American Journal of Computational Mathematics, 2012, 2, 341-344
doi:10.4236/ajcm.2012.24047 Published Online December 2012 (http://www.SciRP.org/journal/ajcm)

A New Block-Predictor Corrector Algorithm for the Solution of $y''' = f(x, y, y', y'')$

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Received May 10, 2012; revised September 16, 2012; accepted September 25, 2012

ABSTRACT

We consider direct solution to third order ordinary differential equations in this paper. Method of collection and interpolation of the power series approximant of single variable is considered to derive a linear multistep method (LMM) with continuous coefficient. Block method was later adopted to generate the independent solution at selected grid points. The properties of the block viz: order, zero stability and stability region are investigated. Our method was tested on third order ordinary differential equation and found to give better result when compared with existing methods.

Keywords: Collection; Interpolation; Power Series; Approximant; Linear Multistep; Continuous Coefficient; Block Method

1. Introduction

This paper considers the general third order initial value problems of the form

$$y''' = f\left(x, y, y', y''\right), y\left(x_0\right) = y_0,\quad y'\left(x_0\right) = y'_0, y''\left(x_0\right) = y''_0. \tag{1}$$

Conventionally, higher order ordinary differential equations are solved directly by the predictor-corrector method where separate predictors are developed to implement the correct and Taylor series expansion adopted to provide the starting values. Predictor-corrector methods are extensively studied by [1-5]. These authors proposed linear multistep methods with continuous coefficient, which have advantage of evaluation at all points within the grid over the proposed method in [6] The major setbacks of predictor-corrector method are extensively discussed by [7].

Lately, many authors have adopted block method to solve ordinary differential equations because it addresses some of the setbacks of predictor-corrector method discussed by [6]. Among these authors are [8-10].

According to [6], the general block formula is given by

$$\boldsymbol{Y}_m = \boldsymbol{e}y_n + h^{\mu}\boldsymbol{df}\left(y_n\right) + h^{\mu}\boldsymbol{bF}\left(y_m\right). \tag{2}$$

where $\boldsymbol{e}$ is $s \times s$ vector, d is $r-$vector and b is $r \times r$ vector, s is the interpolation points and r is the collection points. F is a k-vector whose j^{th} entry is $f_{n+j} = f\left(t_{n+j,} y_{n+j}\right)$, μ is the order of the differential equation.

Given a predictor equation in the form

$$\boldsymbol{Y}_m^{(0)} = \boldsymbol{e}y_n + h^{\mu}\boldsymbol{df}\left(y_n\right). \tag{3}$$

Putting(3) in (2) gives

$$\boldsymbol{Y}_m = \boldsymbol{e}y_n + h^{\mu}\boldsymbol{df}\left(y_n\right) + h^{\mu}\boldsymbol{bF}\left(\boldsymbol{e}y_n + h^{\mu}\boldsymbol{df}y_n\right). \tag{4}$$

Equation (4) is called a self starting block-predictor-corrector method because the prediction equation is gotten directly from the block formula as claimed by [11,12].

In this paper, we propose an order six block method with step length of four using the method proposed by [11] for the solution of third order ordinary differential equation.

2. Methodology

2.1. Derivation of the Continuous Coefficient

We consider monomial power series as our basis function in the form

$$y(x) = \sum_{j=0}^{(s+r+1)} a_j x^j. \tag{5}$$

The third derivative of (5) gives

$$y'''(x) = \sum_{j=3}^{(s+r+1)} j(j-1)(j-2)a_j x^{j-3}. \tag{6}$$

The solution to (1) is soughted on the partition π_N : $a = x_0 < x_1 < x_2 < \cdots < x_n < x_{n+1} < \cdots < x_N = b$ within the integration interval $[a,b]$ with constant step length h given as $h = x_{n+i} - x_n, i = 0(1)N$.

Substituting (6) into (1) gives

$$f\left(x, y(x), y'(x), y''(x)\right) = \sum_{j=3}^{(s+r+1)} j(j-1)(j-2) a_j x^{j-3}. \tag{7}$$

Interpolating (5) at $x_{n+i}, j = 1(1)3$; collocating (7) $x_{n+i}, j = 0(1)4$, gives a system of equations

$$\sum_{j=0}^{(s+r+1)} a_j x^j = y_{n+s}. \tag{8}$$

$$\sum_{j=3}^{(s+r+1)} j(j-1)(j-2) a_j x^{j-3} = f_{n+r}. \tag{9}$$

Solving (8) and (9) for a_j's and substituting back into (5) gives a LMM with continuous coefficients of the form

$$\sum_{j=1}^{3} \alpha_j^{(t)} y_{n+1} = h^3 \sum_{j=0}^{4} \beta_j^{(t)} f_{n+j}. \tag{10}$$

where a_j's and β_j's are given as

$$\alpha_1(t) = \frac{1}{2}\left(t^2 + t\right); \alpha_2(t) = -\left(t^2 + 2t\right);$$

$$\alpha_3(t) = \frac{1}{2}\left(t^2 + 3t + 2\right);$$

$$\beta_0(t) = \frac{1}{10080}\left(2t^7 + 7t^6 - 11t^5 - 35t^4 + 49t^2 + 26t\right);$$

$$\beta_1(t) = \frac{1}{5040}\left(4t^7 + 21t^6 - 14t^5 - 10t^4 + 126t^2 + 52t\right);$$

$$\beta_2(t) = \frac{1}{1680}\left(2t^7 + 14t^6 + 7t^5 - 105t^4 + 532t^2 + 432t\right);$$

$$\beta_3(t) = \frac{1}{5040} \cdot \left(4t^7 + 35t^6 + 70t^5 - 175t^4 - 840t^3 - 1078t^2 + 452t\right);$$

$$\beta_4(t) = \frac{1}{10080}\left(2t^7 + 21t^6 + 77t^5 + 105t^4 - 105t^3 - 58t\right).$$

where $t = \dfrac{x - x_{n+3}}{h}$.

2.2. Derivation of the Block Method

The general block formula proposed by [6] in the normalized form is given by

$$\boldsymbol{A}^{(0)} \boldsymbol{Y}_m = \boldsymbol{e} y_n + h^{\mu-\lambda} \boldsymbol{df}\left(y_n\right) + h^{\mu-\lambda} \boldsymbol{bF}\left(y_m\right). \tag{11}$$

Evaluating (10) at $x = x_{n+1}, i = 0,4$; the first and second derivative at $x_{n+i}, i = 0(1)4$; and substituting into (10) gives the coefficient of (11) as

$$\boldsymbol{d} = \begin{bmatrix} \frac{113}{1120} & \frac{331}{630} & \frac{1431}{1120} & \frac{248}{105} & \frac{367}{1440} & \frac{53}{90} & \frac{147}{160} & \frac{56}{45} & \frac{251}{720} & \frac{29}{90} & \frac{25}{80} & \frac{14}{45} \end{bmatrix}^{T},$$

$$\boldsymbol{e} = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 3 & 4 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 2 & \frac{9}{2} & 8 & 1 & 2 & 3 & 4 & 1 & 1 & 1 & 1 \end{bmatrix}^{T},$$

$A^0 = 12 \times 12$ identity matrix,

$$b = \begin{bmatrix} \frac{107}{1008} & \frac{332}{315} & \frac{1863}{560} & \frac{2176}{315} & \frac{3}{8} & \frac{8}{5} & \frac{117}{40} & \frac{64}{15} & \frac{232}{360} & \frac{62}{45} & \frac{51}{40} & \frac{64}{45} \\ -\frac{103}{1680} & -\frac{8}{21} & -\frac{243}{560} & \frac{32}{105} & -\frac{47}{240} & -\frac{1}{3} & \frac{27}{80} & \frac{16}{15} & -\frac{11}{30} & \frac{4}{15} & \frac{9}{10} & \frac{8}{15} \\ \frac{43}{168} & \frac{52}{315} & \frac{45}{112} & \frac{128}{105} & \frac{39}{360} & \frac{8}{45} & \frac{3}{8} & \frac{64}{45} & \frac{53}{360} & \frac{2}{45} & \frac{21}{40} & \frac{64}{45} \\ -\frac{47}{10080} & \frac{19}{630} & -\frac{81}{1120} & -\frac{8}{63} & -\frac{7}{480} & -\frac{1}{30} & -\frac{1}{30} & -\frac{9}{160} & 0 & -\frac{19}{720} & -\frac{7}{120} & \frac{14}{45} \end{bmatrix}^{T}.$$

3. Analysis of the Properties of the Block

3.1. Order of the Method

We define a linear operator on the block (11) to give

$$\mathcal{L}\left\{y(x) : h\right\} = \boldsymbol{Y}_m - \boldsymbol{e} y_m + h^{\mu-\lambda} \boldsymbol{df}\left(y_m\right) + h^{\mu-\lambda} \boldsymbol{bF}\left(y_m\right) \tag{12}$$

Expanding $y\left(x_n + ih\right)$ and $f\left(x_n + jh\right)$ in Taylor series , (12) gives

$$\mathcal{L}\left\{y(x) : h\right\} = C_0 y(x) + C_1 h y'(x) + C_2 h^2 y''(x) + \cdots + C_p h^p y^p(x) \tag{13}$$

The block (11) and associated linear operator are said to have order p if $C_0 = C_1 = \cdots = C_{p+1} = 0, C_{p+2} \neq 0$.

The term C_{p+2} is called the error constant and implies that the local truncation error for the block is given by

$$t_{n+k} = C_{p+2} h^{(p+2)} y^{(p+2)}(x_n) + 0\left(h^{p+3}\right) \qquad (14)$$

Hence the block (11) has order 6, $0\left(h^{6+2}\right)$ with error constant

$$C_{p+2} = \left[\frac{139}{40320} \quad \frac{1}{45} \quad \frac{243}{4480} \quad \frac{32}{315} \quad \frac{107}{10080} \quad \frac{8}{315} \quad \frac{9}{224} \quad \frac{16}{315} \quad \frac{3}{160} \quad \frac{1}{90} \quad \frac{3}{160} \quad -\frac{8}{945}\right]$$

3.2. Zero Stability of the Block

The block (11) is said to be zero stable if the roots $z_s = 1, 2, \cdots, N$ of the characteristic polynomial $\rho(z) = \det(zA - E)$, satisfies $|z| \le 1$ and the root $|z| = 1$ has multiplicity not exceeding the order of the differential equation. moreover as $h^{\mu} \to 0, \rho(z) = z^{r-\mu}(\lambda - 1)^{\mu}$, where μ is the order of the differential equation, $r = \dim\left(A^{(0)}\right)$

For the block (11), $r = 12, \mu = 3$

$$\rho(z) = \lambda^{9}(\lambda - 1)^{3}$$

Hence our method is zero stable.

3.3. Convergence

A method is said to be convergent if it is zero stable and has order $p \ge 1$

From the theorem above, our method is convergent.

4. Numerical Experiments

4.1. Test Problem

We test our schemes with third order initial value problems:

Problem 1. *Consider a special third order initial value problem*

$$y''' = 3\sin x$$

$$y(0) = 1, y'(0) = 0, y'''(0) = -20 \le x \le 1$$

Exact solution: $y(x) = 3\cos x + \left(\frac{x^2}{2}\right) - 2$

This problem was solved by [13] using self-starting predictor-corrector method for special third order differential equations where a scheme of order six was proposed.

Problem 2. *Consider a linear third order initial value problem*

$$y''' + y' = 0$$

$$y(0) = 1, y'(0) = 1, y'''(0) = 1, x \in [0,1]$$

Exact solution: $y(x) = 2(1 - \cos x) + \sin x$

This problem was solved by [14] where a method of order six was proposed. They adopted predictor corrector method in their implementation. Our result is shown in **Table 1**.

4.2. Numerical Results

The following notations are used in the table.

XVAL: Value of the independent variable where numerical value is taken

ERC: Exact result at XVAL

NRC: Numerical result of the new result at XVAL

ERR: Magnitude of error of the new result at XVAL

5. Discussion

We have proposed a new block method for solving third order initial value problem in this paper. It should be noted that the method performs better when the step-size is chosen within the stability interval. The **Tables 1** and **2** had shown our new method is more efficient in terms of accuracy when compared with the self starting predictor

Table 1. Showing result of problem 1, h = 0.01.

XVAL	ERC	NRC	NRC	ERR IN [13]
0.1	0.990012495834077	0.990012495834077	0.0000+00	9.992007(−16)
0.2	0.960199733523725	0.960199733523724	9.99200(−16)	7.660538(−15)
0.3	0.911009467376818	0.911009467376816	1.55431(−15)	2.287059(−14)
0.4	0.843182982008655	0.843182982008652	3.10862(−15)	5.906386(−14)
0.5	0.757747685671118	0.757747685671113	4.66293(−15)	1.153521(−13)
0.6	0.656006844729035	0.656006844729028	6.88338(−15)	1.982858(−13)
0.7	0.539526561853465	0.539526561853456	9.10382(−15)	3.127498(−13)
0.8	0.410120128041496	0.410120128041484	1.14908(−14)	4.635736(−13)
0.9	0.269829904811992	0.269829904811978	1.42108(−14)	6.542544(−13)
1.0	0.120906917604418	0.120906917604401	1.74582(−14)	8.885253(−13)

Table 2. Showing result of problem 2, h = 0.1.

XVAL	ERC	NRC	ERR	ERR IN [14]
0.1	0.004987516654767	0.004987518195317	1.54055(–09)	1.189947(–11)
0.2	0.019801063624459	0.019801073469968	9.84550(–09)	3.042207(–09)
0.3	0.043999572204435	0.043999595857285	2.36528(–08)	7.779556(–08)
0.4	0.076867491997406	0.076867535270603	4.32732(–08)	7.749556(–07)
0.5	0.117443317649723	0.117443356667842	3.90181(–08)	3.398961(–06)
0.6	0.164557921035623	0.164557928005710	6.97008(–08)	9.501398(–06)
0.7	0.216881160706204	0.216881108673223	5.20329(–08)	1.756558(–06)
0.8	0.272974910431491	0.272974775207245	1.35224(–07)	2.745889(–05)
0.9	0.331350392754953	0.331349917920840	4.74834(–07)	3.888082(–05)
1.0	0.390527531852589	0.390526462491195	1.06936(–06)	5.137153(–05)

corrector method proposed by [11] and [15]. It should be noted that this method performs better when the step size (h) is within the stability interval.

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