On Aleksandrov-Fenchel Inequalities for k-Convex Domains

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1. Preface

These are the lecture notes based on a course which the first named author has given at the second congress organized by the Riemann International School of Mathematics in Verbania, Italy from September 26 to October 1, 2010.

The topic of the school is Nonlinear Analysis and Nonlinear PDE, with Louis Nirenberg served as the Director of the school for the year. The first named author has greatly enjoyed the lectures given by the participants and the warm hospitality of the organizers. The lectures she gave were mostly based on the ongoing joint work of her with Yi Wang, which have since been written up in [9] and [10].

In these notes we will mainly describe the work in [9]. We will do so by providing more backgrounds and also by adding some new material in section 3 as examples in which these inequalities fail.

These notes are organized as follows: In section 2, we describe the classical Aleksandrov-Fenchel inequalities for convex domains and describe the Question we are interested in–how to extend these inequalities to classes of non-convex domains? In section 3, we discuss examples which indicate why one can not expect to adopt the proof of the original Aleksandrov-Fenchel inequalities for convex domains to this class of non-convex domains. This part of the work is motivated by some recent work of De Lellis-Topping [13] on the almost Schur inequality, and also the subsequent interpretation by Ge-Wang [16] of their work. In section 4, we summarize some known results related to the Question and describe a new approach to a special case of the Question–namely the recent work of Castillon [8] applying the method of optimal transport to give a reproof of the Michael-Simon inequality. In section 5, we describe how to generalize the proof of Castillon to give a partial answer to the Question we are interested in, we will provide more background of the subject, and outline the proof of the main result in [9].

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2. Background

The Aleksandrov-Fenchel inequalities ([1], [2]) for the quermassintegrals for convex domains are fundamental inequalities in classical geometry. For a bounded smooth, convex domain $A \subset \mathbb{R}^{n+1}$, and B the unit ball in R^{n+1}, Minkowski considered the volume expansion of the set $A + tB$ with the expansion

$$Vol(A + tB) = \sum_{k=0}^{n+1} V_{n+1-k}(A)t^k \tag{1}$$

Thus

$$\begin{aligned} When\ \ k = 0,\ V_{n+1}(A) \ &= \ \text{volume of } A. \\ When\ \ k = 1,\ V_n(A) \ &= \ \text{surface area of the boundary of A}. \end{aligned} \tag{2}$$

There are several equivalent expressions of the Minkowski's $(n+1-k)-$volume or the quermassintegral, $V_{n+1-k}(A)$. One of them is the following. Denote $M = \partial A$. Let

$$\kappa(x) = (\kappa_1(x), \kappa_2(x), ...\kappa_n(x))$$

be the principal curvatures at $x \in M$, and let $\sigma_k(\lambda)$ denote the k-th elementary symmetric function of $\lambda = (\lambda_1, \lambda_2, \ldots \lambda_n)$ i.e. $\sigma_0 = 1$ and

$$\sigma_k(\lambda) = \sum_{i_1 < i_2 < \cdots < i_k} \lambda_{i_1}\lambda_{i_2}...\lambda_{i_k}.$$

Thus

$$\begin{aligned} \sigma_1(\kappa) &= \sum_{i=1}^{n} \kappa_i = H \ \text{ the mean curvature of the boundary of A} \\ \sigma_2(\kappa) &= \sum_{i<j} \kappa_i\kappa_j \end{aligned} \tag{3}$$

We now introduce a new notation. Denote by $L = L_M$ the second fundamental form on the boundary of A. Then the principal curvatures on the boundary of A are the eigenvalues of L. Thus

$$\sigma_k(\kappa(x)) = \sigma_k(L(x))$$

for all $x \in M$, where $\sigma_k(L)$ is the k-th elementary symmetric function of the eigenvalues of L.

Then for $k \geq 1$,

$$V_{n+1-k}(A) = C(n,k) \int_M \sigma_{k-1}(L) d\mu_M \tag{4}$$

for some constant $C(n,k) = \frac{1}{n+1-k}$, where $d\mu_M$ denotes the surface measure on M.

A special case of the classical Aleksandrov-Fenchel inequality for a convex domain A is:

$$V_{n+1-k}(A)V_{n-1-k}(A) \leq (V_{n-k}(A))^2. \tag{5}$$

As we will see in the argument in section 3, this inequality implies the following inequality

$$(V_{n+1-k}(A))^{\frac{1}{n+1-k}} \leq \overline{C(n,k)}(V_{n-k}(A))^{\frac{1}{n-k}}, \tag{6}$$

where $\overline{C(n,k)}$ is a constant depending only on n,k and is only achieved when A is a ball. An equivalent way of writing (6) is:

$$(\int_M \sigma_{k-1}(L)d\mu_M)^{\frac{1}{n-(k-1)}} \leq \overline{C(n,k)}(\int_M \sigma_k(L)d\mu_M)^{\frac{1}{n-k}}, \tag{$(*)_k$}$$

where $\overline{C(n,k)}$ again denotes a constant depending only on n,k and is only achieved when A is a ball.

We remark that when $k=0$, $(*)_0$ is interpreted as the classical isoperimetric inequality

$$\mathrm{Vol}(A)^{\frac{1}{n+1}} \leq \bar{C_n}|\partial A|^{\frac{1}{n}}$$

which holds for <u>all</u> bounded domains A in $\mathbb{R}^{n+1}$.

The <u>Question</u> we are interested in these notes is: Does $(*)_k$ hold for a class of domains which are less restrictive than the convex domains?

First we recall some known results of this question. We start with a definition.

<u>Definition</u> We call a domain $A \subset \mathbb{R}^{n+1}$ a k-convex domain if $\sigma_i(L(x)) > 0$ for $1 \leq i \leq k$ for all $x \in \partial A$. In this case, we say the second fundamental form L of M is in the positive k-cone, denoted by $L \in {\Gamma_k}^+$; or we sometimes use the language that $M = \partial A$ is k-convex.

Theorem 2.1. *(Guan-Li [18]) Suppose A is a smooth star-shaped domain in $\mathbb{R}^{n+1}$ with k-convex boundary, then the inequality $(*)_m$ is valid for all $1 \leq m \leq k$; with the equality holds if and only if A is a ball.*

The main idea in the proof of the theorem above is to apply a fully nonlinear flow to study the inequality $(*)_k$ for k-convex domains. Namely, one evolves the hypersurface $M := \partial A \subset \mathbb{R}^{n+1}$ along the flow

$$\vec{X}_t = \frac{\sigma_{k-1}}{\sigma_k}(L)\vec{n}, \tag{7}$$

where $\vec{n}$ is the unit outer normal of hypersurface M.

In earlier articles Gerhardt [17] and Urbas [28] have independently proved that the flow (7) exists for all t and converges to the round sphere when either A is a convex domain or A is a k-convex domain AND star-shaped.

The key observation made in [18] is that the ratio

$$\frac{\left(\int_M \sigma_{k-1}(L)d\mu_M\right)^{\frac{1}{n-k+1}}}{\left(\int_M \sigma_k(L)d\mu_M\right)^{\frac{1}{n-k}}} \tag{8}$$

is monotonically increasing along the flow (7). Therefore if the solution of the flow (7) exists for all $t > 0$ and converges to a round sphere (or up to a rescaling), one can establish the inequality $(*)_k$.

We remark that the flow (7) when $k = 1$ is the inverse mean curvature flow studied earlier by Evans-Spruck [14], Huisken-Illmanen [20] and others.

We would also like to mention that a special case of a sharp inequality between V_{n+1} and V_{n-k} was established by Trudinger for k-convex domains. (See section 3 in [27].)

Another result which is related to a special case of the inequality $(*)_k$ for $k = 1$ is the Michael-Simon inequality.

Theorem 2.2. *(Michael-Simon [24]) Let $i : M^n \to \mathbb{R}^N$ be an isometric immersion $(N > n)$. Let U be an open subset of M. For a nonnegative function $u \in C_c^\infty(U)$, there exists a constant $C = Cn$, such that*

$$\left(\int_M u^{\frac{n}{n-1}} dv_M\right)^{\frac{n-1}{n}} \leq C \int_M (|\mathfrak{H}|u + |\nabla u|) dv_M, \tag{9}$$

where $\mathfrak{H}$ is the mean curvature vector of the immersion.

In the special case, when we take $A \subset \mathbb{R}^{n+1}$ and $M = \partial A$, then $\mathfrak{H} = H\vec{n}$, $\vec{n}$ the outward normal of the domain and H the mean curvature on the boundary of A, and take $U \equiv 1$, we get

Corollary 2.3. *There is a dimensional constant $C = C(n)$, so that*

$$|\partial A|^{\frac{n-1}{n}} \leq C \int_{\partial A} H d\mu.$$

or equivalently:

$$|\partial A|^{\frac{1}{n}} \leq C(\int_{\partial A} H d\mu)^{\frac{1}{n-1}}. \tag{10}$$

We remark that it is an open question what is the best constant in inequality (10). Huisken has indicated that when the domain is "outward minimizing" in the sense as is defined in his earlier work with Illmanen [20], the inequality is sharp with the constant as in the case when A is the ball.

The main result in A. Chang and Y. Wang [9] is the following, which we will call <u>Main Theorem</u> in these notes.

Theorem 2.4. *Let us denote $M = \partial A$. For $k = 2, ..., n-1$, if A is a $(k+1)$-convex domain, then there exists a constant C depending only on n and m, such that for $1 \leq m \leq k$,*

$$\left(\int_M \sigma_{m-1}(L) d\mu_M\right)^{\frac{1}{n-m+1}} \leq C \left(\int_M \sigma_m(L) d\mu_M\right)^{\frac{1}{n-m}}.$$

If $k = n$, then the inequality holds when A is n-convex. If $k = 1$, then the inequality holds when A is 1-convex.

From now now we will denote the m-th inequality in above theorem $(**)_m$, thus $(**)_k$ denotes the inequality

$$\left(\int_M \sigma_{k-1}(L) d\mu_M\right)^{\frac{1}{n-k+1}} \leq C \left(\int_M \sigma_k(L) d\mu_M\right)^{\frac{1}{n-k}}. \tag{$(**)_k$}$$

for some constant $C = C(n, k)$

We remark the method of proof in the theorem above actually extends to prove a generalized class of Michael-Simon inequalities:

Theorem 2.5. *Denote $M = \partial A$ for some bounded domain A in $\mathbb{R}^{n+1}$. Suppose U is an open subset of M and $u \in C_c^\infty(U)$ is a nonnegative function. For $k = 2, ..., n-1$, if A is a $(k+1)$-convex domain, then there exists a constant C depending only on n and m, such that for $1 \le m \le k$*

$$\left(\int_M \sigma_{m-1}(L) u^{\frac{n-m+1}{n-m}} d\mu_M\right)^{\frac{n-m}{n-m+1}} \le C \int_M (\sigma_m(L)u + \sigma_{m-1}(L)|\nabla u| +, ..., +|\nabla^m u|) d\mu_M.$$

If $k = n$, then the inequality is true when A is an n-convex domain. If $k = 1$, then the inequality holds when A is a 1-convex domain. ($k = 1$ case is a corollary of the Michael-Simon inequality.)

For the rest of the notes, we will describe the proof of the Main Theorem.

3. The classical Aleksandrov-Fenchel inequality

In this section, we are going to discuss the classical Aleksandrov-Fenchel inequalities for convex domains A, and derive as a consequence the sharp inequality $(*)_k$. We will then proceed to explain why one cannot use the same proof to obtain $(*)_k$ for k-convex domains. In fact, we will construct examples of k-convex domains such that the Aleksandrov-Fenchel inequality (at level k) is not valid when $k = 2$.

Let A_1,..., A_{n+1} be convex domains. By a result of Minkowski [23] the volume of the linear combination of A_1,..., A_{n+1} with nonnegative coefficients $\lambda_1, ..., \lambda_{n+1}$ is a homogeneous polynomial of degree $n+1$ with respect to $\lambda_1, ..., \lambda_{n+1}$:

$$V(\sum_{i=1}^{n+1} \lambda_i A_i) = \sum_{i_1=1}^{n+1} \cdots \sum_{i_{n+1}=1}^{n+1} V(A_{i_1}, ..., A_{i_{n+1}}) \lambda_{i_1} \cdots \lambda_{i_{n+1}}, \tag{11}$$

In this way, one defines the generalized Minkowski mixed volumes $V(A_1, ..., A_{n+1})$. When $A_1 = \cdots = A_{n+1}$, $V(A, ..., A) = Vol(A)$. When $A_1 = \cdots = A_{n+1-k} = A$, $A_j = B$ for $j > n+1-k$,

$$V(\overbrace{A, ..., A}^{n+1-k}, \overbrace{B, ..., B}^{k}) = C_{n,k} \int_{\partial A} \sigma_{k-1}(L) d\mu \tag{12}$$

Aleksandrov-Fenchel inequality is an inequality relating different mixed volumes:

$$V^2(A_1, A_2, A_3, ..., A_{n+1}) \ge V(A_1, A_1, A_3, ..., A_{n+1}) V(A_2, A_2, A_3, ..., A_{n+1}). \tag{13}$$

In the special case when $A_1 = ... = A_{n-k} = A$, $A_j = B$ for $j > n-k$, we obtain

$$(\int_{\partial A} \sigma_k(L) d\mu)^2 \ge \bar{C}_k (\int_{\partial A} \sigma_{k+1}(L) d\mu)(\int_{\partial A} \sigma_{k-1}(L) d\mu), \tag{$(AF)_k$}$$

where $\bar{C}_k$ is the constant that is attained when A is a ball. We will now derive the inequality $(*)_k$ as a consequence of the string of inequalities $((AF)_l)$ for $l \geq k$. To see this, we first notice that for a convex domain A, $\sigma_n(L)$ is the Jacobian of the Gauss map from A to the unit sphere, hence the integral $\int_{\partial A} \sigma_n(L) d\mu$ is a constant which is equal to the area of the unit sphere. Also, one can prove inductively that

$$(\int_{\partial A} \sigma_k(L) d\mu)^{p(l)} \geq (\int_{\partial A} \sigma_{k-l}(L) d\mu)^{q(l)} (\int_{\partial A} \sigma_{k+1}(L) d\mu) \tag{14}$$

for $k+1 \leq n$; and

$$(\int_{\partial A} \sigma_k(L) d\mu)^{p(l)} \geq (\int_{\partial A} \sigma_{k+l}(L) d\mu)^{q(l)} (\int_{\partial A} \sigma_{k-1}(L) d\mu); \tag{15}$$

for $k+l \leq n$. Here $p(l)$ and $q(l)$ are two positive indices depending only on l. For example, $p(1) = 2$, $q(1) = 1$; $p(2) = 3/2$, $q(2) = 1/2$ etc. Thus we have

$$(\int_{\partial A} \sigma_{k+1}(L) d\mu)^{p(n-k-1)} \geq (\int_{\partial A} \sigma_n(L) d\mu)^{q(n-k-1)} (\int_{\partial A} \sigma_k(L) d\mu); \tag{16}$$

and

$$(\int_{\partial A} \sigma_k(L) d\mu)^2 \geq (\int_{\partial A} \sigma_{k+1}(L) d\mu)(\int_{\partial A} \sigma_{k-1}(L) d\mu). \tag{17}$$

Taking $\frac{1}{p(n-k-1)}$-th power of the inequality (16) into (17), one gets

$$\begin{aligned}(\int_{\partial A} \sigma_k(L) d\mu)^{2-\frac{1}{p(n-k-1)}} \geq & (\int_{\partial A} \sigma_{k-1}(L) d\mu)(\int_{\partial A} \sigma_n(L) d\mu)^{\frac{q(n-k-1)}{p(n-k-1)}} \\ \geq & C(\int_{\partial A} \sigma_{k-1}(L) d\mu).\end{aligned} \tag{18}$$

Therefore the inequalities $(AF)_l$ for $l \geq k$ imply $(*)_k$.

From the above argument, it is natural to consider that whether it would be possible to prove $((AF)_k)$ for $k+1$-convex domains first, then obtain $(*)_k$ for such domains. In the following we will indicate an example of a 2-convex domain in $\mathbb{R}^{n+1}$ when $n \geq 4$ while the inequality $(AF)_1$ fails.

The construction of the example is inspired by a recent work of De Lellis and Topping [13], in which they establish an almost Schur theorem on manifolds under the assumption that Ricci curvature of the manifold being positive; they also went on to construct a counterexample of their inequality on manifolds of dimensions bigger than four when the Ricci curvature is not positive. In a later work, Ge and Wang [16] reformulated the inequality in [13] in terms of an inequality similar to the $(AF)_1$ inequality above but with the integrand under the integration the σ_k of the Schouten tensor on manifolds.

Here we will run a similar program for domains in $\mathbb{R}^{n+1}$ and for σ_k of the second fundamental form of the domains. Following that of [16], we first notice that we can reformulate inequality

$$(\int_{\partial A} \sigma_0(L) d\mu)(\int_{\partial A} \sigma_2(L) d\mu) \leq \frac{n-1}{2n}(\int_{\partial A} \sigma_1(L) d\mu)^2 \tag{19}$$

in an equivalent form as

$$\int_{\partial A} |H - \bar{H}|^2 d\mu \leq \frac{n}{n-1} \int_{\partial A} |\mathring{L}| d\mu^2, \tag{20}$$

where $\mathring{L}_{ij} = L_{ij} - \frac{H}{n} g_{ij}$ is the traceless part of L. To see this, (20) is equivalent to

$$\int_{\partial A} H^2 d\mu - |\partial A| \bar{H}^2 \leq \frac{n}{n-1} \int_{\partial A} (|L|^2 - \frac{H^2}{n}) d\mu, \tag{21}$$

or equivalently

$$\frac{n}{n-1} \int_{\partial A} (H^2 - |L|^2) d\mu \leq |\partial A| \bar{H}^2 = \frac{(\int_{\partial A} H d\mu)^2}{|\partial A|}. \tag{22}$$

Since $2(H^2 - |L|^2) = \sigma_2(L)$, $\sigma_1(L) = H$, and $\sigma_0 = 1$, we have

$$(\int_{\partial A} \sigma_0(L) d\mu)(\int_{\partial A} \sigma_2(L) d\mu) \leq \frac{n-1}{2n} (\int_{\partial A} \sigma_1(L) d\mu)^2, \tag{23}$$

which is $((AF)_1)$. Therefore to construct a counterexample of $(AF)_1$ is equivalent to construct a counterexample of (20).

In the following, we present a construction which is motivated by the work of De Lellis and Müller [12] as well as a recent work of De Lellis and Topping [13]. In $\mathbb{R}^{n+1}$, denote each point by the coordinate (x, x') with $x \in \mathbb{R}$ and $x' \in \mathbb{R}^n$. Consider a positive even function $y = f(x)$, whose definition will be specified later, and a family of positive functions $f_\epsilon : \mathbb{R} \to \mathbb{R}$ given by $f_\epsilon(x) := \epsilon f(\frac{x}{\epsilon})$. Let Σ_ϵ denote the hypersurface generated by the revolution of f_ϵ along the x-axis. Denote $h(y) = f^{-1}(y)$ the inverse function of f; then the second fundamental form on $\Sigma = \Sigma_1$ at the point (x, x') with $x' = (x_2, ..., x_{n+1})$ is equal to

$$(g^{il} L_{lj})(x, x') = \operatorname{diag}(\frac{h''(y)}{(\sqrt{1 + h'^2(y)}^3}, \frac{h'(y)}{y\sqrt{1 + h'^2(y)}}, ..., \frac{h'(y)}{y\sqrt{1 + h'^2(y)}}). \tag{24}$$

Here $y = |x'|$. Thus

$$\sigma_1(L) = \frac{h''(y)}{(1 + h'^2(y))^{3/2}} + \frac{(n-1)h'(y)}{y\sqrt{1 + h'^2(y)}},$$

$$\sigma_2(L) = (\frac{(n-1)h''(y)}{yh'(y)(1 + h'^2(y))} + \frac{(n-1)(n-2)}{2y^2}) \frac{h'^2(y)}{1 + h'^2(y)}.$$

One can always find a unique function f such that $f(0) = 1$, $\sigma_2(L) = 0$, and $\sigma_1(L) > 0$. It turns out the unique inverse function of such an f can be explicitly written as $h(y) = \pm \int_1^y \frac{ds}{\sqrt{s^{n-2} - 1}}$. For example, when $n = 4$, $f(x) = \cosh^{-1} x$. we remark by the same computation we have $\sigma_1(L_\epsilon) > 0$ and $\sigma_2(L_\epsilon) = 0$, where L_ϵ is the second fundamental form on the surface Σ_ϵ.

Consider a round n-sphere of radius 1 centered at some point in the x-axis which is tangent to the surface Σ_ϵ whose intersection with Σ_ϵ is an $(n-1)$-sphere centered at some point $(a_\epsilon, 0, ..., 0) \in \mathbb{R}^{n+1}$ with $a_\epsilon > 0$; then consider another round n-sphere of radius $1/2$ centered at some point in the negative x-axis and tangent to

Σ_ϵ whose intersection with Σ_ϵ is an $(n-1)$-sphere centered at $(b_\epsilon, 0, ..., 0) \in \mathbb{R}^{n+1}$ with $b_\epsilon < 0$.

Consider the domain that is bounded by these two n-spheres together with the hypersurface Σ_ϵ in between them. Denote it as our domain A_ϵ. Then ∂A_ϵ consists of three parts: the part of the sphere lying on the right side of the $(n-1)$-sphere $\Sigma_\epsilon \cap \{x = a_\epsilon\}$; the part of the sphere lying on the left side of the $(n-1)$-sphere $\Sigma_\epsilon \cap \{x = b_\epsilon\}$; and Σ_ϵ in between of these two $(n-1)$-spheres.

In the following we will show that the inequality (20) is not valid for ∂A_ϵ. First, we claim that ∂A_ϵ converges to two n-spheres S_1 and S_2 with radius 1 and 1/2 respectively, and the area of the neck Σ_ϵ goes to 0.

To see this, by some plane geometry computation coupled with the explicit formula for f^{-1}, we see that a_ϵ satisfies the equation

$$\epsilon^2 f^n(\frac{a_\epsilon}{\epsilon}) = 1.$$

Thus, $a_\epsilon = \epsilon \cdot f^{-1}(\frac{1}{\epsilon^{2/n}})$. Using the explicit formula for f^{-1} again, one obtains $a_\epsilon = O(\epsilon \log \epsilon)$, if $n = 4$; $a_\epsilon = O(\epsilon)$, if $n > 4$. Therefore $a_\epsilon \to 0$ as $\epsilon \to 0$. By a similar argument we also have $b_\epsilon \to 0$ as $\epsilon \to 0$. Thus

$$|\Sigma_\epsilon| \le 2\omega_{n-1}[\epsilon f(\frac{a_\epsilon}{\epsilon})]^{n-1} \cdot \max\{a_\epsilon, -|a_\epsilon|\} \to 0.$$

Also the above estimates on a_ϵ and $\epsilon f(\frac{a_\epsilon}{\epsilon})$ together with the explicit formula of the second fundamental form (24) imply

$$\lim_{\epsilon \to 0} \int_{\Sigma_\epsilon} |\mathring{L}|^2 d\mu = 0, \tag{25}$$

and

$$\int_{\Sigma_\epsilon} |H - \bar{H}|^2 d\mu \to 0 \tag{26}$$

for $n \ge 4$. Combining (25) together with the fact that $\mathring{L} = 0$ on the sphere, we get

$$\int_{\partial A_\epsilon} |\mathring{L}|^2 d\mu \to 0.$$

On the other hand, since S_1 and S_2 are spheres with different radius, we obtain from (26)

$$\int_{\partial A_\epsilon} |H - \bar{H}|^2 d\mu \to \int_{S_1 \cup S_2} |H - \bar{H}|^2 d\mu \ne 0.$$

Thus A_ϵ is a counterexample of the inequality (20).

Notice that although A_ϵ is on the boundary of Γ_2^+ cone, one can easily perturb A_ϵ so that it is in Γ_2^+ cone. (More precisely, keep the two n-spheres to be the same while perturb Σ_ϵ, such that $\sigma_2(L) = \delta > 0, \sigma_1(L) = 0$ on Σ_ϵ.) And following a similar argument as above, inequality (20) does not hold for this class of perturbed domains in the Γ_2^+ cone.

4. Castillon's proof of the Michael-Simon inequality

In the following, we will begin to outline a new proof by Castillon ([8]) of the Michael-Simon inequality ([24]). Before we do so, we will describe the main tool Castillon has used in his proof–the method of optimal transport.

4.1. Optimal transport map and regularity

Optimal transportation problem is one of the fundamental subject which has been extensively studied in the literature. The readers are referred to the books of Villani ([29],[30]) for a comprehensive review of the subject. In recent years, the method of optimal transport has been applied to study various inequalities which includes the study of Minkowski's inequality for convex body, Aleksandrov-Fenchel inequality for convex domains, sharp Sobolev inequalities, etc. for examples by ([22], [3] and [11]) and the list is fast growing. Here we will just mention some basic, elementary fact of the subject for the purpose of describing the application below in these notes.

Consider two normed spaces X_1 and X_2, with probability measures μ and ν defined on them respectively. Given a cost function $c : X_1 \times X_2 \to \mathbb{R}$, the problem of Monge consists in finding a map $T : X_1 \to X_2$ such that its cost $C(T) := \int_{X_1} c(x, Tx) d\mu$ attains the minimum of the costs among all the maps that push forward μ to ν. In general, the problem of Monge may have no solution, however in the special case when X_1 and X_2 are bounded domains defined on the Euclidean space with quadratic cost function, Brenier ([4]) proved the existence and uniqueness result. More precisely,

Theorem 4.1. *Suppose that $X_i = D_i$ (i=1,2) are bounded domains in $\mathbb{R}^n$ with $|\partial D_i| = 0$ and that the cost function is defined by $c(x,y) := d^2(x,y)$, where d is the Euclidean distance. Given two nonnegative functions F, G defined on D_1, D_2 respectively with $\int_{D_1} F(x)dx = \int_{D_2} G(y)dy = 1$. Then there exists a unique optimal transport map (solution of the problem of Monge) $T : D_1 \to D_2$. Also T is the gradient of some convex potential function V.*

Since the optimal map $T = \nabla V$ pushes forward $F(x)dx$ to $G(y)dy$, it satisfies the Monge-Ampère equation in the weak sense.

$$\int_{D_2} \eta(y)G(y)dy = \int_{D_1} \eta(\nabla V(x))F(x)dx, \tag{27}$$

for any continuous function η.

Thus the potential function V satisfies the Monge-Ampère type equation

$$F(x) = det(D^2_{ij}V(x))G(\nabla V(x)) \quad \text{(Optimal transport equation)} \tag{28}$$

for all $x \in D_1$ in a weak sense. However, under the additional assumptions on the convexity of D_i, as well as on the smoothness of F and G, Caffarelli has addressed in his papers ([5], [6], [7]) the regularity problem when the Brenier function V satisfies the equation (28) in the classical sense. We now state these results of Caffarelli here as we shall apply them later in the proof of our main theorem.

Theorem 4.2. *([6]) If D_2 is a convex domain and F, G, $1/F$, $1/G$ are bounded, then V is strictly convex and $C^{1,\beta}$ for some β.*
If F and G are C^{k,α_0}, then $V \in C^{k+2,\alpha}$ for any $0 < \alpha < \alpha_0$.

For the boundary regularity, one needs to assume D_1 to be convex as well:

Theorem 4.3. *([7]) If both D_i are C^2 and strictly convex, and F, $G \in C^\alpha$ are bounded away from zero and infinity, then the convex potential function V is $C^{2,\beta}$ up to ∂D_i for some $\beta > 0$. Both β and $\|V\|_{C^{2,\beta}}$ depend only on the maximum and minimum diameter of D_i and the bounds on F, G. Higher regularity of V follows from assumptions on the higher regularity of F and G by the standard elliptic theory.*

From these two theorems, we know that if D_i are smooth and strictly convex, and F, G are both smooth and bounded away from zero and infinity up to the boundary, then the potential function is smooth up to the boundary as well. For more results on the regularity of optimal transport maps between manifolds, we refer the readers to [25], [21], [29], etc.

4.2. Restriction of convex functions to submanifolds

Consider an isometric embedding $i : M^n \to \mathbb{R}^{n+1}$. Let $\vec{n}(\xi)$ be the outer unit normal at $\xi \in M$. Let ∇ and D^2 (resp. $\bar{\nabla}$ and $\bar{D}^2$) be the gradient and Hessian on M (resp. on $\mathbb{R}^{n+1}$); let $\vec{L}(\cdot,\cdot)(\xi) = L(\cdot,\cdot)(\xi)\vec{n}(\xi)$ be the second fundamental form at $\xi \in M$. Suppose $\bar{V} : \mathbb{R}^{n+1} \to \mathbb{R}$ is a smooth function and $v = \bar{V}|_M$ is its restriction to M. Then the Hessian of v with respect to the metric on M relates to the Hessian of $\bar{V}$ on the ambient space $\mathbb{R}^{n+1}$ in the following way: for all $\xi \in M$ and all $\alpha, \beta \in T_\xi M$,

$$\begin{aligned} D^2 v(\alpha,\beta)(\xi) =& \bar{D}^2\bar{V}(\alpha,\beta)(\xi) - \langle(\bar{\nabla}\bar{V}), \vec{L}(\alpha,\beta)\rangle(\xi) \\ =& \bar{D}^2\bar{V}(\alpha,\beta)(\xi) + b(\xi)\cdot L(\alpha,\beta)(\xi), \end{aligned} \quad \text{(Structure equation)} \tag{29}$$

where $b(\xi) := -\langle(\bar{\nabla}\bar{V}), \vec{n}\rangle(\xi)$. We remark in general the function $b(\xi)$ changes sign on M.

Finally we recall the well known Gauss equation and Codazzi equation that are satisfied by the curvature tensors defined on the embedded submanifolds. Denote the curvature tensor of M by R_{ijkl} and the curvature tensor of the ambient space $\mathbb{R}^{n+1}$ by $\bar{R}_{ijkl}$. Then

$$0 = \bar{R}_{ijkl} = R_{ijkl} - L_{jl}L_{ik} + L_{il}L_{jk}, \quad \text{(Gauss equation)} \tag{30}$$

and

$$L_{ij,k} = L_{ik,j}. \quad \text{(Codazzi equation)} \tag{31}$$

4.3. Outline of Castillon's proof

We now outline the proof of Castillon ([8]) of the Michael-Simon inequality for the special case when the submanifold $M = \partial\Omega$ is the boundary of a simply connected bounded domain $\Omega \subset R^{n+1}$.

Assuming M is 1-convex with its mean curvature $H > 0$, we will show that

$$(\int_M u^{\frac{n}{n-1}} d\mu_M)^{\frac{n-1}{n}} \leq C \int_M (|\nabla u| + uH) d\mu_M \tag{32}$$

for all positive smooth function u defined on M and for some dimensional constant C.

Wherein the notation above we recall $d\mu_M$ is the surface measure induced on M by the plane Euclidean measure on Ω.

Now given a positive function f defined on M so that $\mu =: f d\mu_M$ is a probability measure on M, and fixed an n dimensional linear subspace E in $\mathbb{R}^{n+1}$, denote $p_E : \mathbb{R}^{n+1} \to E$ the orthogonal projection, Then the pushforward measure $p_{E\#}\mu$ is a probability measure on E. It is absolutely continuous with respect to the Lebesgue measure on E with density $F(x)$ given by

$$F(x) = \sum_{\xi \in {p_E}^{-1}(x) \cap Spt(\mu)} \frac{f(\xi)}{J_E(\xi)}, \tag{33}$$

where J_E is the Jacobian of the map p_E.

Applying Brenier's theorem, there exists a convex potential V such that ∇V is the solution of Monge problem on E between $(D_E, F(x)dx)$ and $(B_E(0,1), \frac{\chi_{B_E(0,1)}}{\omega_n} dy)$, where $D_E := Spt(p_{E\#}\mu)$ and $B_E(0,1)$ is the unit ball in E. $\frac{\chi_{B_E(0,1)}}{\omega_n} dy$ is the normalized Lebesgue measure on $B_E(0,1)$. Since $\nabla V(Spt(p_E \# \mu)) \subset B_E(0,1)$, we have $|\nabla V| \leq 1$ on D_E.

In general, the convex potential V is only a Lipschitz function, let us ASSUME V is C^3 for a moment to finish the proof of the theorem. If V is C^3, then V satisfies the Structure equation of the Monge-Ampère type:

$$\omega_n F(x) = det(D^2 V(x)) \tag{34}$$

in the classical sense. Define the extension of V by $\bar{V} := V \circ p : \mathbb{R}^{n+1} \to \mathbb{R}$ and its restriction to M by $v(\xi) := \bar{V}|_M(\xi) = V \circ p|_M(\xi)$. Denote the gradient and Hessian on M by ∇ and D^2 respectively, and denote the gradient and Hessian on $\mathbb{R}^{n+1}$ by $\bar{\nabla}$ and $\bar{D}^2$ respectively. By (33), (34), for $\xi \in M$

$$\omega_n \frac{f(\xi)}{J_E} \leq \omega_n F(p(\xi)) = det(D^2 V(p_E(\xi))). \tag{35}$$

By the change of variable formula,

$$det(\bar{D}^2 \bar{V}(\xi)|_{T_\xi M}) = J_E^2(\xi) det(D^2 V(p_E(\xi))).$$

Thus for $\xi \in M$ we have:

$$\omega_n f(\xi) J_E(\xi) \leq det(\bar{D}^2 \bar{V}(\xi)|_{T_\xi M}). \tag{36}$$

Since $\bar{D}^2 \bar{V}(\xi)|_{T_\xi M}$ is a nonnegative matrix, we take the n-th root on both sides of (36), and apply the inequality between the geometric and arithmetic mean followed by the Structure equation (29) to obtain

$$\begin{aligned}(\omega_n f(\xi) J_E(\xi))^{\frac{1}{n}} &\leq \left(det(\bar{D}^2\bar{V}(\xi)|_{T_\xi M})\right)^{\frac{1}{n}} \\ . &\leq \frac{1}{n}(\bar{\Delta}\bar{V}(\xi)) \\ &= \frac{1}{n}\Delta v \,+\, \langle \bar{\nabla}\bar{V}, H\,\vec{n}\rangle.\end{aligned} \tag{37}$$

For any positive function u defined on M, we then choose f to be the function

$$\frac{u^{\frac{n}{n-1}}}{\int_M u^{\frac{n}{n-1}} d\mu_M},$$

multiply the equation (37) by u and integrate it over M, to obtain

$$\begin{aligned}\left(\int_M J_E{}^{\frac{1}{n}} u^{\frac{n}{n-1}} d\mu_M\right)^{\frac{n-1}{n}} &\leq C\left(\int_M ((-\Delta v)u + Hu) d\mu_M\right. \\ &\leq C\int_M (|\nabla u| + Hu) d\mu_M.\end{aligned} \tag{38}$$

We remark that in the work of Castillon, as the Brenier potential function V is only Lipschitz, the Laplacian of V and v in the Structure equation (29) are only satisfied in the “Aleksandrov sense”. Castillon has rigirously justified the step of integration by part in the proof of (38) above. As we will describe below, in the proof of our Main Theorem, we have overcome this difficulty by an approximation process using the regularity results of Caffarelli’s Theorem 4.2 and Theorem 4.3 on the optimal transport map.

To finish the proof, we then integrate both sides of the inequality (38) above on the Grassmannian $G_{n,n+1}$ of n-planes in R^{n+1}, since integration of $\int_{G_{n,n+1}} J_E^{\frac{1}{n}} dE$ is finite and invariant in $\xi \in M$, we obtain the desired inequality (32).

5. Proof of the Main Theorem

In this section, we will outline the proof of the Main Theorem, which we re-state below:

Theorem 5.1. *For $k = 2, ..., n-1$, if Ω is a $(k+1)$-convex domain, denote $M = \partial\Omega$ then there exists a constant C depending only on n and m, such that for $1 \leq m \leq k$,*

$$\left(\int_M \sigma_{m-1}(L) d\mu_M\right)^{\frac{1}{n-m+1}} \leq C\left(\int_M \sigma_m(L) d\mu_M\right)^{\frac{1}{n-m}}. \tag{39}$$

If $k = n$, then the inequality holds when Ω is n-convex. If $k = 1$, then the inequality holds when Ω is 1-convex.

Proof. of Theorem 5.1: It is clear that we only need to prove the inequality for $m = k$ when M is $k+1$-convex, that is we will establish $(**)_k$:

$$\left(\int_{M^n} \sigma_{k-1}(L) d\mu_M\right)^{\frac{1}{n-(k-1)}} \leq C(n,k) \left(\int_{M^n} \sigma_k(L) d\mu_M\right)^{\frac{1}{n-k}}. \tag{40}$$

Let $E \subset \mathbb{R}^{n+1}$ be an n-dimensional linear subspace, and $\mu := f d\mu_M$ is a probability measure on M, following the same steps as in Castillon's proof outlined in section 4 before, we reached the position

$$\omega_n f(\xi) J_E(\xi) \leq det(\bar{D}^2 \bar{V}(\xi)|_{T_\xi M}). \tag{41}$$

Since $\bar{D}^2\bar{V}(\xi)|_{T_\xi M}$ is a nonnegative matrix, we take the $(n-k+1)$-th root on both sides of (41).

$$(\omega_n f(\xi) J_E(\xi))^{\frac{1}{n-k+1}} \leq \left(det(\bar{D}^2\bar{V}(\xi)|_{T_\xi M})\right)^{\frac{1}{n-k+1}}. \tag{42}$$

To simplify the notation, from now on we will denote $\bar{D}^2\bar{V}(\xi)|_{T_\xi M}$ by $\bar{D}^2\bar{V}(\xi)$.

For each positive constant $a > 1$, multiplying the previous inequality by $\frac{\sigma_{k-1}(\bar{D}^2\bar{V}+(a-1)L)}{\sigma_{k-1}(\bar{D}^2\bar{V})^{\frac{1}{n-k+1}}}$, we get

$$\begin{aligned} &(\omega_n f(\xi) J_E(\xi))^{\frac{1}{n-k+1}} \cdot \frac{\sigma_{k-1}(\bar{D}^2\bar{V} + (a-1)L)}{\sigma_{k-1}(\bar{D}^2\bar{V})^{\frac{1}{n-k+1}}} \\ \leq &\left(det(\bar{D}^2\bar{V}(\xi)\right)^{\frac{1}{n-k+1}} \cdot \frac{\sigma_{k-1}(\bar{D}^2\bar{V} + (a-1)L)}{\sigma_{k-1}(\bar{D}^2\bar{V})^{\frac{1}{n-k+1}}}. \end{aligned} \tag{43}$$

Denote the left hand side (resp. right hand side) of this inequality by LHS (resp. RHS). Then

$$RHS = \left(\frac{det(\bar{D}^2\bar{V})}{\sigma_{k-1}(\bar{D}^2\bar{V})}\right)^{\frac{1}{n-k+1}} \sigma_{k-1}(\bar{D}^2\bar{V} + (a-1)L). \tag{44}$$

We then apply the concavity properties of $(\frac{\sigma_k(A)}{\sigma_j(A)})^{\frac{1}{k-j}}$ for matrix $A \in \Gamma_k^+$ and $j < k$ to get

$$RHS \lessapprox \sigma_k(\bar{D}^2\bar{V} + (a-1)L). \tag{45}$$

Next we apply the Structure equation (29), and the fact that $|\nabla V(\xi)| \leq 1$:

$$RHS \lessapprox \sigma_k(D^2 v + aL), \tag{46}$$

where the symbol $\lessapprox$ means that the inequality holds up to a constant.

We now begin to lower estimate the LHS of (43). To do so, we first observe that we can apply Garding's inequality to obtain $\sigma_{k-1}(\bar{D}^2\bar{V} + (a-1)L) \geq \sigma_{k-1}(\bar{D}^2\bar{V})$. Therefore,

$$LHS \geq (\omega_n f(\xi) J_E(\xi))^{\frac{1}{n-k+1}} \cdot (\sigma_{k-1}(\bar{D}^2\bar{V} + (a-1)L))^{1-\frac{1}{n-k+1}}. \tag{47}$$

We apply Garding's inequality again to get

$$\sigma_{k-1}(\bar{D}^2\bar{V} + (a-1)L)^{1-\frac{1}{(n-k+1)}} \geq C(a)\sigma_{k-1}(L)^{1-\frac{1}{(n-k+1)}}.$$

This together with the choice of f as

$$f := \frac{\sigma_{k-1}(L) J_E^{\frac{1}{n-k}}}{\int_M \sigma_{k-1}(L) J_E^{\frac{1}{n-k}} d\mu_M} \tag{48}$$

implies that

$$\begin{aligned} LHS \geq & (\omega_n f(\xi) J_E(\xi))^{\frac{1}{n-k+1}} \cdot C(a) \sigma_{k-1}(L)^{1-\frac{1}{n-k+1}} \\ = & \frac{C_{n,a} \sigma_{k-1}(L) J_E(\xi)^{\frac{1}{n-k}}}{(\int_M \sigma_{k-1}(L) J_E^{\frac{1}{n-k}} d\mu_M)^{\frac{1}{n-k+1}}} \end{aligned} \tag{49}$$

By integrating LHS and RHS in (43) over M, we obtain

$$\begin{aligned} & \frac{C_{n,a} \int_M \sigma_{k-1}(L) J_E^{\frac{1}{n-k}} d\mu_M}{(\int_M \sigma_{k-1}(L) J_E^{\frac{1}{n-k}} d\mu_M)^{\frac{1}{n-k+1}}} \\ \leq & C_{n,k} \int_M \sigma_k(D^2 v + aL) d\mu_M. \end{aligned} \tag{50}$$

Thus

$$\left(\int_M \sigma_{k-1}(L) J_E^{\frac{1}{n-k}} d\mu_M \right)^{1-\frac{1}{n-k+1}} \leq C_{n,k,a} \int_M \sigma_k(D^2 v + aL) d\mu_M. \tag{51}$$

□

The following Proposition then provides the key step to finish the proof of the Main theorem.

Proposition 5.2. *Let $E \subset \mathbb{R}^{n+1}$ be an n-dimensional linear subspace, and p_E be the orthogonal projection from $\mathbb{R}^{n+1}$ to E. Suppose $V : E \to \mathbb{R}$ is a C^3 convex function that satisfies $|\nabla V| \leq 1$. Define its extension to $\mathbb{R}^{n+1}$ by $\bar{V} := V \circ p_E$, and define the restriction of $\bar{V}$ to the closed hypersurface M by v. Suppose also that M is $(k+1)$-convex if $2 \leq k \leq n-1$, i.e. the second fundamental form $L_{ij} \in \Gamma_{k+1}^+$ and suppose that M is n-convex if $k = n$. Then for each k, there exists a constant $a > 1$, which depends only on k and n, such that*

$$\int_M \sigma_k(D^2 v + aL) d\mu_M \leq C \int_M \sigma_k(L) d\mu_M, \tag{52}$$

where the constant C depends on k, n and a.

For the rest of this section, we will describe the proof of Proposition 5.2 for the special cases $k = 2$ and $k = 3$.

In order to describe the proof, we will introduce some new concept–namely the polarized form of the $\sigma_k(A)$ for any $n \times n$ matrix A. We will also state some basic properties of these polarized forms.

5.1. The polarization of σ_k

Notice that $\sigma_n(A) = \det(A)$. An equivalent definition of $\det(A)$ is

$$\det A = \frac{1}{n!}\delta^{i_1,...,i_n}_{j_1,...,j_n} A_{i_1 j_1} \cdots A_{i_n j_n}, \tag{53}$$

where $\delta^{i_1,...,i_n}_{j_1,...,j_n}$ is the generalized Kronecker delta; it is zero if $\{i_1,...,i_k\} \neq \{j_1,...,j_k\}$, equals to 1 (or -1) if $(i_1,...,i_k)$ and $(j_1,...,j_k)$ differ by an even (or odd) permutation. Inspired by (53), an equivalent way of writing σ_k is that

$$\sigma_k(A) := \frac{1}{k!}\delta^{i_1,...,i_k}_{j_1,...,j_k} A_{i_1 j_1} \cdots A_{i_k j_k}.$$

The Newton transformation tensor is defined as

$$[T_k]_{ij}(A_1,...,A_k) := \frac{1}{k!}\delta^{i,i_1,...,i_k}_{j,j_1,...,j_k} (A_1)_{i_1 j_1} \cdots (A_k)_{i_k j_k}. \tag{54}$$

Definition 5.3. *With the notion of $[T_k]_{ij}$, one may define the polarization of σ_k by*

$$\Sigma_k(A_1,...,A_k) := A_{1ij} \cdot [T_{k-1}]_{ij}(A_2,...,A_k) = \frac{1}{(k-1)!}\delta^{i_1,...,i_k}_{j_1,...,j_k} (A_1)_{i_1 j_1} \cdots (A_k)_{i_k j_k}. \tag{55}$$

It is called the polarization of σ_k because if we take $A_1 = \cdots = A_k = A$, then $\Sigma_k(A,...,A)$ is equal to $\sigma_k(A)$ up to a constant. Namely,

$$\sigma_k(A) = \frac{1}{k}\Sigma_k(A,...,A).$$

Also, from the right hand side of Definition 5.3, we see that Σ_k is symmetric and linear in each component. Also for simplicity, we denote

$$[T_k]_{ij}(A) := [T_k]_{ij}(\overbrace{A,...,A}^{k}).$$

Some relations between the Newton transformation tensor T_k and σ_k are listed below. For any symmetric matrix A, if we denote the trace by Tr, then

$$\sigma_k(A) = \frac{1}{n-k} Tr([T_k]_{ij})(A), \tag{56}$$

and

$$\sigma_{k+1}(A) = \frac{1}{k+1} Tr(A_{im} \cdot [T_k]_{mj}(A)). \tag{57}$$

On the other hand, one can write $[T_k]_{ij}$ in terms of σ_k by the formula

$$[T_{k-1}]_{ij}(A) = \frac{\partial \sigma_k(A)}{\partial A_{ij}},$$

and

$$[T_k]_{ij}(A) = \sigma_k(A)\delta_{ij} - [T_{k-1}]_{im}(A)A_{mj}. \tag{58}$$

We now list some algebraic properties for matrix A in ${\Gamma_k}^+$ cone which can be found for example in [26]:

(i) if $A = (a_{ij})$ in Γ_k^+, then

$$[T_{k-1}]_{ij} = \frac{\partial \sigma_k(A)}{\partial a_{ij}} > 0,$$

(ii) if $A_1, ..., A_k \in \Gamma_{k+1}^+$, then $([T_k]_{ij})$ is a positive matrix, i.e.

$$[T_k]_{ij}(A_1, ..., A_k) > 0;$$

(iii) if $A_1, ..., A_k \in \Gamma_k^+$, then

$$\Sigma_k(A_1, ..., A_k) > 0;$$

(iv) if $A - B \in \Gamma_k^+$ and $A_2, ..., A_k \in \Gamma_k^+$, then

$$\Sigma_k(B, A_2..., A_k) < \Sigma_k(A, A_2, ..., A_k).$$

5.2. Proof of $k = 2$ case of the proposition

In this section, we are going to prove Proposition 5.2 for the special case $k = 2$. For this special case, only $|\nabla V| \leq 1$ property of the Brenier map is relevant, and one can also simply choose $a = 2$.

Proof. We first recall that $\frac{1}{2}\Sigma_2(A, A) = \sigma_2(A)$, thus

$$\begin{aligned}
\int_M \sigma_2(D^2 v + aL) d\mu_M &= \int_M \frac{1}{2}\Sigma_2(D^2 v + aL) d\mu_M \\
&= \int_M \frac{1}{2}[\Sigma_2(D^2 v, D^2 v) + 2a\Sigma_2(D^2 v, L) + a^2\Sigma_2(L, L)] d\mu_M \\
&= \int_M \sigma_2(D^2 v) + a\Sigma_2(D^2 v, L) + a^2\sigma_2(L) d\mu_M \\
&:= I \, + \, a\, II \, + \, a^2 III.
\end{aligned} \tag{59}$$

By the integration by parts formula,

$$I := \int_M \sigma_2(D^2 v) d\mu_M = \int_M (v_{ii}v_{jj} - v_{ij}v_{ij}) d\mu_M = \int_M -v_i(v_{jji} - v_{ijj}) d\mu_M. \tag{60}$$

If we apply the curvature equation

$$v_{ijk} - v_{ikj} = R_{mijk} v_m. \text{ curvature equation} \tag{61}$$

then

$$I = \int_M v_i (Rc)_{mi} v_m d\mu_M, \tag{62}$$

where Rc is the Ricci curvature tensor of g, g the induced metric (i.e. the surface measure) on M. By the Gauss equation (30), the Ricci curvature tensor satisfies $(Rc)_{ik} = L_{jj}L_{ik} - L_{ij}L_{jk}$. If we diagonalize $L_{ij} \sim diag(\lambda_1, ..., \lambda_n)$, then $Rc \sim$

$diag(\lambda_1(H-\lambda_1), ..., \lambda_n(H-\lambda_n))$. Note that

$$\lambda_i(H-\lambda_i) + \frac{\partial \sigma_3(L)}{\partial \lambda_i} = \sigma_2(L) \tag{63}$$

for each $i = 1, ..., n$. Also by our assumption $L \in \Gamma_3^+$ (we remark that this is the only place in the proof where we have used the property that more than $L \in \Gamma_2^+$, it is in $L \in \Gamma_3^+$), thus $\frac{\partial \sigma_3(L)}{\partial \lambda_i} > 0$ for each i by the Garding's inequality. Thus by (63), $\lambda_i(H-\lambda_i) < \sigma_2(L)$ for each i, i.e. $Rc < \sigma_2(L) \cdot g$. Applying this formula to the inequality (62), we get

$$I \leq \int_M \sigma_2(L)|\nabla v|^2 d\mu_M \leq \int_M \sigma_2(L) d\mu_M, \tag{64}$$

where $|\nabla v| \leq 1$ because $|\bar{\nabla} \bar{V}| \leq 1$. Thus

$$I \leq \int_M \sigma_2(L) d\mu_M. \tag{65}$$

For the term II, by definition $\Sigma_2(D^2 v, L) = v_{ii}L_{jj} - v_{ij}L_{ij}$. Thus

$$\begin{aligned} II :=& \int_M \Sigma_2(D^2 v, L) d\mu_M \\ =& \int_M (v_{ii}L_{jj} - v_{ij}L_{ij}) d\mu_M \\ =& \int_M (-v_i L_{jj,i} + v_i L_{ij,j}) d\mu_M = 0, \end{aligned} \tag{66}$$

due to the Codazzi equation (31).
Finally,

$$III := \int_M \sigma_2(L) d\mu_M. \tag{67}$$

Hence from (59), we get

$$\int_M \sigma_2(D^2 v + aL) d\mu_M \leq C(a) \int_M \sigma_2(L) d\mu_M. \tag{68}$$

This finishes the proof of Proposition 5.2 when $k = 2$. Note that in this case, (52) holds for any constant $a > 1$. □

5.3. $k = 3$ case of the proposition

In this section, we will prove Proposition 5.2 when $k = 3$. We will see that in this case convexity property of $\bar{V}$ together with the size estimate $|\bar{\nabla} V| \leq 1$ both play a role in the proof. Again we denote $\bar{D}^2 \bar{V}|_{T_\xi M}$ by $\bar{D}^2 \bar{V}$.

First we use the polarized form of $\sigma_3(D^2v+aL)$ and expand it into four parts

$$
\begin{aligned}
3\int_M \sigma_3(D^2v+aL) = &\int_M \Sigma_3(D^2v+aL, D^2v+aL, D^2v+aL)\\
&= 3a^2\int_M \Sigma_3(D^2v, L, L) + 3a\int_M \Sigma_3(D^2v, D^2v, L)\\
&+ \int_M \Sigma_3(D^2v, D^2v, D^2v) + a^3\int_M \Sigma_3(L, L, L)\\
&= 3a^2 I + 3aII + III + a^3 IV.
\end{aligned} \tag{69}
$$

We will begin the estimates of each of $I, IIandIII$ by proving the following three lemmas.

Lemma 5.4. *Suppose v and M satisfy the same conditions as in Proposition 5.2, then*

$$\int_M \Sigma_3(D^2v, L, L)d\mu_M = 0. \tag{70}$$

Proof.

$$
\begin{aligned}
I = \int_M \Sigma_3(D^2v, L, L)d\mu_M &= \int_M \frac{1}{2!}v_{ij}\delta^{i,i_1,i_2}_{j,j_1,j_2}L_{i_1j_1}L_{i_2j_2}d\mu_M\\
&= \int_M \frac{-1}{2!}v_i\delta^{i,i_1,i_2}_{j,j_1,j_2}\left(L_{i_1j_1,j}L_{i_2j_2} + L_{i_1j_1}L_{i_2j_2,j}\right)d\mu_M.
\end{aligned} \tag{71}
$$

Since $\delta^{i,i_1,i_2}_{j,j_1,j_2}L_{i_1j_1,j}L_{i_2j_2} = \delta^{i,i_1,i_2}_{j,j_1,j_2}L_{i_1j_1}L_{i_2j_2,j}$, we have

$$\int_M \Sigma_3(D^2v, L, L)d\mu_M = \int_M -v_i\delta^{i,i_1,i_2}_{j,j_1,j_2}L_{i_1j_1,j}L_{i_2j_2}d\mu_M. \tag{72}$$

Also, since $\delta^{i,i_1,i_2}_{j,j_1,j_2}L_{i_1j_1,j}L_{i_2j_2} = \delta^{i,i_1,i_2}_{j_1,j,j_2}L_{i_1j,j_1}L_{i_2j_2}$, and by definition, $\delta^{i,i_1,i_2}_{j_1,j,j_2} = -\delta^{i,i_1,i_2}_{j,j_1,j_2}$, we have

$$
\begin{aligned}
\delta^{i,i_1,i_2}_{j,j_1,j_2}L_{i_1j_1,j}L_{i_2j_2} =& -\delta^{i,i_1,i_2}_{j,j_1,j_2}L_{i_1j,j_1}L_{i_2j_2}\\
=&\frac{1}{2}\delta^{i,i_1,i_2}_{j,j_1,j_2}(L_{i_1j_1,j} - L_{i_1j,j_1})L_{i_2j_2},
\end{aligned} \tag{73}
$$

which in turn implies that

$$\int_M \Sigma_3(D^2v, L, L)d\mu_M = \int_M -\frac{1}{2}v_i\delta^{i,i_1,i_2}_{j,j_1,j_2}(L_{i_1j_1,j} - L_{i_1j,j_1})L_{i_2j_2}d\mu_M = 0 \tag{74}$$

by the Codazzi equation (31). □

Lemma 5.5. *Suppose v and M satisfy the same conditions as in Proposition 5.2. Then*

$$II = \int_M \Sigma_3(D^2v, D^2v, L)dv_M \le \int_M \sigma_3(L)d\mu_M. \tag{75}$$

Proof. We perform the integration by parts to get

$$
\begin{aligned}
II &= \int_M \Sigma_3(D^2v, D^2v, L)d\mu_M \\
&= \int_M \frac{1}{2!} v_{ij}\delta^{i,i_1,i_2}_{j,j_1,j_2} v_{i_1j_1}L_{i_2j_2}d\mu_M \\
&= \int_M \frac{-1}{2!} v_i\delta^{i,i_1,i_2}_{j,j_1,j_2} \left(v_{i_1j_1j}L_{i_2j_2} + v_{i_1j_1}L_{i_2j_2,j}\right) d\mu_M := II_a + II_b.
\end{aligned}
\tag{76}
$$

By the same argument as in (73) and the curvature equation $v_{ijk} - v_{ikj} = R_{mijk}v_m$,

$$
\begin{aligned}
II_a &:= \int_M \frac{-1}{2!} v_i\delta^{i,i_1,i_2}_{j,j_1,j_2} v_{i_1j_1j}L_{i_2j_2}d\mu_M \\
&= \int_M \frac{-1}{4} v_i\delta^{i,i_1,i_2}_{j,j_1,j_2} (v_{i_1j_1j} - v_{i_1jj_1})L_{i_2j_2}d\mu_M \\
&= \int_M \frac{1}{4} v_i\delta^{i,i_1,i_2}_{j,j_1,j_2} R_{mi_1jj_1}v_mL_{i_2j_2}d\mu_M.
\end{aligned}
\tag{77}
$$

Using the Gauss equation (30) in (77), we get

$$
\begin{aligned}
II_a &= \int_M \frac{1}{4} v_iv_m\delta^{i,i_1,i_2}_{j,j_1,j_2} (L_{mj}L_{i_1j_1} - L_{mj_1}L_{i_1j})L_{i_2j_2}d\mu_M \\
&= \int_M \frac{1}{2} v_iv_m\delta^{i,i_1,i_2}_{j,j_1,j_2} L_{mj}L_{i_1j_1}L_{i_2j_2}d\mu_M \\
&= \int_M [T_2]_{ij}(L,L)L_{mj}v_iv_md\mu_M.
\end{aligned}
\tag{78}
$$

Now, we use the formula (58) for $k = 3$, i.e.

$$
\begin{aligned}
&\int_M \Sigma_3(D^2v, D^2v, L)d\mu_M \\
&= \int_M \frac{1}{2!} v_{ij}\delta^{i,i_1,i_2}_{j,j_1,j_2} v_{i_1j_1}L_{i_2j_2}d\mu_M \\
&= \int_M \frac{-1}{2!} v_i\delta^{i,i_1,i_2}_{j,j_1,j_2} ((v_{i_1j_1j}L_{i_2j_2} + v_{i_1j_1}L_{i_2j_2,j}) d\mu_M := II_a + II_b.
\end{aligned}
\tag{79}
$$

By the same argument as in (73) and the curvature equation $v_{ijk} - v_{ikj} = R_{mijk}v_m$,

$$
\begin{aligned}
II_a &:= \int_M \frac{-1}{2!} v_i\delta^{i,i_1,i_2}_{j,j_1,j_2} v_{i_1j_1j}L_{i_2j_2}d\mu_M \\
&= \int_M \frac{-1}{4} v_i\delta^{i,i_1,i_2}_{j,j_1,j_2} (v_{i_1j_1j} - v_{i_1jj_1})L_{i_2j_2}d\mu_M \\
&= \int_M \frac{1}{4} v_i\delta^{i,i_1,i_2}_{j,j_1,j_2} R_{mi_1jj_1}v_mL_{i_2j_2}d\mu_M.
\end{aligned}
\tag{80}
$$

Apply the Gauss equation (30) to (80), we get

$$\begin{aligned} II_a &= \int_M \frac{1}{4} v_i v_m \delta^{i,i_1,i_2}_{j,j_1,j_2} (L_{mj}L_{i_1j_1} - L_{mj_1}L_{i_1j}) L_{i_2j_2} d\mu_M \\ &= \int_M \frac{1}{2} v_i v_m \delta^{i,i_1,i_2}_{j,j_1,j_2} L_{mj}L_{i_1j_1}L_{i_2j_2} d\mu_M \\ &= \int_M [T_2]_{ij}(L,L) L_{mj} v_i v_m d\mu_M. \end{aligned} \tag{81}$$

Now, we use the formula (58) for $k = 3$, i.e.

$$[T_2]_{ij}(L,L)L_{mj} = \sigma_3(L)\delta_{im} - [T_3]_{im}(L), \tag{82}$$

and note that when $M \in \Gamma_4^+$, $[T_3]_{im}(L,L,L) \geq 0$. Thus

$$\begin{aligned} II_a &= \int_M \sigma_3(L)|\nabla v|^2 - [T_3]_{im}(L,L,L) v_i v_m d\mu_M \\ &\leq \int_M \sigma_3(L) d\mu_M. \end{aligned} \tag{83}$$

Also,

$$\begin{aligned} II_b &:= \int_M \frac{-1}{2!} v_i \delta^{i,i_1,i_2}_{j,j_1,j_2} v_{i_1j_1} L_{i_2j_2,j} d\mu_M \\ &= \int_M \frac{-1}{4} v_i \delta^{i,i_1,i_2}_{j,j_1,j_2} v_{i_1j_1} \left(L_{i_2j_2,j} - L_{i_2j,j_2}\right) d\mu_M = 0, \end{aligned} \tag{84}$$

by the Codazzi equation (31). In conclusion, (83) and (84) imply that

$$II = \int_M \Sigma_3(D^2v, D^2v, L) d\mu_M = II_a + II_b \leq \int_M \sigma_3(L) d\mu_M. \tag{85}$$

This completes the proof of (75). □

Lemma 5.6. *Suppose v and M satisfy the same conditions as in Proposition 5.2. Then*

$$III = \int_M \Sigma_3(D^2v, D^2v, D^2) d\mu_M \leq C(n) \int_M \sigma_3(L) dv_M. \tag{86}$$

Proof.

$$\begin{aligned} \frac{1}{3} III &:= \int_M \sigma_3(D^2 v) d\mu_M \\ &= \int_M \frac{1}{3!} v_{ij} \delta^{i,i_1,i_2}_{j,j_1,j_2} v_{i_1j_1} v_{i_2j_2} d\mu_M \\ &= \int_M \frac{-1}{3!} v_i \delta^{i,i_1,i_2}_{j,j_1,j_2} \left(v_{i_1j_1j} v_{i_2j_2} + v_{i_1j_1} v_{i_2j_2j}\right) d\mu_M. \end{aligned} \tag{87}$$

For the same reason as we present in the proof of Lemma 5.4,

$$\delta^{i,i_1,i_2}_{j,j_1,j_2} v_{i_1j_1j} v_{i_2j_2} = \delta^{i,i_1,i_2}_{j,j_1,j_2} v_{i_1j_1} v_{i_2j_2j}.$$

Thus

$$\frac{1}{3}III = \int_M \frac{-2}{3!} v_i \delta^{i,i_1,i_2}_{j,j_1,j_2} v_{i_1 j_1 j} v_{i_2 j_2} d\mu_M. \tag{88}$$

Also

$$\begin{aligned}\delta^{i,i_1,i_2}_{j,j_1,j_2} v_{i_1 j_1 j} v_{i_2 j_2} =& - \delta^{i,i_1,i_2}_{j,j_1,j_2} v_{i_1 j j_1} v_{i_2 j_2} \\ =& \frac{1}{2} \delta^{i,i_1,i_2}_{j,j_1,j_2} (v_{i_1 j_1 j} - v_{i_1 j j_1}) v_{i_2 j_2}.\end{aligned} \tag{89}$$

This together with the curvature equation (61) gives

$$\begin{aligned}\frac{1}{3}III =& \int_M \frac{-1}{3!} v_i \delta^{i,i_1,i_2}_{j,j_1,j_2} (v_{i_1 j_1 j} - v_{i_1 j j_1}) v_{i_2 j_2} d\mu_M \\ =& \int_M \frac{1}{3!} v_i \delta^{i,i_1,i_2}_{j,j_1,j_2} R_{m i_1 j j_1} v_m v_{i_2 j_2} d\mu_M.\end{aligned} \tag{90}$$

By the Gauss equation (30),

$$\begin{aligned}\frac{1}{3}III =& \int_M \frac{1}{3!} v_i \delta^{i,i_1,i_2}_{j,j_1,j_2} (L_{mj} L_{i_1 j_1} - L_{m j_1} L_{i_1 j}) v_m v_{i_2 j_2} d\mu_M \\ =& \int_M \frac{2}{3!} v_i v_m \delta^{i,i_1,i_2}_{j,j_1,j_2} L_{mj} L_{i_1 j_1} v_{i_2 j_2} d\mu_M.\end{aligned} \tag{91}$$

We now recall by the definition of Newton transformation tensor (54)

$$[T_2]_{ij}(D^2 v, L) = \frac{1}{2!} \delta^{i,i_1,i_2}_{j,j_1,j_2} L_{i_1 j_1} v_{i_2 j_2}. \tag{92}$$

Thus

$$\begin{aligned}\frac{1}{3}III =& \int_M \frac{2}{3!} v_i v_m \delta^{i,i_1,i_2}_{j,j_1,j_2} L_{mj} L_{i_1 j_1} v_{i_2 j_2} d\mu_M \\ =& \int_M \frac{2}{3!} v_i v_m \delta^{i,i_1,i_2}_{j,j_1,j_2} L_{mj} L_{i_1 j_1} v_{i_2 j_2} d\mu_M \\ =& \int_M \frac{4}{3!} v_i v_m [T_2]_{ij}(D^2 v, L) L_{mj} d\mu_M.\end{aligned} \tag{93}$$

Using the inductive formula (58)

$$[T_2]_{ij}(D^2 v, L) L_{mj} = \frac{1}{2}\Sigma_3(D^2 v, L, L)\delta_{im} - \frac{3}{2}[T_3]_{im}(D^2 v, L, L) - \frac{1}{2}[T_2]_{ij}(L, L) v_{mj}, \tag{94}$$

in (93), we get

$$\begin{aligned}\frac{1}{3}III =& \int_M \frac{4}{3!} v_i v_m \left(\frac{1}{2}\Sigma_3(D^2 v, L, L)\delta_{im} - \frac{3}{2}[T_3]_{im}(D^2 v, L, L) - \frac{1}{2}[T_2]_{ij}(L, L) v_{mj} \right) d\mu_M \\ :=& III_a + III_b + III_c.\end{aligned} \tag{95}$$

To estimate III_a, we will use properties that $|\nabla v|$, $|b(x)| \leq 1$. We will also use the fact that $\Sigma_3(\bar{D}^2 \bar{V}, L, L) \geq 0$ since $\bar{D}^2 \bar{V} \geq 0$ and $L \in \Gamma_3^+$. Therefore if we replace

D^2v by $\bar{D}^2\bar{V}+b(x)L$ in III_a, then

$$\begin{aligned}
III_a :=& \int_M \frac{2}{3!}|\nabla v|^2\Sigma_3(D^2v,L,L)d\mu_M\\
=& \int_M \frac{2}{3!}|\nabla v|^2\Sigma_3(\bar{D}^2\bar{V}+b(x)L,L,L)d\mu_M\\
\leq& \frac{2}{3!}\int_M (\Sigma_3(\bar{D}^2\bar{V},L,L)+\Sigma_3(L,L,L))d\mu_M\\
=& \frac{2}{3!}\int_M (\Sigma_3(D^2v-b(x)L,L,L)+\Sigma_3(L,L,L))d\mu_M\\
\leq& \frac{2}{3!}\int_M (\Sigma_3(D^2v,L,L)+2\Sigma_3(L,L,L))d\mu_M\\
=& \frac{2}{3!}\int_M (\Sigma_3(D^2v,L,L)+6\sigma_3(L))d\mu_M.
\end{aligned} \tag{96}$$

By Lemma 5.4,

$$\int_M \Sigma_3(D^2v,L,L)d\mu_M = 0.$$

So

$$III_a \leq 2\int_M \sigma_3(L)d\mu_M. \tag{97}$$

This finishes the estimate of III_a.

To analyze the term III_b, we use $D^2v=\bar{D}^2\bar{V}+b(x)L$ to get

$$\begin{aligned}
III_b :=& \int_M -v_iv_m[T_3]_{im}(D^2v,L,L)d\mu_M\\
=& \int_M (-v_iv_m[T_3]_{im}(\bar{D}^2\bar{V},L,L)-v_iv_m[T_3]_{im}(L,L,L)b(x))d\mu_M.
\end{aligned} \tag{98}$$

Again $\bar{D}^2\bar{V}$ is positive definite and $L\in\Gamma_4^+$ (again, this is the only place we have used the property that $L\in\Gamma_4^+$ instead of being in Γ_3^+ in the proof), thus $[T_3]_{im}(\bar{D}^2\bar{V},L,L)\geq 0$ and $[T_3]_{im}(L,L,L)\geq 0$. Also, $|\nabla v|\leq 1$. Therefore

$$\begin{aligned}
III_b \leq& \int_M Tr([T_3]_{ij})(L,L,L)d\mu_M\\
=& \int_M (n-3)\sigma_3(L)d\mu_M.
\end{aligned} \tag{99}$$

For the last term III_c,

$$III_c := -\frac{1}{3}\int_M v_iv_m[T_2]_{ij}(L,L)(\bar{D}^2_{mj}\bar{V}+b(x)L_{mj})d\mu_M. \tag{100}$$

Notice that $v_iv_m\bar{D}^2_{mj}\bar{V}\geq 0$. Thus $[T_2]_{ij}(L,L)\bar{D}^2_{mj}\bar{V}v_iv_m\geq 0$. This together with the formula (58)

$$[T_2]_{ij}(L,L)L_{mj} = \sigma_3(L)\delta_{im}-[T_3]_{im}(L) \tag{101}$$

implies that

$$\begin{aligned}
III_c \leq & -\frac{1}{3}\int_M v_i v_m [T_2]_{ij}(L,L) L_{mj} b(x) d\mu_M \\
\leq & -\frac{1}{3}\int_M b(x)(\sigma_3(L)\delta_{im} - [T_3]_{im}(L,L,L)) v_i v_m d\mu_M \\
= & -\frac{1}{3}\int_M (b(x)\sigma_3(L)|\nabla v|^2 - b(x)[T_3]_{im}(L,L,L) v_i v_m) d\mu_M \\
\leq & \frac{1}{3}\int_M \sigma_3(L) dv_M + \frac{1}{3}\int_M Tr([T_3]_{ij})(L) d\mu_M.
\end{aligned} \tag{102}$$

Using (56) we get

$$III_c \leq \frac{n-2}{3}\int_M \sigma_3(L) d\mu_M. \tag{103}$$

In conclusion $III = III_a + III_b + III_c \lessapprox \int_M \sigma_3(L) d\mu_M$. This finishes the proof of Lemma 5.6. □

Combining our estimates of I, II, III and IV to (69), we have established the inequality in Proposition 5.2 for the case $k = 3$.

The proof of Proposition 5.2 for general k follows a complicated multi-layer inductive process, we will not present the proof here and instead refer the interested readers to the proof of the result in the paper ([9]).

5.4. Regularity of the function V

As Proposition 5.2, hence our main theorem is proved under the assumption that the potential function V is C^3 (thus we can integrate by part freely in the proof of the inequalities $(**)_k$), now we still need to justify this assumption. The key observation is that the constant $C(n,k)$ in the inequalities $(**)_k$ which we have established is an a priori constant depending only on n, k and a and independent of the estimate of the smoothness of V (beyond the estimate that $|\nabla V| \leq 1$ and the fact that V is a convex function), thus we can apply an approximation process to construct a sequence of C^3 functions V_ϵ which satisfy the conditions in Caffarelli's regularity results Theorem 4.2 and Theorem 4.3; and thus we can apply the method of the proof above to establish inequalities $(**)_k$ to a sequence of corresponding approximate domains and functions, we then let ϵ tend to zero to establish our result. In the following we will describe this process in more details.

If the density $F(x)$ is bounded away from zero and infinity, and also if Ω is a strictly convex domain, then for each n-linear space E, $D_E = p_E(\Omega)$ is convex, and by Caffarelli's result, $V = V_E$ is a smooth convex potential. In general (when Ω is not necessarily convex), we now describe an approximation process to obtain a sequence of smooth convex potentials V_ϵ.

For each fixed n-plane E, we first observe that there exists a constant $R > 0$, such that $D_E = p_E(\Omega)$ is contained in $B_E(0,R)$, the ball centered at the origin with radius R in E. For each $\epsilon > 0$, define the subset $D_\epsilon := \{x \in D_E | \epsilon \leq F(x) \leq 1/\epsilon\}$.

Since F is integrable on D_E and $F \geq 0$, we know $D_\epsilon \to Spt(F)$, as $\epsilon \to 0$. One can extend $F|_{D_\epsilon}$ to $F_\epsilon : B_E(0,R) \to \mathbb{R}$, such that $\frac{\epsilon}{2} \leq F_\epsilon(y) \leq \frac{2}{\epsilon}$ on $B_E(0,R)$, and

$$\int_{B_E(0,R)\setminus D_\epsilon} F_\epsilon(y)dy \leq 2\epsilon \cdot \omega_n R^n.$$

Such an extension exists because $\epsilon \leq F|_{D_\epsilon} \leq \frac{1}{\epsilon}$, and $Vol(B_E(0,R) \setminus D_\epsilon) \leq Vol(B_E(0,R)) \leq \omega_n R^n$. Therefore

$$m_\epsilon := \int_{B_E(0,R)} F_\epsilon(x)dx = \int_{B_E(0,R)\setminus D_\epsilon} F_\epsilon(x)dx + \int_{D_\epsilon} F_\epsilon(x)dx \leq c_0\epsilon + 1, \tag{104}$$

where $c_0 = 2\omega_n R^n$. Also

$$m_\epsilon \geq \int_{D_\epsilon} F_\epsilon(x)dx \to 1, \tag{105}$$

as $\epsilon \to 0$. Hence $m_\epsilon \to 1$, as $\epsilon \to 0$. Now for each sufficiently small ϵ, $m_\epsilon > 0$. Thus $\frac{F_\epsilon(x)}{m_\epsilon}dx$ is a probability measure on $B_E(0,R)$, such that $0 < \frac{\epsilon}{4} < \frac{F_\epsilon(x)}{m_\epsilon} \leq \frac{4}{\epsilon}$ on $B_E(0,R)$. As before, Brenier's theorem implies that there exists a convex potential V_ϵ such that ∇V_ϵ is the solution of Monge problem between $(B_E(0,R), \dfrac{F_\epsilon(x)}{m_\epsilon}dx)$ and $(B_E(0,1), \dfrac{\chi_{B_E(0,1)}}{\omega_n}dx)$. By Theorem 4.3, V_ϵ is a smooth convex potential. Obviously $|\nabla V_\epsilon(x)| \leq 1$ for $x \in B_E(0,R)$. Also V_ϵ satisfies the optimal tranport equation $\omega_n \dfrac{F_\epsilon(x)}{m_\epsilon} = det(D^2 V_\epsilon(x))$ in the classical sense. We now apply the procedure we have described in the proof above to obtain the sequences of functions v_ϵ and to apply Proposition 5.2 to establish the inequality for each ϵ,

$$\int_M \sigma_k(D^2 v_\epsilon + aL)d\mu_M \leq C \int_M \sigma_k(L)d\mu_M$$

with some constant C depending only on k and n and independent of ϵ.

Since $m_\epsilon \to 1$ and $M \cap p^{-1}(D_\epsilon) \to M \cap p^{-1}(Spt(F))$ as $\epsilon \to 0$. Also by (33), $M \cap Spt(f) \subset M \cap p^{-1}(Spt(F))$, we can let ϵ tend to zero and integrate over all E in the Grassmannian $G_{n,n+1}$ to establish the inequality $(**)_k$ and to finish the proof of the theorem.

6. Concluding Remarks

It is obvious that the result reported in these lecture notes is a work in progress. It left open many questions in this research area. Chiefly among them are the questions if the assumption of $k+1$-convexity of the domain is a necessary condition for the inequality $(**)_k$ to hold or if k-convexity is enough? Also is the best constant C in the inequality $(**)_k$ the same as the sharp constant $\bar{C}$ as in $(*)_k$ and only obtained when the domain is a ball? All these problems are open as of this date. The authors would also like to point out that it would be interesting to study the Minkowski's mixed volume on Riemannian manifolds and to study the class of generalized isoperimetric inequalities of the type like $(**)_k$ on domains of such manifolds. In general, the classical work of Minkowski, Aleksandrov-Fenchel etc. is a rich area of research. One feels that the connection of their work to curvatures, non-linear analysis and other concepts in modern geometry is waiting to be further explored.

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