ON A CONFORMAL GAP AND FINITENESS THEOREM FOR A CLASS OF FOUR MANIFOLDS

SUN-YUNG A. CHANG[1], JIE QING[2] AND PAUL YANG[3]

ABSTRACT. In this paper we develop a bubble tree structure for a degenerating class of Riemannian metrics satisfying some global conformal bounds on compact manifolds of dimension 4. Applying the bubble tree structure, we establish a gap theorem, a finiteness theorem for diffeomorphism type for this class, and make a comparison of the solutions of the σ_k equations on a degenerating family of Bach flat metrics.

1. Introduction

Given a Riemannian four-manifold (M^4, g), let Rm denote the curvature tensor, W the Weyl curvature tensor, Ric the Ricci tensor, and R the scalar curvature. The usual decomposition of Rm under the action of $O(4)$ can be written

$$Rm = W + \tfrac{1}{2} E \bigcirc\!\!\!\!\wedge\, g + \tfrac{1}{24} Rg \bigcirc\!\!\!\!\wedge\, g, \tag{1.1}$$

where $E = Ric - \frac{1}{4}Rg$ is the trace-free Ricci tensor and $\bigcirc\!\!\!\!\wedge$ denotes the Kulkarni-Nomizu product. We also recall the Gauss-Bonnet-Chern formula:

$$8\pi^2\chi(M^4) = \int_{M^4} \big(\tfrac{1}{4}|W|^2 + \tfrac{1}{24}R^2 - \tfrac{1}{2}|E|^2\big) dvol. \tag{1.2}$$

In [CGY-2], based on an earlier work of [Ma], we have the following sharp form of "sphere" theorem:

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[1]Princeton University, Department of Mathematics, Princeton, NJ 08544-1000, supported by NSF Grant DMS–0245266.

[2]University of California, Department of Mathematics, Santa Cruz, CA 95064, supported in part by NSF Grant DMS–0103160.

[3]Princeton University, Department of Mathematics, Princeton, NJ 08544-1000, supported by NSF Grant DMS–0245266.

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Theorem 1.1. *Let (M^4, g) be a smooth, closed manifold for which*

(i) *the Yamabe constant $Y(M^4, [g]) > 0$, and*

(ii) *the curvature satisfies*

$$\int_{M^4} \big(\tfrac{1}{24}R^2 - \tfrac{1}{2}|E|^2\big) dvol > \int_{M^4} \tfrac{1}{4}|W|^2 dvol \geq 0. \tag{1.3}$$

Then M^4 is diffeomorphic to either S^4 or $\mathbb{RP}^4$. Furthermore, if M is not diffeomorphic to (S^4, g_c) or $(\mathbb{RP}^4, g_c)$ and the inequality in (1.3) becomes an inequality; then either
(1) (M, g) is conformally equivalent to $(\mathbb{CP}^2, g_{FS})$, or
(2) (M, g) is conformal equivalent to $((S^3 \times S^1)/\Gamma, g_{prod})$ for a finite group Γ.

Remarks.

1. Recall that the Yamabe constant is defined by

$$Y(M^4, [g]) = \inf_{\tilde{g} \in [g]} vol(\tilde{g})^{-\frac{1}{2}} \int_{M^4} R_{\tilde{g}} dvol_{\tilde{g}},$$

   where $[g]$ denotes the conformal class of g. Positivity of the Yamabe invariant implies that g is conformal to a metric of strictly positive scalar curvature.
2. In the statement of Theorem 1.1 the norm of the Weyl tensor is given by $|W|^2 = W_{ijkl}W^{ijkl}$;

The proof of the second part of the Theorem above relies on the vanishing of the Bach tensor. If g satisfies the condition of Theorem 1.1 and (M^4, g) is not diffeomorphic to either S^4 or $\mathbb{RP}^4$, then g is a critical point (actually, a local minimum) of the *Weyl functional* $g \mapsto \int |W|^2 dvol$. The gradient of this functional is called the *Bach tensor*; which we shall define in Section 2 below, and we will say that critical metrics are *Bach flat*. Note that the conformal invariance of the Weyl functional in dimension four implies that Bach-flatness is a conformally invariant property.

We now define some notations to state results in this paper. Given a Riemannian four–manifold (M^4, g), the *Weyl–Schouten* tensor is defined by

$$A = Ric - \frac{1}{6}Rg$$

In terms of the Weyl–Schouten tensor, the decomposition (1.1) can be written as

$$\mathrm{Rm} = W + \frac{1}{2} A \owedge g\,. \tag{1.4}$$

This splitting of the curvature tensor induces a splitting of the Euler form. To describe this, we introduce the elementary symmetric polynomials

$$\sigma_\kappa(\lambda_1, \dots, \lambda_n) = \sum_{i_1 < \cdots < i_k} \lambda_{i_1} \cdots \lambda_{i_\kappa} ;$$

where $\lambda_i{}'s$ denote the eigenvalues of the (contracted) tensor $g^{-1}A$. To simplify notations, we denote $\sigma_\kappa(A_g) = \sigma_\kappa(g^{-1}A)$. We note for a manifold of dimension 4,

$$\sigma_2(A_g) = \tfrac{1}{24}R^2 - \tfrac{1}{2}|E|^2. \tag{1.5}$$

Sometimes we will also denote $\sigma_2(A_g)$ as σ_2 when the metric g is fixed.

The Gauss–Bonnet–Chern formula (1.2) may be written as

$$8\pi^2\chi(M^4) = \int_{M^4} \frac{1}{4}|W|^2 dvol + \int_{M^4} \sigma_2 dvol \, . \tag{1.6}$$

Note that the conformal invariance of the Weyl functional implies that the quantity

$$\int_{M^4} \sigma_2 dvol$$

is conformally invariant as well.

For a four manifold with a positive Yamabe constant, it follows from the solution of the Yamabe problem ([Au], [S]) that we may assume that g is the Yamabe metric which attains the Yamabe constant, then R_g is a constant and

$$\begin{aligned} \int_M \sigma_2(A_g) dv_g \le \int_M \frac{1}{24} R_g^2 dv_g \; &= \; \frac{1}{24} \frac{(\int_M R_g dv_g)^2}{vol(g)} \\ \le \frac{1}{24} \frac{(\int_M R_c dv_{g_c})^2}{(vol(g_c))} \; &= \; 16\pi^2; \end{aligned} \tag{1.7}$$

and equality holds if and only if (M^4, g) is conformally equivalent to the standard 4-sphere (S^4, g_c) with $R_c = R_{g_c} = 12$ and $vol(g_c) = \frac{8\pi^2}{3}$.

In view of the inequality (1.7), it is natural to ask whether a stronger form of rigidity holds in the statement of Theorem 1.1 in which the integral of σ_2 is compared directly to the constant $16\pi^2$ instead to the L^2 integral of the Weyl tensor. Such a possibility is suggested by the recent remarkable rigidity result of Bray-Neves [BN] on compact manifolds of dimension 3. To state their result, recall the Yamabe invariant is defined as $Y(M) = sup_g Y(M, [g])$, and denote the Yamabe constants $Y_1 = Y(S^3, [g_c])$, $Y_2 = Y(RP^3, [g_c])$.

Theorem 1.2 ([BN]). *A closed 3-manifold with Yamabe invariant $Y > Y_2$ is either S^3 or a connect sum with a S^2 bundle over S^1.*

The gap theorem in this paper is a first step in this direction:

Theorem A. *Suppose that (M^4, g) is a Bach flat closed 4-manifold with positive Yamabe constant and that*

$$\int_M |W|_g^2 dv_g \leq \Lambda_0. \tag{1.8}$$

for some fixed positive number Λ_0. Then there is a positive constant ϵ_0 such that if

$$\int_M \sigma_2(A_g) dv_g \geq (1-\epsilon)16\pi^2 \tag{1.9}$$

holds for some constant $\epsilon < \epsilon_0$, then (M^4, g) is conformally equivalent to the standard 4-sphere.

The analysis that is developed to prove the gap theorem above can be adopted to prove a result of finiteness of diffeomorphism classes of manifolds satisfying some conformally invariant conditions and L^2 bounds:

Theorem B. *Suppose that $\mathcal{A}$ is a collection of Bach flat Riemannian manifolds (M^4, g) with positive Yamabe constants, satisfying inequality (1.8) for some fixed positive number Λ_0, and that*

$$\int_M (\sigma_2(A_g) dv_g \geq a_0, \tag{1.10}$$

for some fixed positive number a_0. Then there are only finitely many diffeomorphism types among manifolds in $\mathcal{A}$.

It is known that in each conformal class of metrics belonging to the family $\mathcal{A}$, there is a metric $\bar{g} = e^{2w} g$ such that $\sigma_2(A_{\bar{g}}) = 1$, which we shall call the σ_2 metric. The recent work of Gursky and Viaclovsky [GV] also showed that if in addition, the positive cone Γ_k^+ is nonempty for $k = 3$ or 4, then there exists a conformal metric with $\sigma_k(A_{\bar{g}}) = 1$. We shall call these metrics σ_k metrics. As an application of the bubbling tree analysis, we will show:

Theorem C. *For the conformal classes $[g_0] \in \mathcal{A}$ the conformal metrics $g = e^{2w} g_0$ satisfying the equation $\sigma_2(g) = 1$ has a uniform bound for the diameter.*

In cases where the $\sigma_k = 1$ metrics exists for $k > 2$, the Ricci tensor has a positive apriori lower bound, hence the diameter bound follows immediately.

Remarks:

1. Our proof of the theorems above builds upon some estimates in the recent work of Tian-Viaclovsky ([TV-1], [TV-2]; see also [An-2]) on the compactness of Bach-flat metrics on 4-manifolds; but our proof relies on a finer analysis of the concentration phenomenon near points of curvature concentrations. To do this, we build a bubble tree consisting of vertices which are bubbles near points of concentrations, and edges consisting of neck regions connecting different vertexes (see section 4 for a more precise definition of the bubble trees.) The idea of using bubble tree construction to achieve finite diffeomorphism types for classes of manifolds under suitable curvature conditions was developed in the work of Anderson-Cheeger [AC]. Our construction is modeled after this work but differs in the way that our bubble tree is built from the bubbles at points with the smallest scale of concentration, that is around those points p with the smallest radius r so that the geodesic ball $B_r(p)$ centered around p with radius r achieved some fixed energy $\int_{B_r(p)} |Rm|^2 dvol$, to bubbles with larger scale; while the tree in [AC] is constructed from bubbles of large scale to bubbles with smaller scales. The inductive method of construction of our bubble tree is modeled on earlier work of [BC], [Q] and [St] on the study of concentrations of energies in harmonic maps and the scalar curvature equations.

2. Our proof does not use the stronger volume estimates, that is, the uniform volume growth for any geodesic ball in a Bach flat 4-manifold with positive Yamabe constant, L^2 bounds of curvature, and bounded first Betti number developed in [TV-2] and [An-2]. Instead we obtain as a corollary (see Corollary 4.6 in section 4 below) of our bubble tree construction some uniform estimates for the *intrinsic* diameter of the geodesic spheres near points of curvature concentrations for this class of manifolds. Indeed one can derive the uniform volume growth as a consequence of the uniform bound of the diameters.

3. Since the neck theorem and the bubble tree construction which we have derived in this paper are not "uniform" in scale, we cannot derive our version of the result of finite diffeomorphism type (Theorem B) by directly applying the arguments in [AC]. Instead we have established the proof of Theorem B by an argument of contradiction.

4. Since smooth points of the limit space may possibly be points of curvature concentration, the proof of Lemma 2.16 in [AC] is not valid in our setting.

This paper is organized as follows. In section 2, we discuss some preliminaries and recall some results in [CGY-2] and [G], and some key estimates in [TV-1]. In section 3, we present a neck theorem, which is a variant form of the neck theorem in [AC]. In section 4 we describe our bubble tree construction; which is the major part of the paper. In section 5 we apply the bubble tree construction to prove our main results Theorem A and Theorem B. In section 6, we develop the analysis of the σ_2 equation on an ALE bubble.

In a subsequent paper, we will apply the result in Theroem C of this paper to further study some classification problem of metrics in $\mathcal{A}$.

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2. Preliminaries

In this section we recall some known facts and quote some recent work in [TV-1] which forms the basis for our construction of the bubble tree in section 4.

Recall our setting is a compact, closed four manifold (M^4, g) with positive Yamabe constant $Y(M, [g])$. One basic fact ([Au], [S]) is that on such a manifold, $Y(M, [g])$ is achieved by a metric, called the Yamabe metric and which we denote again by g, with constant scalar curvature R_g. We also have the following Sobolev inequality:

Lemma 2.1. *Suppose that (M^4, g) is a closed, compact 4-manifold with positive Yamabe constant and suppose that g is a Yamabe metric. Then*

$$Y(M, [g])(\int_M u^4 dv_g)^{\frac{1}{2}} \leq 6 \int_M |\nabla_g u|^2 dv_g + \int_M R_g u^2 dv_g, \tag{2.1}$$

for all function $u \in W^{1,2}(M)$.

Notice that the inequality (2.1) is invariant under the scaling of metrics $g \to cg$. Thus we may consider 4-manifolds which are complete, non-compact and have infinite volume and are limits of rescaled manifolds which satisfy conditions in Lemma 2.1 with a common lower bound on the Yamabe constants. On such a manifold, the Sobolev inequality (2.1) takes the following form:

$$(\int_M u^4 dv_g)^{\frac{1}{2}} \leq C_s \int_M |\nabla_g u|^2 dv_g \tag{2.2}$$

for any $u \in C^1(M)$ with compact support inside M.

As a by-product of the Sobolev inequality (2.2), one can derive a lower bound on the volume of all geodesic balls defined on M. (cf. Lemma 3.2 in [He].)

Lemma 2.2. *Suppose that either the Sobolev inequality (2.1) or (2.2) hold on (M^4, g), then there exists a positive number v_0 depending only on C_s such that*

$$vol(B_r(p)) \geq v_0 r^4 \tag{2.3}$$

for all $p \in M$ and $r < dist(p, \partial M)$, where $B_r(p)$ is the geodesic ball of radius r centered at p.

Recall (cf [De]) in local coordinate, the Bach tensor is defined as

$$B_{ij} \;=\; \nabla^k \nabla^\ell W_{kij\ell} \;+\; \frac{1}{2}\, R^{k\ell}\, W_{kij\ell}. \tag{2.4}$$

Using the Bianchi identities, this can be rewritten as

$$\begin{aligned} B_{ij} \;=&\; -\,\frac{1}{2}\Delta E_{ij} \;+\frac{1}{6}\,\nabla_i\nabla_j R - \frac{1}{24}\,\Delta R g_{ij} - E^{k\ell} W_{ikj\ell} \\ &+ E_i^k E_{jk} \;-\frac{1}{4}\,|E|^2 g_{ij} \;+\; \frac{1}{6}\, R E_{ij}, \end{aligned} \tag{2.5}$$

where $\Delta E_{ij} = g^{kl}\nabla_k\nabla_l E_{ij}$.

A Bach flat metric is a metric in which the Bach tensor vanishes–this happens for example when the metric a critical metric for the functional which is the L^2 norm of the Weyl tensor. The following are some basic identities for Bach flat metrics on manifolds of dimension 4.

Lemma 2.3. *(See [De], and Proposition 3.3 in [CGY-2]). If (M^4, g) is Bach flat, then*

$$0 = \int_{M^4} \left\{ 3 \left(|\nabla E|^2 - \frac{1}{12}|\nabla R|^2 \right) + 6 trE^3 + R|E|^2 - 6 W_{ijk\ell} E_{ik} E_{j\ell} \right\} dvol\,, \tag{2.6}$$

where $trE^3 = E_{ij}E_{ik}E_{jk}$, and

$$\int_{M^4} |\nabla W|^2 dvol = \int \left\{ 72 \det W^+ + 72 \det W^- - \frac{1}{2} R|W|^2 + 2 W_{ij\kappa\ell} E_{i\kappa} E_{j\ell} \right\} dvol, \tag{2.7}$$

where W^+ and W^- are respectively the self-dual and the anti-self dual part of the Weyl tensor.

We remark that the second identity is a consequence of Stokes' theorem, the Bianchi identities, and the definition of the Bach tensor in (2.4). In this paper we do not need the precise form of the identity (2.7) but only the fact that the terms in $\det W^+$ and $\det W^-$ are of the form $W * W * W$ –a contraction of three Weyl tensor terms.

In the recent work of Tian-Viaclovsky–which is obtained from an iteration process by applying equation (2.5) and the the Sobolev inequality (2.2)- they have derived the following "ϵ -regularity" theorem.

Theorem 2.4. *([TV-1], Theorem 3.1) Suppose that (M^4, g) is a Bach flat 4-manifold with the Yamabe constant $Y(M^4,[g]) > 0$ and let g denotes its Yamabe metric. Then there exist some positive numbers τ_k and C_k depending on $Y(M^4,[g])$ such that, for each geodesic ball $B_{2r}(p)$ centered at $p \in M$, if*

$$\int_{B_{2r}(p)} |Rm|^2 dv \leq \tau_k, \tag{2.8}$$

then

$$\sup_{B_r(p)} |\nabla^k Rm| \leq \frac{C_k}{r^{2+k}} (\int_{B_{2r}(p)} |Rm|^2 dv)^{\frac{1}{2}}. \tag{2.9}$$

Built on the estimate in Theorem 2.4, the main result in [TV-1] –which plays a crucial role in this paper – is the following volume estimates (2.11) of geodesic balls:

Theorem 2.5. *Let (X, g) be a complete, non-compact, four manifold with base point p, and let $r(x) = d(p, x)$, for $x \in X$. Assume that there exists a constant $v_0 > 0$ so that*

$$vol(B_r(q)) \geq v_0 r^4 \tag{2.3}$$

holds for all $q \in X$, assume furthermore that as $r \to \infty$,

$$sup_{S(r)} |Rm_g| = o(r^{-2}), \tag{2.10}$$

where $S(r) = \partial B_r(p)$. Assume further that the first Betti number $b_1(X) < \infty$, then (X, g) is an ALE space, and there exists a constant v_1 (depending on (X, g)) so that

$$vol(B_r(p)) \leq v_1 r^4. \tag{2.11}$$

We remark that the constant v_1 in (2.11) obtained above may depend on the given manifold (X, g).

We return to the class of metrics g with positive Yamabe constant and which satisfy the inequality (1.10) in the statement of Theorem B. First, we make some simple observations:

Lemma 2.6. *Suppose that (M^4, g) is a closed, compact 4-manifold with positive Yamabe constant and which satisfies the inequality (1.10). Suppose further that g is a Yamabe metric with $vol(g) = \frac{8\pi^2}{3}$, then*

$$\frac{3}{\pi}\sqrt{a_0} \leq R[g] \leq 12, \tag{2.12}$$

and

$$\int_M |E|^2 dv_g \leq (32\pi^2 - 2a_0). \tag{2.13}$$

We remark that if the metric g satisfies the inequality (1.9) in the statement of the gap Theorem A, then $a_0 = (1-\epsilon)16\pi^2$ and we have

$$12\sqrt{(1-\epsilon)} \leq R[g] \leq 12. \tag{2.12'}$$

and

$$\int_M |E|^2 dv_g \leq 32\pi^2\epsilon. \tag{2.13'}$$

A deeper result for this class of metrics is following result of Gursky:

Theorem 2.7. *([G]) Suppose that (M^4, g) is a closed 4-manifold with positive Yamabe constant and $\int_M \sigma_2(A_g)dv_g > 0$. Then $b_1(M^4) = 0$.*

We remark that under the same assumptions as in Theorem 2.7, in [CGY-1], we obtained the stronger result that the fundamental group of M^4 is finite.

In summary, we have:

Corollary 2.8. *Suppose that $\mathcal{A}$ is a collection of Bach flat Riemannian manifolds (M^4, g) with positive Yamabe constants, satisfying (1.8) and either (1.9) or (1.10), then there is a lower bound of the Yamabe constants and an upper bound of the L^2 norm of the curvature tensor Rm_g for the whole collection; also $b_1(M^4) = 0$. Thus one may apply results in Theorem 2.4 and and Theorem 2.5 to metrics in the collection $\mathcal{A}$.*

3. The Neck Theorem

The neck theorem which we are going to establish in this section is modeled after the result of Anderson-Cheeger (Theorem 1.18 in [AC]), in which the theorem was established under the assumption that there is a point-wise Ricci curvature bound on the manifold.

Suppose (M^4, g) is a Riemannian manifold. For a point $p \in M$, we denote $B_r(p)$ the geodesic ball with radius r centered at p, $S_r(p)$ the geodesic sphere of radius r centered at p. We consider the geodesic annulus centered at p as:

$$\bar{A}_{r_1,r_2}(p) = \{q \in M : r_1 \leq \mathrm{dist}(q,p) \leq r_2\}. \tag{3.1}$$

In general, $\bar{A}_{r_1,r_2}(p)$ may have more than one connected components. We will denote by

$$A_{r_1,r_2}(p) \subset \bar{A}_{r_1,r_2}(p)$$

any component of $\bar{A}_{r_1,r_2}(p)$ which meets $S_{r_2}(p)$:

$$A_{r_1,r_2}(p) \bigcap S_{r_2}(p) \neq \emptyset. \tag{3.2}$$

Since the distance function is Lipschitz, we may consider the 3 dimensional Hausdorff measure for the geodesic sphere $S_r(p)$ and denote it by $\mathcal{H}^3(S_r(p))$.

Theorem 3.1. *Suppose that (M, g) is a Bach flat Riemannian 4-manifold with a Yamabe metric with a positive Yamabe constant, and suppose the first Betti number $b_1(M) = 0$. Assume that $p \in M$, and $\alpha \in (0, 1)$, $\epsilon > 0$, $v_2 > 0$ and $a < dist(p, \partial M)$ are given constants. Then there exist positive numbers δ_0, c_2, n depending on ϵ, α, C_s, v_2 and a such that the following statements hold. Let $A_{r_1,r_2}(p)$ be a connected component of the geodesic annulus which satisfies the condition (3.2), with*

$$r_2 \leq c_2 a, \qquad r_1 \leq \delta_0 r_2, \tag{3.3}$$

$$\mathcal{H}^3(S_r(p)) \leq v_2 r^3, \quad \forall r \in [r_1, 100r_1], \tag{3.4}$$

and

$$\int_{A_{r_1,r_2}(p)} |Rm|^2 dv \leq \delta_0. \tag{3.5}$$

Then $A_{r_1,r_2}(p)$ is the only component which satisfies (3.2). In addition for this unique component

$$A_{(\delta_0^{-\frac{1}{4}}-\epsilon)r_1,(\delta_0^{\frac{1}{4}}+\epsilon)r_2}(p),$$

which intersects with $S_{(\delta_0^{\frac{1}{4}}+\epsilon)r_2}(p)$, there exist some finite group $\Gamma \subset O(4)$, acting freely on S^3, with $|\Gamma| \leq n$, and a quasi isometry Ψ, so that

$$A_{(\delta_0^{-\frac{1}{4}}+\epsilon)r_1,(\delta_0^{\frac{1}{4}}-\epsilon)r_2}(p) \subset \Psi(C_{\delta_0^{-\frac{1}{4}}r_1,\delta_0^{\frac{1}{4}}r_2}(S^3/\Gamma)) \subset A_{(\delta_0^{-\frac{1}{4}}-\epsilon)r_1,(\delta_0^{\frac{1}{4}}+\epsilon)r_2}(p)$$

such that for all $C_{\frac{1}{2}r,r}(S^3/\Gamma) \subset C_{\delta_0^{-\frac{1}{4}}r_1,\delta_0^{\frac{1}{4}}r_2}(S^3/\Gamma)$, in a local coordinate, one has

$$|(\Psi^*(r^{-2}g))_{ij} - \delta_{ij}|_{C^{1,\alpha}} \leq \epsilon. \tag{3.6}$$

We now begin the proof of Theorem 3.1; the first part of our arguments are in large part modifications of the arguments used in the proof of Theorem 4.1 in [TV-1] (see Theorem 2.5 for the statement of the theorem); the second part some modification of the proof of Theorem 1.18 in [AC]. We will be brief in presenting our proof here. We begin with a lemma to prove the uniqueness of the connected component satisfying condition (3.2).

Lemma 3.2. *Under the same assumptions of the theorem, there is only one connected component in the annulus $\bar{A}_{r_1,r_2}(p)$ which intersects with $S_{r_2}(p)$.*

Proof. Assume otherwise, that is, assume there is another component $B_{r_1,r_2}(p)$ in $\bar{A}_{r_1,r_2}(p)$ which also intersects with $S_{r_2}(p)$. It then follows from the assumption $b_1(M) = 0$ that these two components can not be joined by a curve from their intersections with the components of $S_{r_2}(p)$. Thus the region $A_{r_1,2r_1}(p) \subset A_{r_1,r_2}(p)$ separates $M - A_{r_1,2r_1}(p)$ into two disjoint components, so that the volume of each components at least Cr_2^4 for some constant C. We now observe that under the assumption that the Yamabe constant $Y = Y(M,[g])$ be positive, the Sobolev inequality (2.1) holds by Lemma 2.1. If we define a function u to be equal to 1 on the component with a smaller volume and 0 on the other component and set it to be a smooth function which varies from 0 to 1 with its first derivative bounded by $\frac{2}{r_1}$ on $A_{r_1,2r_2}(p)$, then apply the Sobolev inequality (2.1) and the assumption (3.4) we have

$$\begin{aligned} Y(Cr_2^4)^{\frac{1}{2}} &\leq Y(\int_M u^4 dv)^{\frac{1}{2}} \leq 6\int_M |\nabla u|^2 dv + \frac{Y}{\sqrt{\mathrm{vol}(M)}}\int_M u^2 dv \\ &\leq \frac{24}{r_1^2} v_2(16r_1^4 - r_1^4) + \frac{Y}{\sqrt{\mathrm{vol}(M)}} v_2 r_1^4 + \frac{Y}{\sqrt{2}}(\int_M u^4 dv)^{\frac{1}{2}}. \end{aligned} \tag{3.7}$$

This implies

$$r_2^4 \leq C' r_1^4, \tag{3.8}$$

for some suitable constant C' and contradicts with the assumption (3.3) of the Lemma.

Proof of Theorem 3.1.

Part I : First we apply Theorem 2.4 in section 2 to obtain

$$\sup_{S_r(p)\bigcap A_{r_1,r_2}(p)} |\nabla^k Rm| \leq \frac{C_k \epsilon(r)}{r^{2+k}} \delta_0^{\frac{1}{2}}, \tag{3.9}$$

for each $r \in [2r_1, \frac{1}{2}r_2]$, provided that $\delta_0 \leq \tau_k$ for $k = 0,1$, where $\epsilon(r) \leq 1$ and goes to 0 as $r/r_1 \to \infty$.

We then observe that by arguments similar to that in the proof of Lemma 3.2 above, there is a unique connected component which intersects with $S_{\frac{1}{2}r_2}(p)$, which we denote by $A_{2r_1,\frac{1}{2}r_2}(p)$. This component $A_{2r_1,\frac{1}{2}r_2}(p)$ has a tree structure in the sense that while the intersections of $S_s(p)$ with the connected components $A_{r,s}(p)$

may have more than one connected components; the intersections of $S_r(p)$ with the connected components $A_{r,s}(p)$ are always connected for all $2r_1 < r < s < \frac{1}{2}r_2$. Following the argument in section 4 in [TV-1], we choose a constant s with $2 \leq s \leq 10$, and denote by N the integer determined by the condition $r_2 \in [2s^N r_1, 2s^{N+1} r_1)$. Let $\{A_{2s^j r_1, 2s^{j+1} r_1}(p)\}_{j=0}^N$ denote the set of annuli such that

$$S_{2s^{j+1}r_1}(p) \bigcap A_{2s^{j+1}r_1, 2s^{j+2}r_1}(p) \subset S_{2s^{j+1}r_1}(p) \bigcap A_{2s^j r_1, 2s^{j+1} r_1}(p).$$

and call

$$D(s) = \bigcup_{j=0}^{N} A_{2s^j r_1, 2s^{j+1} r_1}(p) \tag{3.10}$$

a direction in the tree $A_{2r_1, \frac{1}{2}r_2}(p)$. We claim that there exists some constants $\delta_1 > 0$ and $v_3 > 0$ such that

$$\mathcal{H}^3(S_r(p) \bigcap D(s)) \leq v_3 r^3, \quad for \quad all \quad r \in [2r_1, \frac{1}{4}r_2], \tag{3.11}$$

provided that $\delta_0 \leq \delta_1$. Note that as a consequence of (3.11), we have

$$\begin{aligned} \mathcal{H}^3(S_r(p) \bigcap A_{2s^j r_1, 2s^{j+1} r_1}(p)) &\leq C_4 r^3 \\ \mathrm{vol}(A_{2s^j r_1, 2s^{j+1} r_1}(p)) &\leq C_4 s^{4j} r_1^4 \end{aligned} \tag{3.12}$$

for some constant C_4.

The proof to establish the claim (3.11) in [TV-1] is rather complicated. One of the key step there is to show that given a direction, there exists some maximal subsequence of annuli $A_{2s^j r_1, 2s^{j+1} r_1}(p)$, such that

(3.13)

$$\mathcal{H}^3(S_{2s^{j+1}r_1}(p) \bigcap A_{2s^{j+1}r_1, 2s^{j+2}r_1}(p)) \geq (1-\eta_j)\mathcal{H}^3(S_{2s^j r_1}(p) \bigcap A_{2s^j r_1, 2s^{j+1} r_1}(p))$$

for some sequence $\eta_j \to 0$. One then establishes the estimates in (3.12) using a proof by contradiction for this subsequence of annuli. We observe that the same strategy of proof works in our case. That is, assuming that (3.11) does not hold, then there is a sequence of of (M_i, g_i) and directions in the annuli

$$D^i(s) \subset A^i_{2r_1^i, \frac{1}{2}r_2^i}(p_i) \subset M_i,$$

satisfying (3.3), (3.4) and (3.5) for some sequence $r_1^i, r_2^i \to 0$, $\delta^i \to 0$, and a maximal subsequence of annuli $A^i_{2s^j r_1^i, 2s^{j+1} r_1^i}(p_i)$ and numbers $\eta_{i,j} \to 0$ such that

$$\begin{aligned}\mathcal{H}^3(S^i_{2s^{j+1}r_1^i}(p_i) \bigcap A^i_{2s^{j+1}r_1^i, 2s^{j+2}r_1^i}(p_i)) \\ \geq (1-\eta_{i,j})\mathcal{H}^3(S^i_{2s^j r_1^i}(p_i) \bigcap A^i_{2s^j r_1^i, 2s^{j+1}r_1^i}(p_i))\end{aligned}$$

due to the assumption (3.4) in the statement in Theorem 3.1. So that

$$\mathrm{vol}(A^i_{2s^j r_1^i, 2s^{j+1} r_1^i}(p_i)) \to \infty$$

when both i and j tend to infinity. We then follow the same line of argument as in [TV-1] to get a contradiction and establish the claims (3.11) and (3.12).

We may now repeat the argument similar to that used in the proof of Lemma 3.2 above to show that the direction $D(s)$ established does not further split into more branches at the end sufficient far away from the initial sphere $S_r(p)$ of the annulus. More precisely:

(3.14) There is a constant K, which only depends on the Sobolev constant in (2.2), such that there is only one component in the geodesic annulus $\bar{A}_{r,t}(p)$ which intersects with $S_t(p)$, when $2r_1 \leq r < Kr \leq t \leq \frac{1}{2s} r_2$.

(3.15) Moreover the estimates in (3.12) hold for all geodesic annuli in between $2r_1$ and $\frac{1}{2s} r_2$.

Part II : We will now modify the argument in the proof of Theorem 1.18 in [AC] to finish the rest of the proof of Theorem 3.1.

Again, we argue by contradiction. Suppose the statement of the theorem is not true, then for some ϵ_0, there exist a sequence of (M_i, g_i) and annuli

$$A^i_{r_1^i, r_2^i}(p_i) \subset M_i$$

satisfying (3.3), (3.4) and (3.5) for some sequence $r_1^i, r_2^i \to 0$, and $\delta^i \to 0$, such that some sub-annuli $A^i_{\frac{1}{2}r_i, r_i}(p_i)$ with $r_i \in [(\delta^i)^{-\frac{1}{4}} r_1^i, (\delta^i)^{\frac{1}{4}} r_2^i]$ with rescaled metric $\tilde{g}_i = r_i^{-2} g_i$, are not ϵ_0-close in $C^{1,\alpha}$ topology to annuli in any cone $C(S^3/\Gamma)$. We note that by our choice of r_i, we have $\frac{r_2^i}{r_i} \to \infty$, thus there is no curvature concentration on the annuli $A^i(\frac{1}{l} r_i, l r_i)$ for any positive integer l.

Consider the sequence $(M_i, r_i^{-2} g_i, q_i)$ of manifolds with base points $q_i \in S_{r_i}(p_i)$. For each integer l, denote $A_{\frac{1}{l}r_i, l r_i}(p_i)$ a connected component in the geodesic annuli $\bar{A}_{\frac{1}{l}r_i, l r_i}(p_i)$ which reaches to $S_{l r_i}(p_i)$; by (3.14) in Part I, such a component $A_{\frac{1}{l}r_i, l r_i}(p_i)$ is unique.

Now fix l and consider the spaces $(A_{\frac{l}{r_i},lr_i}(p_i), r_i^{-2}g_i, q_i)$. We first assert that it follows from our assumption of the uniform positive lower bound of the Yamabe constants on the manifolds M_i, that we may apply Lemma 2.2 to have some lower bound which is uniform in scale for the volume of the geodesic balls (see (2.3)). By applying condition (3.15) in Part I above, we also have a bound for the *intrinsic* diameter of the boundary $S_{\frac{1}{l}r_i}(p_i)$ of $A_{\frac{1}{l}r_i,lr_i}(p_i)$ which is again uniform in scale. Thus (see [An-1]) the spaces converge in the Gromov-Hausdorff topology as $i \to \infty$ to a space which we denote by (B^l_∞, g^l_∞); note that we have $(B^l_\infty, g^l_\infty) \subset (B^{l+1}_\infty, g^{l+1}_\infty)$.

Denote

$$(B_\infty, g_\infty) = \bigcup_{l=k}^{\infty} (B^l_\infty, g^l_\infty).$$

Again, (3.15) in Part I above implies that p_∞ is an isolated singularity for the space (B_∞, g_∞). Thus it follows from the argument of Theorem 1.18 in [AC] that the limiting space B_∞ is a Euclidean cone. Finally we observe that by (3.14), the geodesic annulus $A_{\frac{r_i}{\sqrt{K}},\sqrt{K}r_i}(p_i)$ is contained in $A_{\frac{1}{l}r_i,lr_i}(p_i)$ for l sufficient large thus it tends to part of the Euclidean cone. We can then piece-wisely connect such annuli and conclude the statement in the theorem by the same argument as in ([AC], page 241).

We have thus finished the proof of Theorem 3.1.

4. Bubble tree construction

In this section we consider a sequence of metrics (M_i, g_i) satisfying the conditions stated in the next paragraph. We study the metrics near points of curvature concentration. When there is concentration of curvature, we will blow up the metrics near certain prescribed points in order to extract convergent subsequence of rescaled metrics. The main tool used is the Neck Theorem of the previous section. We follow the procedure used in [BC], [Q] [St] to extract bubbles, proceeding from the smaller scales to larger scales and ending at the the largest bubble. This procedure is different from that of Anderson and Cheeger ([AC]), whch starts from the larger scales to the smaller scales. We find it necessary to proceed in this manner because our Neck theorem is weaker than that of [AC], since we do not have a priori diameter control for the bubble body while in the case considered by [AC] such control is implied by the Ricci curvature bounds.

We will assume in this section that (M_i, g_i) satisfy the following conditions:

(4.1) $(M_i.g_i)$ are Bach flat closed 4-manifolds,

(4.2) There is a positive lower bound for the Yamabe invariants:

$$Y(M_i,[g_i]) = \inf_{g\in[g_i]} \frac{\int_{M_i} R_g dv_g}{(\int_{M_i} dv_g)^{\frac{1}{2}}} \geq Y_0 > 0,$$

for some fixed number Y_0,
(4.3) $b_1(M_i) = 0$,
(4.4) There is a common bound for the curvature tensor:

$$\int_{M_i} |Rm^i|^2 dv^i \leq \Lambda,$$

for some fixed number Λ.

In the following we will ignore the difference between a sequence and its subsequences for simplicity, since it will not cause any problem for the arguments.

Let us start the construction of bubble tree. Set $\delta = \min\{\tau_0, \tau_1, \delta_0\}$ where τ_k are from the ϵ-estimates for curvature (cf. Theorem 2.5) and δ_0 is from the neck Theorem 3.1. This will be an iterative construction. For $p \in M_i$, denote $B^i_r(p)$ a geodesic ball of radius r centered at p in (M_i, g_i),

$$s^1_i(p) = r \text{ such that } \int_{B^i_r(p)} |Rm^i|^2 dv^i = \frac{\delta}{2}. \tag{4.5}$$

We then choose

$$p^1_i = p \text{ such that } s^1_i(p) = \inf_{p\in M_i} s^1_i(p). \tag{4.6}$$

We may assume that $\lambda^1_i = s^1_i(p^1_i) \to 0$, for otherwise there would be no curvature concentration in M_i. We then conclude that $(M_i, (\lambda^1_i)^{-2} g_i, p^1_i)$ converges to $(M^1_\infty, g^1_\infty, p^1_\infty)$, which is a Bach flat, scalar flat, complete 4-manifold satisfying the Sobolov inequality (2.2), and L^2 bound for the Riemann curvature (4.4), and in addition having only one end.

Definition 4.1. *We call a Bach flat, scalar flat, complete smooth 4-manifold with the Sobolev inequality (2.2), L^2-Riemannian curvature (4.4), and a single ALE end a leaf bubble, while we will call such a space with finitely many isolated irreducible orbifold points an intermediate bubble.*

It follows that on such a bubble, the volume growth tends to that of the standard Euclidean cone $C(S^3/\Gamma_1)$ for some Γ_1 near the leaf bubble on (M^1_∞, g^1_∞). Therefore, by Theorem 2.5 for some K^1 (which is allowed to depend on (M^1_∞, g^1_∞) at this point in our approach) such that

$$\int_{M^1_\infty \setminus B_{\frac{K^1}{2}}(p^1_\infty)} |Rm|^2 dv^1_\infty \leq \frac{\delta}{4} \tag{4.7}$$

and

$$\mathcal{H}^3(S_r(p^1_\infty)) \le 2v_4 r^3, \tag{4.8}$$

for all $S_r(p^1_\infty) \subset M^1_\infty$ and $r \in [K^1, 100K^1]$, where v_4 may be chosen to be the one for Euclidean space $(R^4, |dx|^2)$ and $4v_4 = v_2$ in Theorem 3.1.

If there is further curvature concentration, we proceed to extract the next bubble. We define, for $p \in M_i \setminus B^i_{K^1\lambda^1_i}(p^1_i)$,

$$s^2_i(p) = r \text{ such that } \int_{B^i_r(p)\setminus B^i_{K^1\lambda^1_i}(p^1_i)} |Rm^i|^2 dv^i = \frac{\delta}{2}. \tag{4.9}$$

We then choose

$$p^2_i = p \text{ such that } s^2_i(p) = \inf_{M_i\setminus B^i_{K^1\lambda^1_i}(p^1_i)} s^2_i(p). \tag{4.10}$$

Again we may assume that $\lambda^2_i = s^2_i(p^2_i) \to 0$, for if otherwise there would be no more curvature concentration. It follows from (4.7), (cf. (58) in [Q]) that:

Lemma 4.2.

$$\frac{\lambda^2_i}{\lambda^1_i} + \frac{dist(p^1_i, p^2_i)}{\lambda^1_i} \to \infty. \tag{4.11}$$

Proof. Assume to the contrary, there exists a number $M > 0$ such that

$$1 \le \frac{\lambda^2_i}{\lambda^1_i} \le M \text{ and } \frac{\mathrm{dist}(p^1_i, p^2_i)}{\lambda^1_i} \le M.$$

Then

$$p^2_i \in B^i_{M\lambda^1_i}(p^1_i).$$

Hence

$$B^i_{\lambda^2_i}(p^2_i) \setminus B^i_{K^1\lambda^1_i}(p^1_i) \subset B^i_{3M\lambda^1_i}(p^1_i) \setminus B^i_{K^1\lambda^1_i}(p^1_i)$$

and this contradicts (4.9). This finishes the proof of the Lemma.

There are two possible cases which we will discuss separately:

$$\begin{aligned} &\text{Case 1.} \quad \frac{\mathrm{dist}(p^1_i, p^2_i)}{\lambda^2_i} \to \infty \ \textit{as } i \to \infty, \\ &\text{Case 2.} \quad \frac{\mathrm{dist}(p^1_i, p^2_i)}{\lambda^2_i} \le M^1, \ \textit{for some constant } M^1 \textit{ as } i \to \infty. \end{aligned} \tag{4.12}$$

In case 1, applying Lemma (4.2), we certainly also have

$$\frac{\mathrm{dist}(p_i^1, p_i^2)}{\lambda_i^1} \to \infty.$$

Therefore, in the limit of the convergent sequence of metrics $(M_i, (\lambda_i^2)^{-2} g_i, p_i^2)$ one does not see the concentration which produces the bubble (M_∞^1, g_∞^1), and converges to another end bubble (M_∞^2, g_∞^2). Similarly in the limit of the convergent sequence of metrics $(M_i, (\lambda_i^1)^{-2} g_i, p_i^1)$ one does not see the concentration which produces (M_∞^2, g_∞^2). It follows from (4.5) that there are at most $2\Lambda/\delta$ number of such leaf bubbles.

Definition 4.3. *We say two bubbles* $(M_\infty^{j_1}, g_\infty^{j_1})$ *and* $(M_\infty^{j_2}, g_\infty^{j_2})$ *associated with* $(p_i^{j_1}, \lambda_i^{j_1})$ *and* $(p_i^{j_2}, \lambda_i^{j_2})$ *are separable if*

$$\frac{dist(p_i^{j_1}, p_i^{j_2})}{\lambda_i^{j_1}} \to \infty \text{ and } \frac{dist(p_i^{j_1}, p_i^{j_2})}{\lambda_i^{j_2}} \to \infty. \tag{4.13}$$

In Case 2, one starts to introduce intermediate bubbles which will contain in its body a number of previously constructed bubbles. It would be helpful to observe that, in contrast to [BC] [Q] [St], one needs the neck Theorem 3.1 to produce the intermediate bubbles. If the Ricci curvature is bounded as in the case considered in [AC], one would have no problem to take limit in Gromov-Hausdorff topology to be a complete 4-manifold with finitely many possible orbifold points. Instead we will apply our neck Theorem 3.1 to prove that the limit space has only isolated point singularities, which are then orbifold points according to [TV-2].

Lemma 4.4. *Suppose that there are several separable bubbles* $\{(M_\infty^j, g_\infty^j)\}_{j\in J}$ *associated with* $\{(p_i^j, \lambda_i^j)\}_{j\in J}$*,and there is further concentration detected as* (p_i^k, λ_i^k) *after* $\{(p_i^j, \lambda_i^j)\}_{j\in J}$*, such that*

$$\frac{dist(p_i^j, p_i^k)}{\lambda_i^k} \le M^j, \tag{4.14}$$

for some constant M^j *as* i *tends to* ∞*; and hence,*

$$\frac{\lambda_i^k}{\lambda_i^j} \to \infty$$

for each $j \in J$*. Suppose, in addition, that* $\{(p_i^j, \lambda_i^j)\}_{j\in J}$ *is a maximal collection of such bubbles, then* $(M_i, (\lambda_i^k)^{-2} g_i, p_i^k)$ *converges in Gromov-Hausdorff topology to an* *intermediate bubble* (M^k_∞, g^k_∞). *We will call such a bubble a parent or grandparent of the given collection of bubbles* $\{(M^j_\infty, g^j_\infty)\}_{j\in J}$.

Proof. It follows from Theorem 2.5, that there are constant K^j for each bubble such that

$$\int_{M^j_\infty \setminus B_{\frac{K^j}{2}}(p^j_\infty)} |Rm|^2 dv^j < \frac{\delta}{4} \tag{4.15}$$

and

$$\mathcal{H}^3(S_r(p^j_\infty)) \le 2v_4 r^3, \tag{4.16}$$

for all $S_r(p^j_\infty) \subset M^j_\infty$ and $r \in [K^j, 100K^j]$. The claim is clear if J consists of only a single element. If J contains more than one element, we proceed in two steps.

Step 1: We first prove the lemma in the case that there exists some constant $\eta_k > 0$ so that

$$\frac{\mathrm{dist}(p^j_i, p^{j'}_i)}{\lambda^k_i} \ge \eta_k > 0, \ \textit{for all}\ j,\ j' \in\ J. \tag{4.17}$$

For any given number $L >> \eta_k >> 1/L$ and $i >> 1$ with $L \ge M^j$ for each $j \in J$, we have

$$B^i_{\lambda^k_i L}(p^k_i) \supset B^i_{\lambda^k_i/L}(p^j_i) \supset B^i_{\lambda^j_i K^j}(p^j_i) \tag{4.18}$$

for each $j \in J$. It follows from (4.15) and (4.16) that we may take limit in rescaled sequence:

$$(B^i_{\lambda^k_i L}(p^k_i) \setminus (\bigcup_j B^i_{\lambda^k_i/L}(p^j_i)), (\lambda^k_i)^{-2} g_i, p^k_i) \to (M^k_\infty(L), g^k_\infty(L)) \tag{4.19}$$

for each given L, and

$$(M^k_\infty(L), g^k_\infty(L)) \subset (M^k_\infty(L+1), g^k_\infty(L+1)) \subset \bigcup_{l=1}^{\infty}(M^k_\infty(L+l), g^k_\infty(L+l)). \tag{4.20}$$

We will now apply the neck theorem to show that

$$(M^k_\infty, g^k_\infty) = \bigcup_{l=1}^{\infty}(M^k_\infty(L+l), g^k_\infty(L+l))$$

is an intermediate bubble. It follows from (4.17) and the choice of λ_i^k that (4.18) holds and

$$\int_{B^i_{\lambda_i^k \frac{1}{L}}(p_i^j) \setminus B^i_{\lambda_i^j K^j}(p_i^j)} |Rm^i|^2 dv^i \leq \frac{\delta}{2} < \delta_0. \tag{4.21}$$

The Neck theorem then shows that the diameter of $S^i_{\lambda_i^k \frac{1}{L}}(p_i^j)$ in the rescaled space $(B^i_{\lambda_i^k L}(p_i^k) \setminus (\bigcup_j B^i_{\lambda_i^k / L}(p_i^j)), (\lambda_i^k)^{-2} g_i, p_i^k)$ goes to zero.

Step 2: We now deal with the case that there is a subset $J' \subset J$ such that

$$\frac{\text{dist}(p_i^j, p_i^{j'})}{\lambda_i^k} \to 0 \tag{4.22}$$

for all $j, j' \in J'$ as $i \to \infty$. Our strategy will be to combine some elements in J' to create some intermediate bubbles. We remark that this situation does not arise in the case considered in [AC] due to the gap theorem for Ricci flat complete orbifolds (cf. [Ba]). We call those intermediate bubbles thus created the exotic bubbles, since they may carry an arbitrarily small amount of L^2-Riemannian curvature. To start the creation process, let

$$\mu_i^1 = \min\{\text{dist}(p_i^j, p_i^{j'}) : j, j' \in J'\} = \text{dist}(p_i^{j_1}, p_i^{j_2}), \tag{4.23}$$

$$J_1 = \{j \in J' : sup_i\{\frac{\text{dist}(p_i^{j_1}, p_i^j)}{\mu_i^1}\} < \infty\}, \tag{4.24}$$

note that J_1 consists of at least two elements j_1 and j_2; and for some large number N_1

$$\frac{\text{dist}(p_i^{j_1}, p_i^j)}{\mu_i^1} \leq \frac{1}{2} N_1. \tag{4.25}$$

Then we consider the sequence $(M_i, (\mu_i^1)^{-2} g_i, p_i^{j_1})$. In view of Theorem 2.5 and the volume bound we may take limit in the rescaled sequence:

$$(B^i_{(N_1+l)\mu_i^1}(p_i^{j_1}) \setminus (\bigcup_{j \in J_1} B^i_{\frac{1}{N_1+l}\mu_i^1}(p_i^j)), (\mu_i^1)^{-2} g_i, p_i^{j_1}) \ \to (N^1_\infty(l), h^1_\infty(l)).$$

Then

$$(N^1_\infty, h^1_\infty) = \bigcup_{l=1}^{\infty} (N^1_\infty(l), h^1_\infty(l))$$

gives an exotic intermediate bubble by applying the neck Theorem 3.1 to the annuli $B^i_{\frac{1}{N_1}\mu_i^1}(p_i^j)) \setminus B^i_{K^j\lambda_i^j}(p_i^j)$ for all $j \in J_1$. This exotic bubble (N^1_∞, h^1_∞) has at most $|J_1|$ number of orbifold singularities. We use this bubble to replace all bubbles (M^j_∞, g^j_∞) for $j \in J_1$. In other words, we combine all bubbles (M^j_∞, g^j_∞) for $j \in J_1$ into one single bubble (N^1_∞, h^1_∞). Since

$$\frac{\mathrm{dist}(p_i^{j_1}, p_i^j)}{\mu_i^1} \to \infty$$

for all $j \in J' \setminus J_1$ by the definition of J_1 and

$$\frac{\mathrm{dist}(p_i^{j_1}, p_i^j)}{\lambda_i^j} \to \infty$$

as assumed, the collection

$$\{(p_i^{j_1}, \mu_i^1)\} \bigcup \{(p_i^j, \lambda_i^j) : j \in J' \setminus J_1\}$$

constitute a smaller family of separable bubbles. We may then repeat the above combination process for a finite number of steps if necessary, to combine all bubbles $\{(M^j_\infty, g^j_\infty)\}_{j\in J'}$ into one single bubble (N_∞, h_∞) associated with some (p_i^l, μ_i^l) with $l \in J'$ and

$$\frac{\lambda_i^k}{\mu_i^l} \to \infty \text{ and } \frac{\mathrm{dist}(p_i^l, p_i^k)}{\lambda_i^k} \le M^l. \tag{4.26}$$

Let us call the new collection

$$\{(p_i^{j'}, \mu_i^{j'})\} \bigcup \{(p_i^j, \lambda_i^j) : j \in J \setminus J'.\} \tag{4.27}$$

Now we are back to the situation in which the new collection of bubbles (4.27) satisfy hypothesis (4.14) of Lemma 4.4. Thus we may repeat the process of step 1 at the beginning of the proof of the lemma to the new collection and note that after finite many such repetitions, the final collection of bubbles will satisfy (4.17). We have thus finished the proof of Lemma 4.4.

Definition 4.5. *A bubble tree T is defined to be a tree whose vertexes are bubbles and whose edges are necks from Theorem 3.1. At each vertex (M^j_∞, g^j_∞), its ALE end is connected, via a neck, to its parent toward the root bubble of T, while at finitely many isolated possible orbifold points of (M^j_∞, g^j_∞), it is connected, via necks, to its children toward leaf bubbles of T. We say two bubble trees T_1 and T_2 are separable if their root bubbles are separable.*

To finish the bubble tree construction, we describe the following inductive procedure. Suppose that there are m bubbles in $n(\leq m)$ separable bubble trees $\{T^j\}_{j=1}^n$ with their root bubbles $\{(M^j_\infty, g^j_\infty)\}_{j=1}^n$ and the associated centers and scales $\{(p^j_i, \lambda^j_i)\}_{j=1}^n$. We define

$$s_i^{m+1}(p) = r : \text{ such that } \int_{B^i_r(p)\setminus \bigcup_{j=1}^n B^i_{K^j\lambda^j_i}(p^j_i)} |Rm^i|^2 = \frac{\delta}{2} \tag{4.28}$$

and

$$\lambda_i^{m+1} = s_i^{m+1}(p_i^{m+1}) = \inf_{M_i\setminus \bigcup_{j=1}^n B^i_{K^j\lambda^j_i}(p^j_i)} s_i^{m+1}(p). \tag{4.29}$$

Then we repeat the procedure starting at the beginning of section 4. Either $(p_i^{m+1}, \lambda_i^{m+1})$ is separable from all bubble trees $\{T^j\}_{j=1}^n$, or it is a parent of bubble trees from $\{T^j\}_{j\in J} \subset \{T^j\}_{j=1}^n$, which will be called the new root bubble of this tree, according to (4.12). Clearly,

$$m \leq \frac{2\Lambda}{\delta}.$$

Thus the procedure has to stop in finitely many steps. When the procedure stops, we have l number of separable bubble trees $\{T^k\}_{k=1}^l$ with the root bubbles $\{(M^k_\infty, g^k_\infty)\}_{k=1}^l$ and their associated centers and scales $\{(p^k_i, \lambda^k_i)\}_{k=1}^l$. Thus for a fixed number $\nu > 0$,

$$\int_{B^i_\nu(p_i)\setminus \bigcup_{k=1}^l B^i_{K^k\lambda^k_i}(p^k_i)} |Rm^i|^2 < \frac{\delta}{2}. \tag{4.30}$$

After finitely many formation of exotic intermediate bubbles as in the second step of the proof of Lemma 4.4, we finally arrive at bubble trees $\{T^j\}$ whose associated centers are separated from each other by non-zero distances in M_i. Thus the neck that connecting each bubble tree T^j to M_i is given again by applying the neck Theorem 3.1 to $B^i_\sigma(p^j_i) \setminus B^i_{K^j\lambda^j_i}(p^j_i)$ for any small fixed positive number σ, where (M^j_∞, g^j_∞) is the root bubble for T^j. This completes the construction of the bubble tree at any point of curvature concentration in M_i.

One consequence of the bubble tree construction presents an alternative argument for the following important estimates in [An-2] [TV-2]:

Corollary 4.6. *Suppose that (M_i, g_i) is a sequence of Bach flat 4-manifolds (M_i, g_i) with strictly positive scalar curvature Yamabe metrics, i.e.*

$$Y(M_i, [g_i]) = \inf_{g\in[g_i]} \frac{\int_{M_i} R_g dv_g}{(\int_{M_i} dv_g)^{\frac{1}{2}}} \geq Y_0 > 0, \tag{4.31}$$

for some fixed number Y_0, vanishing first homology, and

$$\int_{M_i} |Rm^i|^2 dv_i \leq \Lambda. \tag{4.32}$$

Suppose that $X_i \subset M_i$ including some geodesic ball of radius r_0 satisfies

$$\int_{T_{\eta_0}(\partial X_i)} |Rm^i|^2 dv^i \leq \frac{\delta}{4}, \tag{4.33}$$

for some fixed positive numbers $r_0 > 4\eta_0 > 0$, $\delta = \min\{\tau_0, \tau_1, \delta_0\}$, where τ_k are from Lemma 2.7 and δ_0 is from Theorem 3.1. Assume there exists only one bubble tree T representing the curvature concentration in X_i. Denote the center of the root bubble (M_∞, g_∞) for T by p_i. Then there exist a small number σ_0 such that the intrinsic diameter of $S_\sigma(p_i)$ in $X_i \setminus B^i_\sigma(p_i)$ is bounded by $C_0\sigma$ for any $\sigma \leq \sigma_0$ and some fixed C_0.

Proof. Let (M_∞, g_∞) be the root bubble for the bubble tree T representing the only curvature concentration in X_i and (p_i, λ_i) be the associated center and scale.

Let K be so large that

$$\int_{M_\infty \setminus B^\infty_K(p_\infty)} |Rm^\infty|^2 dv^\infty \leq \frac{\delta}{4}$$

and

$$\mathcal{H}^3(S^\infty_r(p_\infty)) \leq 2v_4 r^3$$

for $S^\infty_r(p_\infty) \subset M_\infty$, $r \geq K$ and $v_2 = 4v_4$, where v_2 is the same as in Theorem 3.1. Then let μ_0 is so small that

$$\int_{B^i_{\mu_0}(p_i)\setminus B^i_{K\lambda_i}(p_i)} |Rm^i|^2 dv^i \leq \frac{\delta}{2}$$

and the neck Theorem 3.1 is applicable to the annulus $B^i_{\mu_0}(p_i)\setminus B^i_{K\lambda_i}(p_i)$. Thus take $\sigma_0 = \delta_0^{-\frac{1}{4}}\mu_0$. The intrinsic diameter of $S_\sigma(p_i)$ for $\sigma \leq \sigma_0$ is then bounded because of the upper bound of volume by Theorem 3.1 and the lower bound of volume given by the Sobolev inequality (2.2).

5. Proof of main theorems

In this section we will apply the bubble tree construction in the previous section and the recent compactness results in [An-2] [TV-2] to prove our main theorems.

We first recall a recent result in [An-2], [TV-2]:

Theorem 5.1. *([An-2] [TV-2]) Suppose that (M_i, g_i) is a sequence of Bach flat, strictly positive Yamabe constants with uniform lower bound, $b_1(M_i) = 0$, uniform L^2-Riemannian curvature bound (4.4), and normalized volume*

$$vol(M_i, g_i) = \frac{8\pi^2}{3}. \tag{5.1}$$

Then a subsequence of (M_i, g_i) converges to a Bach flat manifold (M_∞, g_∞) with finitely many isolated irreducible orbifold points in the Gromov-Hausdorff topology. Moreover, away from a finite set of points of curvature concentration, the convergence is in C^∞.

We now begin the proof of Theorem A in section 1.

Proof. We will first establish the result in the case that M is diffeomorphic to S^4, whose Euler number is 2. In this case, by the Gauss-Bonnet-Chern formula (1.2), we see that under the assumption (1.9), we have

$$\int_M |W|^2 dv \leq 16\pi^2\epsilon.$$

Choose g to be the Yamabe metric on M, apply Kato's inequality and Sobolev inequality (2.2), then there exists some constant C depending on the Yamabe constant, such that

$$(\int_M |E|^4 dv)^{\frac{1}{2}} \leq C \int_M (6|\nabla E|^2 + R|E|^2) dv. \tag{5.2}$$

Thus applying (2.6) in Lemma 2.3, we have

$$\begin{aligned} (\int_M |E|^4 dv)^{\frac{1}{2}} &\leq C(\int_M |W||E|^2 dv + \int_M |E|^3 dv) \\ &\leq C((\int_M |W|^2 dv)^{\frac{1}{2}} + (\int_M |E|^2 dv)^{\frac{1}{2}})(\int_M |E|^4 dv)^{\frac{1}{2}}. \end{aligned} \tag{5.3}$$

Thus, there is a positive number ϵ_0 such that $E = 0$ if $\epsilon \leq \epsilon_0$ in (1.9). When $E = 0$, we may use a similar argument involving Kato's and Sobolev inequalities as above and apply (2.7) in Lemma 2.3 to conclude $W = 0$ provided that ϵ_0 is sufficiently small. Thus (M, g) is isometric to (S^4, g_c).

We will now prove that M has to be diffeomorphic to S^4. To see this, we first first apply Theorem 5.1 above, together with the bubble tree construction in section 4 to conclude:

Lemma 5.2.

$$vol(M_\infty, g_\infty) = \frac{8\pi^2}{3}. \tag{5.4}$$

We will prove M is diffeomorphic to S^4 by a contradiction argument, Assume the contrary, then there is a sequence of Bach flat manifold (M_i, g_i) satisfying all assumptions in Theorem A; and hence the assumptions of Theorem 5.1 by Corollary 2.8, with

$$\int_{M_i} (\sigma_2 dv)[g_i] = 16\pi^2(1 - \epsilon_i) \to 16\pi^2, \tag{5.5}$$

and M_i is not diffeomorphic to S^4. Apply Lemma 2.6 to (5.5), we thus get

$$\int_{M_i} (|E|^2 dv)[g_i] = 32\pi^2 \epsilon_i \to 0. \tag{5.6}$$

Apply Theorem 5.1, we then conclude that some subsequence of (M_i, g_i) converges in the Gromov-Hausdorff topology to a limiting space (M_∞, g_∞) which is an Einstein orbifold satisfying $Ric_\infty = 3g_\infty$. Apply Lemma 5.2 and the Bishop volume comparison result, we then conclude that (M_∞, g_∞) cannot contain any orbifold point, hence it is a smooth Einstein manifold; and thus it is isometric to the standard 4-sphere.

We now trace through the bubble tree construction in section 4. First we note that at each point of curvature concentration, it is the vertex of a bubble tree. Apply (5.6), we conclude that each bubble is a Ricci flat bubble.

Since the limit space is smooth, the neck which connects the limit space and the root bubble has to be a cylinder $[a, b] \times S^3$. Hence the root bubble is Ricci-flat asymptotically Euclidean instead of just being an ALE space. We now follow the proof of the well-known result any Ricci flat asymptotically Euclidean orbifold has to be the Euclidean space (c.f. the proof of Theorem 3.5 in [An-1]). Hence each neck in the bubble tree has to be a cylinder $[a, b] \times S^3$. This in turn implies that the leaf bubbles are Euclidean space, this contradicts the fact that the leaf bubbles carry curvatures. We have thus finished the proof of Theorem A.

Theorem 5.3. *The collection $\mathcal{A}$ of Riemannian 4-manifolds (M^4, g) satisfying the following*

1) Bach flat,

2)

$$Y(M,[g]) = \inf_{g\in[g]} \frac{\int_M R_g dv_g}{(\int_M dv_g)^{\frac{1}{2}}} \geq Y_0, \tag{5.7}$$

for a fixed $Y_0 > 0$,

3)

$$b_1(M) = 0, \tag{5.8}$$

4)

$$\int_M (|W|^2 dv)[g] \leq \Lambda_0$$

for some fixed positive number Λ_0. Then there are at most finitely many diffeomorphism types in the family $\mathcal{A}$.

We will first assume Theorem 5.3 and establish Theorem B in section 1 as a consequence.

Proof of Theorem B. It is easy to see that (1.10) implies (5.7) for manifolds with positive Yamabe constant. Hence (5.8) is a consequence of the vanishing theorem of Gursky [G] (see Theorem 2.6 for the statement of the theorem). This finishes the proof of Theorem B in section 1.

Proof of Theorem 5.4.

Assume otherwise, there is an infinite sequence of manifolds (M_i, g_i) from the collection with pairwise distinct diffeomorphism types. We will show that, at least for a subsequence, manifolds (M_i, g_i) will be diffeomorphic to each other for i sufficient large. Without loss of generality, we may assume that each (M_i, g_i) satisfies all assumptions in Theorem 5.1. So some subsequence of (M_i, g_i) converges to a limit space (M_∞, g_∞) in the Gromov-Hausdorff topology and in C^∞ away from a finite set of points of curvature concentration. In addition, at each point p^k_∞ of curvature concentration of M_∞, as shown in the previous section, there is a bubble tree T^k forming for possibly some subsequence of (M_i, g_i). Since the set of points of curvature concentration is finite and the bubble tree T^k at each point p^k_∞ is a finite tree, we see that all the constants K's (see (4.7) and (4.15)) in the bubble tree construction in section 4 is a fixed finite set for the particular subsequence we are considering. To show that M_i are all diffeomorphic to each other, at least, for i sufficiently large, we consider each of the following three different type of regions:

Body region: Let $\{(p_i^k, \lambda_i^k)\}_{k\in N}$ be all the root bubbles of all the bubble trees. Then

$$(M_i \setminus \bigcup_{k\in N} B^i_{\frac{1}{4}\delta_0^{\frac{1}{4}}}(p_i^k), g_i)$$

tends to

$$(M_\infty \setminus \bigcup_{k\in N} B^\infty_{\frac{1}{4}\delta_0^{\frac{1}{4}}}(p_\infty^k), g_\infty)$$

Neck region: Let $(p_i^{j_2}, \lambda_i^{j_2})$ be the parent for $(p_i^{j_1}, \lambda_i^{j_1})$. Then

$$B^i_{\frac{1}{2}\delta_0^{\frac{1}{4}}\lambda_i^{j_2}}(p_i^{j_1}) \setminus B^i_{2\delta_0^{-\frac{1}{4}}K^{j_1}\lambda_i^{j_1}}(p_i^{j_1})$$

tends to an annulus on an cone $C(S^3/\Gamma)$ for some finite group $\Gamma \subset O(4)$ with a uniform bound on the order of the group Γ according to Theorem 3.1.

Bubble region: Let $\{(p_i^j, \lambda_i^j)\}_{j\in J}$ be all the children of a bubble (p_i^k, λ_i^k) in the bubble tree. Then

$$B^i_{4\delta_0^{-\frac{1}{4}}K^k\lambda_i^k}(p_i^k) \setminus \bigcup_{j\in J} B^i_{\frac{1}{4}\delta_0^{\frac{1}{4}}\lambda_i^k}(p_i^j)$$

tends to

$$B^\infty_{4\delta_0^{-\frac{1}{4}}K^k}(p_\infty^k) \setminus \bigcup_{j\in J} B^\infty_{\frac{1}{4}\delta_0^{\frac{1}{4}}}(p_\infty^j) \subset M_\infty^k \setminus \{p_\infty^j\}_{j\in J}.$$

Notice that the overlap regions of any two of the three different type regions above are also well controlled and M_i is covered by those regions of the above three types. So M_i's are all diffeomorphic to each other for sufficiently large i in the subsequence. This contradicts with our assumption that M_i are pairwise not diffeomorphic to another. We have thus finished the proof of Theorem 5.3.

6. Application to the analysis of the σ_2 equation.

In this section, we study the σ_2 equation for conformal structures in the class $\mathcal{A}$. Given a conformal structure $[g] \in \mathcal{A}$, represented by a metric g_0, a conformal metric $g = e^{2w}g_0$ has its Schouten tensor A_g defined as

$$A_g = \frac{1}{2}(Ric_g - \frac{1}{6}R_g g).$$

Under conformal change of metrics, we have

$$A_g = -\nabla_0^2 w + dw \otimes dw - \frac{|\nabla_0 w|^2}{2} g_0 + A_{g_0}. \tag{6.1}$$

We recall in the following two propositions the local estimate developed in [GW] (see also the recent simplified proof of the estimates in [Ch]) and the classification of entire solutions [CGY-3]. A solution $g = e^{2w} g_0$ to the equation $\sigma_2(g) = \sigma_2(A_g) = 1$ is said to be admissible if the scalar curvature R_g is positive.

Proposition 6.1. *Let $w \in C^3$ be an admissible solution of the equation $\sigma_2(A_g) = 1$ in B_r. There is a constant C depending on $||g_0||_{C^4(B_r)}$ such that*

$$\sup_{B_{r/2}} (|\nabla_0^2 w| + |\nabla_0 w|^2) \leq C(1 + \sup_{B_r} e^{2w}). \tag{6.2}$$

Proposition 6.2. *An entire admissible solution $g = e^{2w}|dx|^2$ to the $\sigma_2(A_g) = 1$ equation on R^4 is the pull back of the spherical metric under the stereographic projection.*

We will also use the following result of Viaclovsky ([V], Proposition 3)

Proposition 6.3. *Suppose $g_1 = e^{2w_1} g_0$ and $g_2 = e^{2w_2} g_0$ are two metrics with $\sigma_2(g_1) \geq c_1$ and $\sigma_2(g_2) \geq c_2$ for some positive numbers c_1 and c_2 and with R_{g_1} and R_{g_2} both positive, then the metric $g_t = e^{2w_t} g_0$ where $e^{-w_t} \equiv (1-t)e^{-w_1} + te^{-w_2}$ has R_{g_t} positive and $\sigma_2(g_1) \geq c_t$ for some positive c_t depends only on c_1, c_2 for all $0 \leq t \leq 1$.*

We need to remark that the same calculation in the argument for the result above also works for a finite convex sum $e^{-w} = \sum t_i e^{-w_i}$ where $\sum t_i = 1$, and thus the argument also works for averaging by integration.

Proof of Theorem C

Proof. We may assume the conformal structures under consideration are distinct from the standard 4-sphere, and hence in view of Theorem A, there is some $\epsilon_0 > 0$ that the following holds:

$$\int \sigma_2(A_g) dV_g \leq 16\pi^2(1 - \epsilon_0). \tag{6.3}$$

We will establish the theorem by a proof of contradiction. Suppose there is a sequence of metrics g_i with $\sigma_2(g_i) = 1$ in the family $\mathcal{A}$ and with the diameter of g_i tends to infinity; we apply the argument in this paper to form a bubble-tree of a subsequence of the underlying Yamabe metrics $(g_i)_Y$ of g_i. We will then prove that the diameter of the metric $g = g_i$ in each of the "body" and "neck" region of this bubble tree is bounded and get a contradiction. We start with a simplest case when the underlying Yamabe metrics of the sequence $\{g_i\}$ is compact in the C^∞ topology.

Lemma 6.4. *Suppose $g = e^{2w} g_Y$ is a family of admissible metrics with $\sigma_2(g) = 1$ satisfying both (6.3) and*

$$\int \sigma_2(A_g) dV_g \geq a_0 > 0 \tag{6.4}$$

with the underlying Yamabe metrics g_Y compact in the C^∞ topology. Then there is a uniform upper and lower bound of the conformal factor w, and the Ricci curvature of the metric is bounded.

Proof. First we claim that w is bounded from above by the following simple blowup argument. Thus we assume that there exists diffeomorphisms $\Psi_i : (X, \bar{g}) \to (X_i, (g_i)_Y)$ so that the sequence of pull back metrics $\Psi_i^*(g_i)_Y$ converges in the C^∞ topology to the limit metric $\bar{g}$. If the sequence w_i were not bounded from above, there is a sequence of points $p_i \in X_i$ such that $e^{w_i(p_i)} = max\ e^{w_i} = \lambda_i$ tends to ∞. The diffeomorphisms Ψ_i^{-1} maps this sequence to a sequence $x_i \in X$ with a convergent subsequence (still denoted by $\{x_i\}$) to a point $x_0 \in X$. Let B be a ball of radius r_0 in a geodesic normal (with respect to the limit metric $\bar{g}$) coordinate system y whose origin correspond to the limit point x_0. Let y_i denote the coordinates of $\Psi_i^{-1}(x_i)$ and $T_i(y) = \lambda_i^{-1} y + y_i$ be a family of dilations, and consider the sequence of metrics $h_i = T_i^* \Psi_i^* (g_i)_Y$. The metrics h_i are isometric to $(g_i)_Y$ but defined on balls of radius $\lambda_i r_0$ in y space. Since the metrics $\Psi_i^*(g_i)_Y$ converges to $\bar{g}$, it follows that h_i converges in C^∞ uniformly on compact subsets in y-space to the flat metric $|dy|^2$. The conformal metrics $T_i^* \Psi_i^* g_i$ are isometric to g_i can be written as $\lambda_i^{-2} e^{2 w_i \circ \Psi_i} h_i = v_i h_i$, where v_i is a bounded function on its domain of definition which includes the ball $|y| < \lambda_i r_0$. Proposition 6.1 then asserts that euclidean y-gradient $|\nabla v_i|$ is uniformly bounded on compact subsets, and hence the functions v_i converges uniformly on compact subsets to a function v on the y-space where the metric $h = v^2 |dy|^2$ is an entire solution of the equation $\sigma_2(A_h) = 1$. Proposition 6.2 asserts such solutions are the standard spherical metric. This means however that the original metrics g_i must satisfy $\limsup \int \sigma_2(A_{g_i}) dV_{g_i} \geq 16\pi^2$. This is a contradiction to (6.3). Thus we have shown that the conformal factor e^w and hence w is bounded from above. Denote $\bar{w} = max\ w$, we now observe as A_g satisfying condition (6.4), we have

$$0 < a_0 \leq \int \sigma_2(A_g) dv_g = \int dv_g = \int e^{4w} dv_{g_Y} \leq e^{4\bar{w}} vol(g_Y), \tag{6.5}$$

Thus $\bar{w}$ is also bounded from below; from this we can apply the local gradient estimate in Proposition 6.1 to conclude that that w is bounded both from above and below, applying Proposition 6.1, we then conclude that the Ricci curvature of g is uniformly bounded. We have finished the proof of the lemma.

We now consider the general situation when the family of conformal structures namely the family of the Yamabe metrics g_Y corresponding to the σ_2 metrics may degenerate. According to the proof of Theorem B, there is at most a bounded number of neck regions and body region in any degenerating family of conformal structures. To establish Theorem C, it suffices to check that in any neck region of an ALE space which is the blown up limit of the corresponding Yamabe metrics in the family $\mathcal{A}$, the restriction of the conformal metric satisfying the equation $\sigma_2 = 1$ has a bounded diameter. At this point it is necessary to make precise the terms body region and neck region of the bubble tree. The bubble tree refers to the pattern of degeneration of a sequence of Yamabe metrics (M_i, h_i) in the family $\mathcal{A}$. Thus there is a finite number of disjoint regions $\Omega_{j,i}, j \in \{1, 2, ., ., K\}$ in each (M_i, h_i) over which the rescaled (with possibly different scaling for each region) metrics h_i' converges uniformly smoothly to a scalar flat, Bach-flat metric which we denote $(\Omega_j,\, h_j')$ and will call the $j - th$ body. There is also a finite number of regions $T_{k,i}$ which overlaps in M_i with a pair of the previously labeled regions $\Omega_{j,i}$ and $\Omega_{j',i}$ such that $(T_{k,i}, h_i')$ converges uniformly on compact and smoothly to a subset of the end of the ALE space, and such regions will be called the neck region. We also refer the readers to the more detailed discussion of the bubble tree construction and the definitions of the "body" and "neck" regions in section 5 of the paper.

We remark that over any "body" region of the bubble tree, the underlying rescaled Yamabe metrics in the family $\mathcal{A}$ is compact in the C^∞ topology. Thus we may apply the argument in the proof of the Lemma above to see that w is bounded above for each body in the family of metrics in $\mathcal{A}$, hence the diameter of the metrics bounded above in each of the body region of the fixed bubble tree.

Our second remark is that over a body region of a bubble tree, if the σ_2 "mass" defined to be the integral of $\sigma_2(A_g)$ over the body is bounded from below; that is if condition (6.5) holds for some constant c_0 replacing a_0, then Lemma 6.4 also applies to metrics in the family in this region. We shall call such a region with the metric g of bounded geometry.

Since the total σ_2 mass of metrics in $\mathcal{A}$ is greater than a_0, and the total number K of body regions and neck regions is finite, we have two cases:

Case I: There is some body region where the mass over there is greater than $\frac{1}{K}a_0$.

Case II: Case I does not happen, while there is a neck region where the mass is greater than $\frac{1}{K}a_0$.

In the following, we will first Analise the situation in case II. In this case, as we are studying the metric g over a region which asymptotically looks like $(S^3/\Gamma)\times R$, where Γ is a finite group with the order $\|\Gamma\|$ uniformly bounded for metrics in the family $\mathcal{A}$, we will use the cylindrical coordinate $x = e^t\nu;\ t\in R;\ \nu\in(S^3/\Gamma)$ and choose the background metric g_0 to be the standard cylindrical metric $g_0 = dt^2+d\nu^2$. Denote $g = e^{2w}g_0$; we may apply argument similar as before and prove that w is uniformly bounded above for $g\in\mathcal{A}$. In the following we will show that in case II, we also have max w is also bounded from below by a constant depending only on the σ_2 mass over the region and K. Applying Proposition 6.1, we then conclude that there is a region (around the point where max w occurs) where the metric g has locally bounded geometry.

To prove such a result for a metric $g = e^{2w}g_0$ in $\mathcal{A}$ over a neck region with $\sigma_2(g) = 1$, we first observe that we may replace $w = w(t,\nu)$ by $\bar{w}(t) = \frac{1}{|S^3|/|\Gamma|}\int_{S^3/\Gamma} w(t,\nu)d\nu$ and apply Proposition 6.3 to conclude that the metric $\bar{g} = e^{2\bar{w}}g_0$ still satisfies that $\sigma_2(\bar{g})\geq c_3 > 0$ and $R_{\bar{g}}$ positive, and where c is a constant. Notice that in this case, we have $|w(t,\nu)-\bar{w}(t)|$ being uniformly bounded, thus

$$\int e^{4\bar{w}(t)}dt \geq C\int e^{4w(t,\nu)}dtd\nu \geq C(K)\sigma_0.$$

Thus it suffices to establish the following Lemma. The idea of the proof of the Lemma follows from the analysis of O.D.E. solutions of $\sigma_k(g) = constant$ solutions on annulus regions in the earlier work of [CHY].

Lemma 6.5. *Suppose $g = e^{2w}g_0$ where $w = w(t)$ is a metric $\sigma_2(\bar{g})\geq c_3 > 0$ and $R_{\bar{g}}$ positive defined on a cylinder $(S^3/\Gamma)\times[t_0,T]$ with*

$$\int_{t_0}^{T} e^{4w(t)}dt = a > 0 \tag{6.6}$$

then

$$maxw \geq \frac{1}{2}loga + c_4$$

for some universal constant c_4 depending only on c_3.

Proof. In the cylindrical coordinate, we have

$$\sigma_2(A_g)(t) = -\frac{2}{3}w''(1-(w')^2)e^{-4w}(t) \geq c_3 \tag{6.7}$$

while $R_g(t) = 6(1-(w')^2 - w'')(t)e^{-2w(t)} > 0$. Thus we have $(1-(w')^2) > 0$ while $w'' < 0$; thus w' is a decreasing function. Suppose $max\, w = w(t_M)$ happens for some point t_M in (t_0, T), then on one of the interval (t_0, t_M), or (t_M, T) the σ_2 mass is greater than $\frac{1}{2}a$. Choose the interval this happens, say (t_M, T). Otherwise, we may assume by reversing the variable t to $-t$ if necessary, that $max\, w = w(t_M)$. We now fix a number $b = \frac{1}{2}\int_{t_M}^{T} \sigma_2(g)dt$, and choose N so that

$$\int_{t_M}^{t_M+N} e^{4w(t)}dt = b.$$

Now for $t > T_M + N$, we have from (6.7)

$$-w''(t) \geq ce^{4w(t)},$$

for $c = \frac{3}{2}c_3$. Thus

$$-w'(t) \geq c\int_{t_M}^{t} e^{4w(s)}ds \geq cb,$$

hence a direct integration yields the bound

$$b = \int_{t_M+N}^{T} e^{4w(t)}dt \leq \frac{e^{4w(t_M)}}{4bc}. \tag{6.8}$$

The assertion of the lemma is a direct consequence of (6.8).

We remark that as a consequence of Lemma 6.5, when condition (6.6) is satisfied for some $a \geq \frac{a_0}{K}$, then the region $B = \{(t,\nu) : |t - t_M| \leq 1, \nu \in S^3/\Gamma\}$ has g-volume bounded from below by a positive constant depending only the data in the family $\mathcal{A}$; and in fact is a region where Ric_g bounded from above and below by positive constant.

Thus we have arrived at the conclusion that in any degenerating sequence of $\sigma_2 = 1$ conformal metrics, either in case I or in case II discussed above, there is a region of bounded geometry namely the body region in case I, the region in the cylinder which has mass in case II which we denote by B. We will now use the volume comparison argument to show that the diameter of each neck is necessarily bounded. We proceed to argue by contradiction. Assume there is one neck region over which the sequence of the $\sigma_2 = 1$ metrics have unbounded diameter. Let d denote the distance function in the g metric to a point p to be specified later. Since $\int_\Omega \sigma_2 dV$ is given by the volume of the neck region Ω and is bounded above by $16\pi^2$, it follows that given any constant $\epsilon_0 > 0$, there is a length L so that the geodesic annulus $A_{r,r+L}$ contains a geodesic sphere S_ρ whose volume is bounded above by ϵ_0. Now as a $\sigma_2 = 1$ metric has nonnegative Ricci curvature, according to the volume comparison result of Bishop-Gromov, we have

$$\frac{Vol(S_{\rho+s})}{(\rho+s)^3} \leq \frac{Vol(S_\rho)}{\rho^3}, \tag{6.9}$$

for each $s > 0$. It is necessary at this point to remark that in the inequality above, the geodesic spheres $S_{\rho+s}$ and S_ρ may be replaced by a component of the geodesic sphere denoted by $S'_{\rho+s}$ and S'_ρ provided the length minimizing geodesic joining p to points in the given component of $S'_{\rho+s}$ passes through the corresponding component in S'_ρ. Thus if $s \leq 2L$ and ρ is much larger than L, we find $Vol(S'_{\rho+s}) \leq 2\epsilon_0$ since each point q in the neck is at g distance no more than L from such a geodesic sphere S'_ρ if we choose to measure distance from a point p located at the far end of the neck from q. Thus we have shown that each component of the geodesic sphere S'_ρ has uniformly small volume as long as the neck has sufficiently large diameter. Now suppose p is chosen to lie in the far end of the neck (away from the region of bounded geometry, Denote $D = d(p, B)$, then the component of spheres $S'_{D+\lambda}$ as λ increases from zero to diameter d of B will sweep out the region B. Using the Fubini theorem, In particular for some choice of λ the set $S'_{D+\lambda}$ will have area bounded from below by $volume(B)/d$. This is in contradiction to the uniform smallness of the volume of all such component of spheres. Thus we have finished the proof of Theorem C.

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