define the jump residual across e:

$$J_e := (A\nabla u_h|_{K_1} - A\nabla u_h|_{K_2}) \cdot \nu_e, \tag{4.6}$$

using the convention that the unit normal vector ν_e to e points from K_2 to K_1. If $e = \Gamma_D \cap \partial K$ for some element $K \in \mathcal{M}_h$, then we define the jump residual

$$J_e := 2(\nabla u_h|_K \cdot \mathbf{n}_D + g). \tag{4.7}$$

For any $K \in \mathcal{M}_h$, we define the local error estimator η_K as

$$\eta_K = \left(\| h_K R_h \|^2_{L^2(K)} + \frac{1}{2} \sum_{e \subset \partial K} h_e \| J_e \|^2_{L^2(e)} \right)^{1/2}. \tag{4.8}$$

The following theorem is the main result of this paper, whose proof will be given in the next section.

Theorem 4.1. *Let (H1)-(H3) and* (3.8) *be satisfied. Then there exists a constant $C > 0$ depending only on γ and the minimum angle of the mesh $\mathcal{M}_h$ such that the following a posterior error estimate is valid*

$$\begin{aligned} \| u - u_h \|_{H^1(\Omega_1)} \le{}& C\hat{C}^{-1} |\alpha_m|^2 (1 + kL) \left(\sum_{K \in \mathcal{M}_h} \eta_K^2 \right)^{1/2} \\ &+ C\hat{C}^{-1} |\alpha_m|^3 (1 + kL)^4 \, e^{-(\gamma k \sigma - 1)} \| u_h \|_{H^{1/2}(\Gamma_1)}. \end{aligned}$$

5. A posteriori error analysis. For any $\varphi \in H^1(\Omega_1)$, let $\tilde{\varphi}$ be its extension in $\Omega^{\rm PML}$ such that

$$\nabla \cdot (\bar{A} \nabla \tilde{\varphi}) + \overline{\alpha_1 \alpha_2} k^2 \tilde{\varphi} = 0 \quad \text{ in } \Omega^{\rm PML}, \tag{5.1a}$$
$$\tilde{\varphi} = \varphi \ \text{ on } \Gamma_1, \quad \tilde{\varphi} = 0 \ \text{ on } \Gamma_2. \tag{5.1b}$$

Lemma 5.1. *Let* (H1) *be satisfied. For any $\varphi, \psi \in H^1(\Omega^{\rm PML})$, we have*

$$\langle \hat{T}\varphi, \psi \rangle_{\Gamma_1} = \langle \hat{T}\bar{\psi}, \bar{\varphi} \rangle_{\Gamma_1}.$$

Proof. By definition, $\hat{T}\varphi = \partial \xi / \partial \mathbf{n}_1$ on Γ_1, where ξ satisfies

$$\nabla \cdot (A \nabla \xi) + \alpha_1 \alpha_2 k^2 \xi = 0 \quad \text{ in } \Omega^{\rm PML},$$
$$\xi = \varphi \ \text{ on } \Gamma_1, \quad \xi = 0 \ \text{ on } \Gamma_2.$$

Similarly, $\hat{T}\bar{\psi} = \partial\zeta/\partial\mathbf{n}_1$ on Γ_1, where ζ satisfies the same equation but $\zeta = \bar{\psi}$ on Γ_1, $\zeta = 0$ on Γ_2. Integrating by parts we know that

$$0 = -\int_{\Omega^{\rm PML}} (A\nabla\xi \cdot \nabla\zeta - \alpha_1\alpha_2 k^2 \xi \cdot \zeta) - \int_{\Gamma_1} \frac{\partial \xi}{\partial \mathbf{n}_1} \cdot \zeta$$
$$0 = -\int_{\Omega^{\rm PML}} (A\nabla\zeta \cdot \nabla\xi - \alpha_1\alpha_2 k^2 \zeta \cdot \xi) - \int_{\Gamma_1} \frac{\partial \zeta}{\partial \mathbf{n}_1} \cdot \xi$$

Since A is a diagonal matrix, the first integrals on the right-hand side of the above two equations are equal. Thus $\langle \hat{T}\varphi, \bar{\zeta}\rangle_{\Gamma_1} = \langle \hat{T}\bar{\psi}, \xi\rangle_{\Gamma_1}$. This completes the proof since $\xi = \varphi, \zeta = \bar{\psi}$ on Γ_1. □

LEMMA 5.2 (Error representation formula). *For any $\varphi \in H^1(\Omega_1)$, which is extended to be a function $\tilde{\varphi} \in H^1(\Omega_2)$ according to* (5.1a)-(5.1b)*, and $\varphi_h \in \overset{\circ}{V}_h$, we have*

$$a(u - u_h, \varphi) = \int_{\Gamma_D} g(\overline{\varphi - \varphi_h}) - b(u_h, \tilde{\varphi} - \tilde{\varphi}_h) + \langle Tu_h - \hat{T}u_h, \varphi\rangle_{\Gamma_1}. \qquad (5.2)$$

Proof. By (2.3) and the definitions (2.2) and (4.1),

$$\begin{aligned} & a(u - u_h, \varphi) \\ = & \int_{\Gamma_D} g\bar{\varphi} - \int_{\Omega_1} (A\nabla u_h \cdot \nabla\bar{\varphi} - \alpha_1\alpha_2 k^2 u_h \bar{\varphi}) + \langle Tu_h, \varphi\rangle_{\Gamma_1} \\ = & \int_{\Gamma_D} g\bar{\varphi} - b(u_h, \tilde{\varphi}) + \int_{\Omega^{\rm PML}} (A\nabla u_h \cdot \nabla\bar{\tilde{\varphi}} - \alpha_1\alpha_2 k^2 u_h \bar{\tilde{\varphi}}) + \langle Tu_h, \varphi\rangle_{\Gamma_1}. \qquad (5.3) \end{aligned}$$

On the other hand, by multiplying (5.1a) by $\bar{u}_h$, integrating by parts, and recalling that $\mathbf{n}_1$ is the unit outer normal to Γ_1 which points outside Ω_1, we deduce that

$$-\int_{\Omega^{\rm PML}} (\bar{A}\nabla\tilde{\varphi} \cdot \nabla\bar{u}_h - \overline{\alpha_1\alpha_2} k^2 \tilde{\varphi}\bar{u}_h) - \left\langle \frac{\partial\tilde{\varphi}}{\partial\mathbf{n}_1}, u_h \right\rangle_{\Gamma_1} = 0,$$

which is equivalent to

$$\int_{\Omega^{\rm PML}} (A\nabla u_h \cdot \nabla\bar{\tilde{\varphi}} - \alpha_1\alpha_2 k^2 u_h \bar{\tilde{\varphi}}) = -\left\langle \frac{\partial\bar{\tilde{\varphi}}}{\partial\mathbf{n}_1}, \bar{u}_h \right\rangle_{\Gamma_1}. \qquad (5.4)$$

Since by the definition of $\hat{T} : H^{1/2}(\Gamma_1) \to H^{-1/2}(\Gamma_1)$,

$$\left.\frac{\partial\bar{\tilde{\varphi}}}{\partial\mathbf{n}_1}\right|_{\Gamma_1} = \hat{T}\bar{\varphi},$$

we obtain by substituting (5.4) into (5.3) that

$$a(u - u_h, \varphi) = \int_{\Gamma_D} g\bar{\varphi} - b(u_h, \tilde{\varphi}) + \langle Tu_h, \varphi\rangle - \langle \hat{T}\bar{\varphi}, \bar{u}_h\rangle.$$

This completes the proof upon using Lemma 5.1 and (4.3). □

Proof of Theorem 4.1: First, we construct a Clément type interpolation operator $\Pi_h : H^1_{(0)}(\Omega_2) \to \overset{\circ}{V}_h$, where $H^1_{(0)}(\Omega_2) = \{v \in H^1(\Omega_2) : v = 0 \text{ on } \Gamma_2\}$. Let $\mathcal{N}_h = \{a_i\}_{i=1}^N$ be the set of the nodes of $\mathcal{M}_h$ which is interior to Ω_2 or on the boundary Γ_D, and $\{\phi_i\}_{i=1}^N$ be the corresponding nodal basis of V_h. Define $\Delta_i = \operatorname{supp}\phi_i \cap \Omega_2$. Then the interpolation operator $\Pi_h : H^1_{(0)}(\Omega_2) \to V_h$ is defined by

$$\Pi_h v(x) = \sum_{i=1}^N \left(\frac{1}{|\Delta_i|} \int_{\Delta_i} v(x)dx \right) \phi_i(x).$$

Since the nodes on Γ_2 are not included in the definition of Π_h, we know that $\Pi_h v \in \overset{\circ}{V}_h$. Moreover, by slightly modifying the argument in Chen and Nochetto [7, Lemmas 3.1-3.2], one can show that the operator Π_h enjoys the following interpolation estimates, for any $v \in H^1_{(0)}(\Omega_2)$,

$$\| v - \Pi_h v \|_{L^2(K)} \le C h_K \| \nabla v \|_{L^2(\tilde{K})}, \quad \| v - \Pi_h v \|_{L^2(e)} \le C h_e^{1/2} \| \nabla v \|_{L^2(\tilde{e})}, \tag{5.5}$$

where $\tilde{K}$ and $\tilde{e}$ are the union of all elements in $\mathcal{M}_h$ having non-empty intersection with $K \in \mathcal{M}_h$ and the side e, respectively.

Now we take $\varphi_h = \Pi_h \tilde{\varphi} \in \overset{\circ}{V}_h$ in the error representation formula (5.2) to get

$$\begin{aligned} a(u - u_h, \varphi) &= \int_{\Gamma_D} g(\overline{\varphi - \Pi_h \varphi}) - b(u_h, \tilde{\varphi} - \Pi_h \tilde{\varphi}) + \langle T u_h - \hat{T} u_h, \varphi \rangle_{\Gamma_1} \\ &:= \mathrm{II}_1 + \mathrm{II}_2 + \mathrm{II}_3. \end{aligned} \tag{5.6}$$

We observe that, by integration by parts and using (4.5)-(4.7),

$$\mathrm{II}_1 + \mathrm{II}_2 = \sum_{K \in \mathcal{M}_h} \left(\int_K R_h (\overline{\tilde{\varphi} - \Pi_h \tilde{\varphi}})\, \mathrm{d}x + \sum_{e \subset \partial K} \frac{1}{2} \int_e J_e (\overline{\tilde{\varphi} - \Pi_h \tilde{\varphi}})\, \mathrm{d}s \right).$$

By using standard argument in the a posteriori error analysis and (5.5) we get

$$\begin{aligned} |\mathrm{II}_1 + \mathrm{II}_2| &\le C \sum_{K \in \mathcal{M}_h} \left(\| h_K R_h \|^2_{L^2(K)} + \frac{1}{2} \sum_{e \subset \partial K} \| h_e^{1/2} J_e \|^2_{L^2(e)} \right)^{1/2} \| \nabla \tilde{\varphi} \|_{L^2(\tilde{K})} \\ &\le C \left(\sum_{K \in \mathcal{M}_h} \eta_K^2 \right)^{1/2} \| \nabla \tilde{\varphi} \|_{L^2(\Omega_2)}. \end{aligned}$$

By the argument in Theorem 3.8, we deduce that

$$\begin{aligned} \| \nabla \tilde{\varphi} \|_{L^2(\Omega^{\mathrm{PML}})} &\le C \hat{C}^{-1} |\alpha_m|^2 (1 + kL) \| \varphi \|_{H^{1/2}(\Gamma_1)} \\ &\le C \hat{C}^{-1} |\alpha_m|^2 (1 + kL) \| \varphi \|_{H^1(\Omega_1)}. \end{aligned}$$

Thus

$$|\mathrm{II}_1 + \mathrm{II}_2| \le C \hat{C}^{-1} |\alpha_m|^2 (1 + kL) \left(\sum_{K \in \mathcal{M}_h} \eta_K^2 \right)^{1/2} \| \varphi \|_{H^1(\Omega_1)}.$$

By Lemma 3.9, we obtain

$$|\mathrm{II}_3| \le C \hat{C}^{-1} |\alpha_m|^3 (1 + kL)^4 \, e^{-(\gamma k \sigma - 1)} \| u_h \|_{H^{1/2}(\Gamma_1)} \| \varphi \|_{H^{1/2}(\Gamma_1)}.$$

Therefore, by the inf-sup condition (2.4), we finally get

$$\begin{aligned}\| u - u_h \|_{H^1(\Omega_1)} &\leq C \sup_{0\neq\varphi\in H^1(\Omega_1)} \frac{|a(u-u_h,\varphi)|}{\| \varphi \|_{H^1(\Omega_1)}} \\ &\leq C\hat{C}^{-1}|\alpha_m|^2(1+kL)\left(\sum_{K\in\mathcal{M}_h}\eta_K^2\right)^{1/2} \\ &\quad + C\hat{C}^{-1}|\alpha_m|^3(1+kL)^4\, e^{-(\gamma k\sigma-1)}\| u_h \|_{H^{1/2}(\Gamma_1)}.\end{aligned}$$

This completes the proof. □

6. Implementation and numerical examples. In this section, we present several numerical examples to illustrate the performance of the adaptive uniaxial PML method. The computations are carried out by using the PDE toolbox of MATLAB. The PML parameters are determined through the a posteriori error estimate in Theorem 4.1. Note that in Theorem 4.1 that the a posteriori error estimate consists of two parts: the PML error and the finite element discretization error. First we choose L_1, L_2 such that $D \subset B_1$ and choose d_1, d_2 such that

$$\frac{d_1}{L_1} = \frac{d_2}{L_2} = \chi, \tag{6.1}$$

where χ is a constant. Then we choose χ and σ such that the exponentially decaying factor:

$$\omega = e^{-(\gamma k\sigma-1)} = e^{-\left(\frac{\chi}{\chi+1}\frac{\min(L_1,L_2)}{\sqrt{L_1^2+L_2^2}}\,k\sigma-1\right)} \leq 10^{-8}, \tag{6.2}$$

which makes the PML error negligible compared with the finite element discretization errors. By (H3),

$$\sigma_j(t) = \tilde{\sigma}_j\left(\frac{|t|-L_j/2}{d_j}\right)^m, \qquad j=1,2, \tag{6.3}$$

where $\tilde{\sigma}_1, \tilde{\sigma}_2$ are determined from σ as follows

$$\tilde{\sigma}_j = \frac{(m+1)\sigma}{d_j}, \qquad j=1,2. \tag{6.4}$$

Once the PML region and the medium property are fixed, we use the standard finite element adaptive strategy to modify the mesh according to the a posteriori error estimate. Now we describe the adaptive uniaxial PML method used in the paper.

Algorithm 6.1. Given tolerance TOL > 0. Let $m = 2$.

- Choose L_1, L_2 such that $D \subset B_1$;
- Choose χ and σ such that the exponentially decaying factor $\omega \leq 10^{-8}$;
- Set d_1, d_2 and $\tilde{\sigma}_1, \tilde{\sigma}_2$ according to (6.1),(6.4);
- Set the computational domain $\Omega_2 = B_2\backslash\bar{D}$ and generate an initial mesh $\mathcal{M}_h$ over Ω_2;
- While $\mathcal{E}_{FEM} = \left(\sum_{K\in\mathcal{M}_h}\eta_K^2\right)^{1/2} > \text{TOL}$ do
  - refine the mesh $\mathcal{M}_h$ according to the strategy:

if $\eta_K > \frac{1}{2}\max_{K\in\mathcal{M}_h}\eta_K$, refine the element $K \in \mathcal{M}_h$

- solve the discrete problem (4.3) on $\mathcal{M}_h$
- compute error estimators on $\mathcal{M}_h$

end while

Now we report two numerical examples to demonstrate the efficiency of the proposed algorithm. We scale the error estimator for determining finite element meshes by a default factor 0.15 as in the PDE toolbox of MATLAB.

Example 1. Let $D = [-\lambda/2, \lambda/2] \times [-\lambda/2, \lambda/2]$. We consider the scattering problem whose exact solution is known: $u = H_0^{(1)}(k|x|)$, where $k = 2\pi/\lambda$ and so λ is the wavelength. Define $\texttt{dist} = \min\{|x - x'|, x \in \Gamma_D, x' \in \Gamma_1\}$ as the minimum distance between the scatterer to the inner boundary of the PML layer. We want to test the influence of the different choices of the size of B_1. Fix $\chi = 1.0$ and take different `dist`, that is, in the first step we choose different L_1 and L_2. Table 6.1 shows the different choices of the PML parameters `dist`, χ and σ determined by the relation (6.2). In the following we simply take $\lambda = 1.0$ to fix the exact solution.

Table 6.1

The PML parameters for Example 1 and Example 2.

Example 1			Example 2		
`dist`	χ	σ	`dist`	χ	σ
0.1λ	1.0	9λ	1.0λ	0.5	20λ
1.0λ	1.0	9λ	1.0λ	1.0	14λ
5.0λ	1.0	9λ	1.0λ	2.0	10λ
			0.1λ	1.0	19λ

Figure 6.1 shows the $\log N_k$-$\log \| u - u_k \|_{H^1(\Omega_1)}$ curves, where N_k is the number of nodes of the mesh $\mathcal{M}_k$ and u_k is the finite element solution of (4.3) over the mesh $\mathcal{M}_k$. It indicates that for different choices of `dist`, the meshes and the associated numerical complexity are quasi-optimal: $\| u - u_k \|_{H^1(\Omega_1)} \approx C N_k^{-1/2}$ is valid asymptotically.

Table 6.2

The number of nodes required to achieve the given relative error of the far field for different choices of `dist` *for Example 1. The exact far field is 0.22507908-0.22507908* **i**.

Error	$\texttt{dist} = 0.1\lambda$	$\texttt{dist} = 1.0\lambda$	$\texttt{dist} = 5.0\lambda$
10%	295	557	5111
1%	997	5556	16845
0.1%	2989	16006	>114848

One of the important quantities in the scattering problems is the far field pattern:

$$u_\infty(\hat{x}) = \frac{e^{i\frac{\pi}{4}}}{\sqrt{8\pi k}} \int_{\partial D} \left(u(y) \frac{\partial e^{-ik\hat{x}\cdot y}}{\partial \nu(y)} - \frac{\partial u(y)}{\partial \nu(y)} e^{-ik\hat{x}\cdot y} \right) ds(y), \qquad \hat{x} = \frac{x}{|x|}.$$

We compute the far field $u_\infty(\hat{x})$, $\hat{x} = (\cos(\theta), \sin(\theta))^T$ in the observation direction $\theta = \pi/4$. Table 6.2 shows the number of nodes required to achieve the given relative

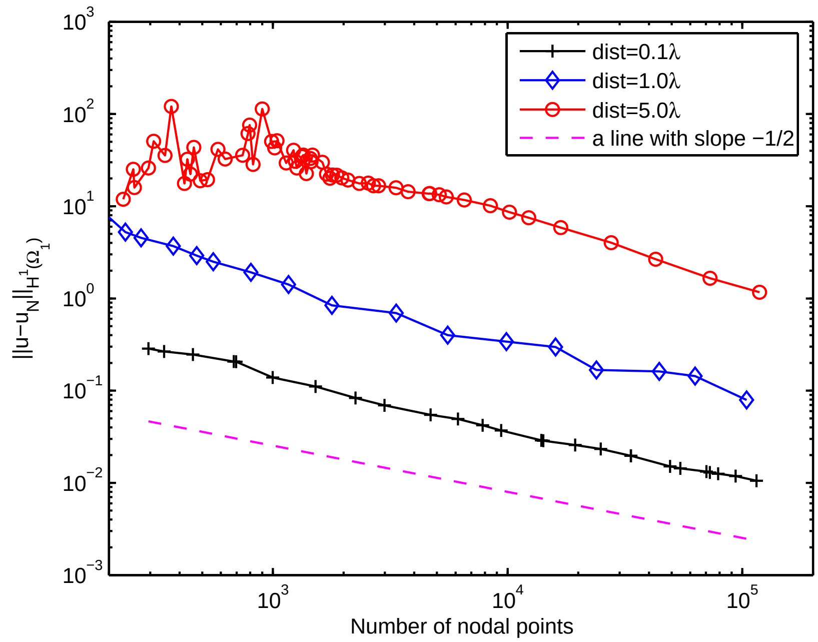

FIG. 6.1. *The quasi-optimality of the adaptive mesh refinements of the error* $\|u - u_N\|_{H^1(\Omega_1)}$ *for Example 1.*

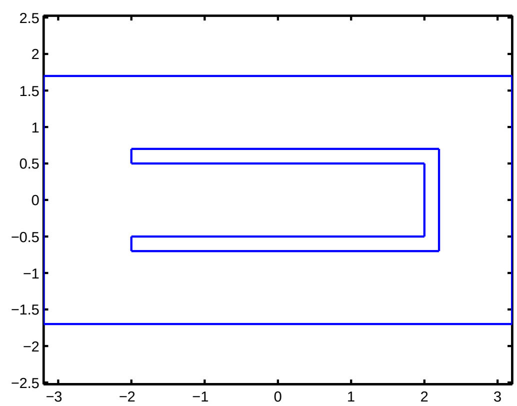

FIG. 6.2. *The geometry of the scatterer for Example 2.*

error of the far field for different choices of PML parameter `dist`. It demonstrates clearly that for given relative error, a smaller `dist` is preferred in terms of number of nodes used. Thus for the same accuracy of the far fields, we can choose small `dist` which will largely save the computational costs.

Example 2. This example is taken from [10] which concerns the scattering of the plane wave $u_I = e^{\mathbf{i}kx_1}$ from a perfectly conducting metal. The scatter D is contained in the box $\{x \in \mathbb{R}^2 : \ -2 < x_1 < 2.2, -0.7 < x_2 < 0.7\}$ as plotted in Figure 6.2. We

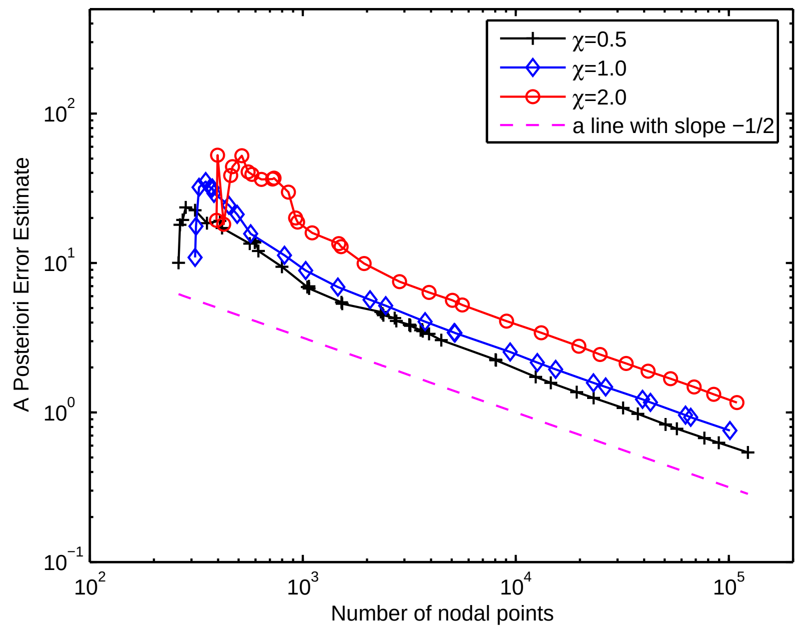

FIG. 6.3. *The quasi-optimality of the adaptive mesh refinements of the a posteriori error estimator for Example 2.*

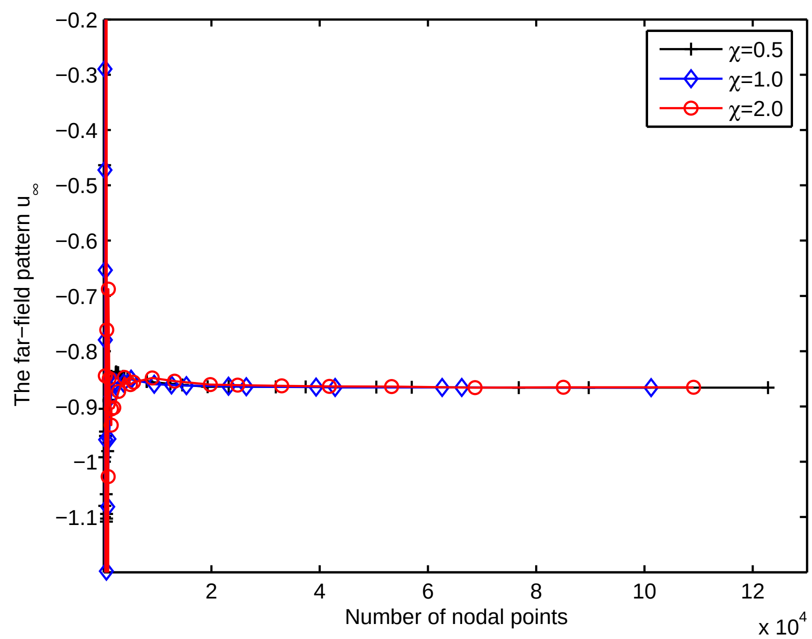

FIG. 6.4. *The real part of the far-field patterns in the incident direction for Example 2.*

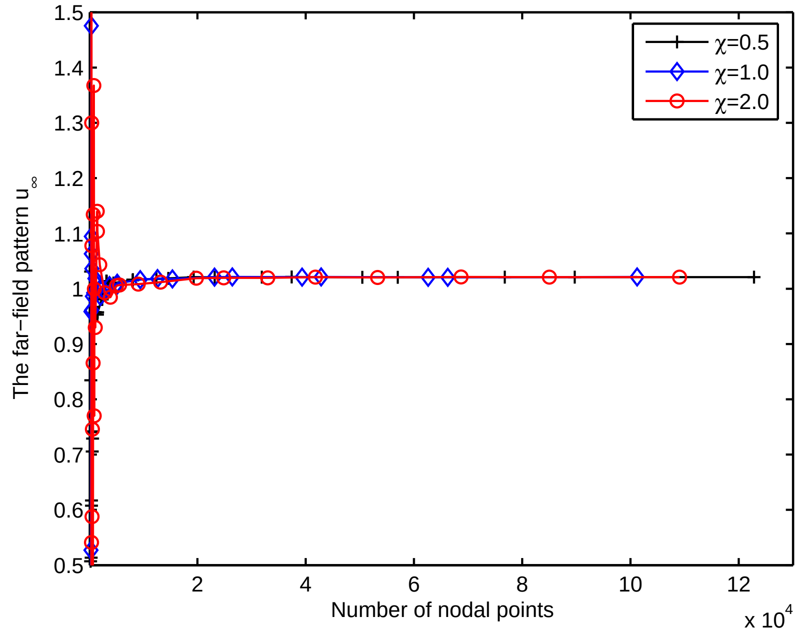

FIG. 6.5. *The real part of the far-field patterns in the reflective direction for Example 2.*

take $k = 2\pi$, that is the wave length $\lambda = 1.0$. Let $\texttt{dist} = 1.0$ and thus the scatterer is contained in the box $\bar{B}_1 = [-3.2, 3.2] \times [-1.7, 1.7]$. The different choices of PML parameters χ and σ determined by the relation (6.2) are shown in Table 6.1.

Figure 6.3 shows the $\log N_k$-$\log \mathcal{E}_k$ curves, where N_k is the number of nodes of the mesh $\mathcal{M}_k$ and the $\mathcal{E}_k = (\sum_{K\in\mathcal{M}_k} \eta_K^2)^{1/2}$ is the associated a posteriori error estimate. It indicates that the meshes and the associated numerical complexity are quasi-optimal: $\mathcal{E}_k \approx C N_k^{-1/2}$ is valid asymptotically.

Figures 6.4 and 6.5 show the far fields in the incident direction $\theta = 0$ and the reflective direction $\theta = \pi$. We observe that the far fields are insensitive to the thickness of the PML layers.

TABLE 6.3
The number of nodes required to achieve the given relative error of the far fields in both incident and reflective directions for different choices of `dist` *for Example 2. The far fields in the last adaptive step when* $\texttt{dist} = 1.0\lambda$ *and* $\chi = 1.0$ *are chosen as the exact far fields.*

Error_inc	$\texttt{dist} = 0.1\lambda$	$\texttt{dist} = 1.0\lambda$	Error_ref	$\texttt{dist} = 0.1\lambda$	$\texttt{dist} = 1.0\lambda$
10%	568	1032	10%	549	492
1%	7315	12614	1%	2871	5183
0.1%	29882	62605	0.1%	9812	23156

Then we take $\chi = 1.0$ and test the influence of different choices of `dist` as in Example 1. Table 6.3 shows the number of nodes required to achieve the given relative error of the far fields in both incident and reflective directions for different choices of PML parameter `dist`. It again shows that for the same accuracy of the far fields, a smaller `dist` is a better choice.

In Figure 6.6 we show the mesh after 16 adaptive iterations when dist $= 1.0$ and $\chi = 1.0$. We observe that the mesh near the boundary Γ_2 is rather coarse, because the solution is rather small there due to the exponential damping of the PML layer. Figure 6.7 shows the real part of the PML solution when dist $= 1.0$ and $\chi = 1.0$.

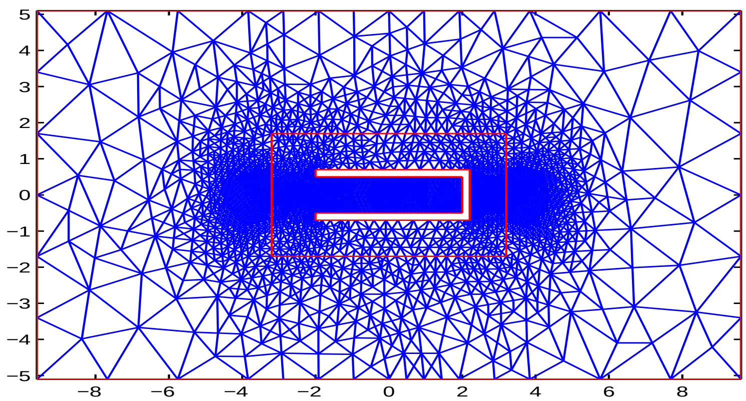

FIG. 6.6. *The mesh of 3754 nodes after 16 adaptive iterations when* dist $= 1.0$ *and* $\chi = 1.0$ *for Example 2.*

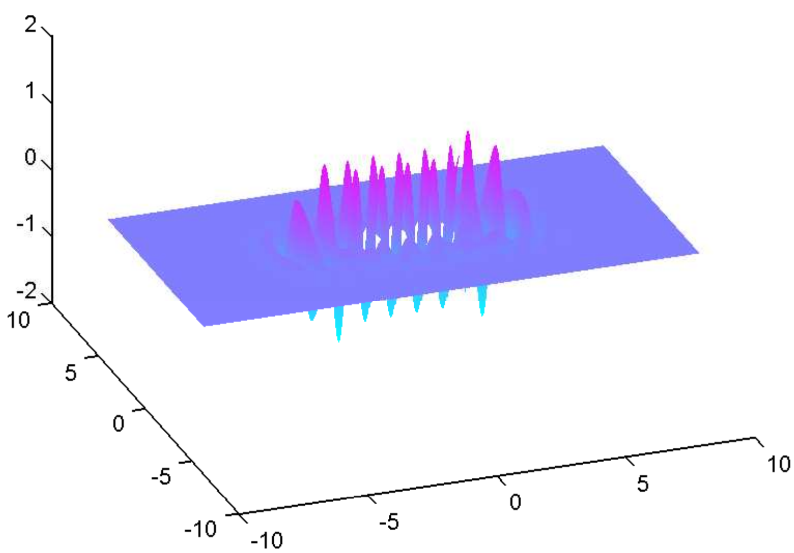

FIG. 6.7. *The real part of the solution of the PML problem for Example 2.*

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