AN EXTENDED MULTISTEP SHANKS TRANSFORMATION AND CONVERGENCE ACCELERATION ALGORITHM WITH THEIR CONVERGENCE AND STABILITY ANALYSIS

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Abstract. The molecule solution of an extended discrete Lotka-Volterra equation is constructed, from which a new sequence transformation is proposed. A convergence acceleration algorithm for implementing this sequence transformation is found. It is proved that our new sequence transformation accelerates some kinds of linear sequences and factorial sequences with good numerical stability. Some numerical examples are also presented.

Key words. convergence acceleration algorithm, integrable system, molecule solution, convergence, stability

1. Introduction. Some intimate relations between certain numerical algorithms and integrable systems have been revealed in recent years, which lead us to reinvestigate both objects from a novel viewpoint.

On the one hand, behind many algorithms in numerical analysis, there imply a variety of interesting dynamical behaviours [16]. One of the intriguing properties is *integrability*, which distinguishes these numerical schemes from others, and attracts experts on integrable systems to study integrable properties of numerical algorithms. In the literature, integrability appears in various guises such as properties of *invariance*, *compatibility* and *identity*. For example, Guass' arithmetic-geometric mean algorithm [3, 5, 28] for computing elliptic integral of the first kind is a discrete-time integrable system with the corresponding elliptic integral as its conserved quantity, which can be viewed as an application of invariance. In addition, compatibility condition of the spectral problem related to the discrete-time Toda equation [20] is nothing but the *qd*-algorithm [19, 34], which plays a significant role in the theory of formal orthogonal polynomials and Padé approximants [4, 8, 31]. For more examples, please consult [25, 26, 27, 32, 42].

On the other hand, some integrable equations can lead to new algorithms. For instance, the discrete Lotka-Volterra equation can be used as an efficient algorithm to compute singular values [23, 24, 43]. Moreover, identity property of integrable systems indicates that, essentially, integrable systems are some kind of determinantal (or pfaffian) identities at so-called τ-function level [35, 21], from which two new convergence acceleration algorithms have been constructed in [12, 18]. Based on the observation that the integrable equation provided by the new algorithm given in [12] is only a special case of the extended Lotka-Volterra equation which was first proposed in [29] (more results in [22]), it is natural to consider whether new convergence acceleration algorithms may be obtained from other cases. This is what we want to do in this article.

Convergence acceleration algorithm is a kind of important numerical algorithm, which is used to accelerate the convergence of a given sequence. In numerical analysis, many methods produce sequences, for example iterative methods, perturbation methods, discretization methods and so on. Sometimes, the convergence of these sequences is so slow as to make the corresponding numerical methods ineffective in

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practice. This is why we study *sequence transformations*, which are based on the idea of *extrapolation* [13, 41]. Let (S_n) be a sequence converging to a limit S, satisfying

$$\lim_{n\to} \frac{S_{n+1}-S}{S_n-S} = \lambda.$$

When $-1 \leq \lambda < 1$ and $\lambda \neq 0$, we say that the sequence (S_n) converges *linearly*; when $\lambda = 1$, we say that this sequence converges *logarithmically*; and when $\lambda = 0$, it is said to be *hyperlinearly* convergent. A sequence transformation $T : (S_n) \to (T_n)$, transforms this sequence to a new sequence (T_n), which converges faster to the same limit S under some assumptions, that is,

$$\lim_{n\to\infty} \frac{T_n-S}{S_n-S} = 0.$$

There are many sequence transformations (see e.g. [9, 10, 11] and the reference therein), among which the most well known is Aitken's Δ^2 process due to Aitken [1], who used it to accelerate the convergence of Bernoulli's method for computing the dominant zero of a polynomial. Furthermore, Pennacchi [33] considered transformations of the form

$$T_n = S_n + \frac{P_m(\Delta S_n, \ldots, \Delta S_{n+p-1})}{Q_{m-1}(\Delta S_n, \ldots, \Delta S_{n+p-1})}, \tag{1.1}$$

where P_m and Q_{m-1} are homogeneous polynomials of degree m and $m-1$ respectively, p and m are positive integers. Such a transformation is called *rational transformation of type* (p, m), denoted by $C_n(p, m)$. In this sense, Aitken's Δ^2 process is a rational transformation of type $(2, 2)$, and Pennacchi proved that any rational transformation of type $(2, m)$ with $m \geq 2$ which accelerates the set of linear converging sequences is equivalent to Aitken's process. He also gave a rational transformation of type $(3, 2)$

$$C_n(3,2) = S_n + \frac{\Delta S_n(\Delta S_n - \Delta S_{n+1}) + (\Delta S_n \Delta S_{n+2} - \Delta S_{n+1}^2)}{\Delta S_n - 2\Delta S_{n+1} + \Delta S_{n+2}}, \tag{1.2}$$

which accelerates the set of linear converging sequences.

For many sequence transformations, new sequences can be expressed as ratios of two determinants. By using some determinantal identities, we can obtain *recursive algorithm* for implementing the corresponding sequence transformation, such an algorithm is called *extrapolation algorithm*, or *convergence acceleration algorithm*. So far, many convergence acceleration algorithms have been found and investigated, such as the famous ε-algorithm proposed by Wynn [46], and some of its generalizations [7, 15]. For more results, please refer to [13, 41, 44, 45].

Then we return to the extended Lotka-Volterra equation, which is expressed as

$$\frac{d}{dt}\left(\prod_{i=0}^{q-1} a_{k-\frac{q-1}{2}+i}\right) = \prod_{i=0}^{N-1} a_{k-\frac{q-1}{2}+i} - \prod_{i=0}^{N-1} a_{k+\frac{q-1}{2}-i}, \quad q, N = 1, 2, \ldots, q \neq N, \tag{1.3}$$

or

$$\frac{d}{dt}\left(\prod_{i=0}^{q-1} a_{k-\frac{q-1}{2}+i}\right) = \prod_{i=0}^{-N-1} a^{-1}_{k-\frac{q+1}{2}-i} - \prod_{i=0}^{-N-1} a^{-1}_{k+\frac{q+1}{2}+i}, \quad q, -N = 1, 2, \ldots. \tag{1.4}$$

In [12], a new convergence acceleration algorithm was obtained from the discretization of (1.4) when $N=-1$. Now we consider equation (1.3), with $N=q+1$. In this case, it can be written as

$$\frac{d}{dt}\left(\prod_{i=0}^{q-1} a_{k+i}\right)=\left(\prod_{i=0}^{q-1} a_{k+i}\right)(a_{k+q}-a_{k-1}),$$

with the following difference equation as its time discretization:

$$\begin{aligned}&\left(\prod_{m=0}^{M_{k-1}}\frac{1+a_{k-mq-1}^{(n+mp+p+1)}}{1+a_{k-mq-1}^{(n+mp+1)}}\right)(1+a_{k-1}^{(n+1)})\prod_{i=0}^{q-1}a_{k+i}^{(n+1)}\\&\quad=\left(\prod_{m=0}^{M_k}\frac{1+a_{k-mq}^{(n+mp+p)}}{1+a_{k-mq}^{(n+mp)}}\right)(1+a_{k+q}^{(n)})\prod_{i=0}^{q-1}a_{k+i}^{(n)},\end{aligned}\tag{1.5}$$

while $p=0,1,\ldots,$ and the nonnegative integer M_k is defined as $M_k=\left[\frac{k}{q}\right]+1$, where $[x]$ stands for the greatest integer not exceeding x.

In this article, we first derive the bilinear form of the discrete equation (1.5), and then construct its molecule solution, from which we obtain a new sequence transformation. We also show that there exists a two-dimensional difference equation, which shares the same bilinear form with equation and can be used as a recursive algorithm for the implementation of the new sequence transformation.

Our article is organized as follows: In section 2, we will derive the molecule solution of equation (1.5) with the help of bilinear method and determinantal identities. In section 3, a new sequence transformation is constructed, and also its corresponding recursive algorithm. In section 4, we will give the convergence and stability analysis of the new sequence transformation. In section 5, some numerical examples are proposed. Section 6 is devoted to conclusion and discussions.

2. Molecule solution of equation (1.5)**.** In this section, we construct the molecule solution of the extended discrete Lotka-Volterra equation by using Hirota's bilinear method (which was invented by Hirota [21] for resolving integrable nonlinear differential or difference evolution equations having soliton solutions) and determinantal identities [2, 14].

It can be proved that under the dependent variable transformation

$$a_k^{(n)}=\frac{f_{k-1}^{(n+p+1)}f_{k+q+1}^{(n)}}{f_k^{(n+p+1)}f_{k+q}^{(n)}},\tag{2.1}$$

with $f_k^{(n)}$ satisfying initial conditions $f_{-q}^{(n)}=\cdots=f_0^{(n)}\equiv 1$, the extended discrete Lotka-Volterra equation (1.5) could be transformed into the following bilinear form

$$f_{k+q}^{(n+1)}f_k^{(n+p)}-f_k^{(n+p+1)}f_{k+q}^{(n)}=f_{k+q+1}^{(n)}f_{k-1}^{(n+p+1)},\ k=-q+1,-q+2,\ldots.\tag{2.2}$$

We now introduce an intermediate bilinear variable $g_k^{(n)}$, and give a class of bi-

linear equations

$$f_{mT+i}^{(n)}g_{mT+i}^{(n+1)} - f_{mT+i}^{(n+1)}g_{mT+i}^{(n)} = f_{mT+i-1}^{(n+1)}f_{mT+i+1}^{(n)}, \tag{2.3}$$

$$f_{mT+1}^{(n)}g_{mT+1}^{(n+1)} - f_{mT+1}^{(n+1)}g_{mT+1}^{(n)} = -f_{mT}^{(n+1)}f_{mT+2}^{(n)}, \tag{2.4}$$

$$f_{(m+1)T+i}^{(n)}g_{mT+i}^{(n+p+1)} - f_{mT+i}^{(n+p+1)}g_{(m+1)T+i}^{(n)} = f_{mT+i+1}^{(n+p)}f_{mT+i+q}^{(n+1)}, \tag{2.5}$$

$$f_{(m+1)T+1}^{(n)}g_{mT+1}^{(n+p+1)} - f_{mT+1}^{(n+p+1)}g_{(m+1)T+1}^{(n)} = -f_{mT+2}^{(n+p)}f_{(m+1)T}^{(n+1)}, \tag{2.6}$$

which can yield (2.2) by eliminating $g_k^{(n)}$, where m is an arbitrary integer, $T = q + 1$ and $i = 2, \ldots, T$. In order to get the molecule solution of equation (1.5), we only need to study bilinear equations (2.3)-(2.6) instead, whose initial conditions are given by

$$f_{-q}^{(n)} = \cdots = f_0^{(n)} = 1, \tag{2.7}$$

$$g_{-q}^{(n)} = 0, g_{-q+1}^{(n)} = \cdots = g_{-1}^{(n)} = n, g_0^{(n)} = S_n. \tag{2.8}$$

In fact, if we set

$$\Psi_m^{(p,q)}(v_n) = \begin{vmatrix} v_{n+(m-1)p} & v_{n+(m-1)p+1} & \cdots & v_{n+(m-1)(p+1)} \\ \Delta^q v_{n+(m-2)p} & \Delta^q v_{n+(m-2)p+1} & \cdots & \Delta^q v_{n+(m-2)p+m-1} \\ \vdots & \vdots & & \vdots \\ \Delta^{(m-1)q} v_n & \Delta^{(m-1)q} v_{n+1} & \cdots & \Delta^{(m-1)q} v_{n+m-1} \end{vmatrix},$$

$$\Psi_{-1}^{(p,q)}(v_n) = 0, \quad \Psi_0^{(p,q)}(v_n) = 1,$$

$$\Phi_m^{(p,q)}(v_n) = \begin{vmatrix} n+(m-1)p & n+(m-1)p+1 & \cdots & n+(m-1)(p+1) \\ v_{n+(m-2)p} & v_{n+(m-2)p+1} & \cdots & v_{n+(m-2)p+m+1} \\ \vdots & \vdots & & \vdots \\ \Delta^{(m-2)q} v_n & \Delta^{(m-2)q} v_{n+1} & \cdots & \Delta^{(m-2)q} v_{n+m-1} \end{vmatrix},$$

$$\Phi_{-1}^{(p,q)}(v_n) = 0, \quad \Phi_0^{(p,q)}(v_n) = 1,$$

then we have the following theorem.

Theorem 1 The molecule solution to bilinear equations (2.3)-(2.6) with intitial conditions (2.7)-(2.8) can be expressed as

$$f_{mT+i}^{(n)} = \Psi_{m+1}(\Delta^i S_n), i = 1, \ldots, q+1,$$
$$g_{mT}^{(n)} = \Psi_{m+1}(S_n), g_{mT+1}^{(n)} = \Psi_m(\Delta^{q+2} S_n),$$
$$g_{mT+i}^{(n)} = \Phi_{m+2}(\Delta^{i-1} S_n), i = 2, \ldots, q.$$

where the upper index (p, q) has been omitted without confusion.

Proof: Firstly, we prove equations (2.3) and (2.4), which are equivalent to the following identities:

$$\Phi_{m+2}(\Delta^{i-1}S_{n+1})\Psi_{m+1}(\Delta^i S_n) - \Phi_{m+2}(\Delta^{i-1}S_n)\Psi_{m+1}(\Delta^i S_{n+1})$$
$$= \Psi_{m+1}(\Delta^{i-1}S_{n+1})\Psi_{m+1}(\Delta^{i+1}S_n), \quad i = 2, \ldots, q, \tag{2.9}$$

$$\Psi_{m+1}(S_{n+1})\Psi_m(\Delta^{q+1}S_n) - \Psi_{m+1}(S_n)\Psi_m(\Delta^{q+1}S_{n+1})$$
$$= \Psi_m(\Delta^q S_{n+1})\Psi_{m+1}(\Delta S_n), \tag{2.10}$$

$$\Psi_{m+1}(\Delta S_n)\Psi_m(\Delta^{q+2}S_{n+1}) - \Psi_{m+1}(\Delta S_{n+1})\Psi_m(\Delta^{q+2}S_n)$$
$$= -\Psi_m(\Delta^{q+1}S_{n+1})\Psi_{m+1}(\Delta^2 S_n). \tag{2.11}$$

Since (2.11) can be obtained in a similar way to (2.10), here we only prove (2.9) and (2.10).

Set

$$D_1 = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ S_{n+mp} & S_{n+mp+1} & \cdots & S_{n+mp+m+1} \\ \vdots & \vdots & & \vdots \\ \Delta^{mq} S_n & \Delta^{mq} S_{n+1} & \cdots & \Delta^{mq} S_{n+m+1} \end{vmatrix},$$

$$D_2 = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ n+(m+1)p & n+(m+1)p+1 & \cdots & n+(m+1)p+m+2 \\ \Delta^{i-1} S_{n+mp} & \Delta^{i-1} S_{n+mp+1} & \cdots & \Delta^{i-1} S_{n+mp+m+2} \\ \vdots & \vdots & & \vdots \\ \Delta^{i-1+mq} S_n & \Delta^{i-1+mq} S_{n+1} & \cdots & \Delta^{i-1+mq} S_{n+m+2} \end{vmatrix}.$$

Applying Jacobi identity [2, 14]

$$D \cdot D \begin{bmatrix} i_1 & i_2 \\ j_1 & j_2 \end{bmatrix} = D \begin{bmatrix} i_1 \\ j_1 \end{bmatrix} \cdot D \begin{bmatrix} i_2 \\ j_2 \end{bmatrix} - D \begin{bmatrix} i_1 \\ j_2 \end{bmatrix} \cdot D \begin{bmatrix} i_2 \\ j_1 \end{bmatrix}, \tag{2.12}$$

where $D \begin{bmatrix} i_1 & \cdots & i_n \\ j_1 & \cdots & j_n \end{bmatrix}$ denotes the determinant with the $i_1 < \cdots < i_n$ *-th* rows and the $j_1 < \cdots < j_n$ *-th* columns removed from the original determinant D, to D_1 and D_2, and noticing that

$$\begin{aligned}
&D_1 = \Psi_{m+1}(\Delta S_n), \quad D_1 \begin{bmatrix} 1 & 2 \\ 1 & m+2 \end{bmatrix} = \Psi_m(\Delta^q S_{n+1}), \\
&D_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \Psi_{m+1}(S_{n+1}), \quad D_1 \begin{bmatrix} 2 \\ m+2 \end{bmatrix} = \Psi_m(\Delta^{q+1} S_n), \\
&D_1 \begin{bmatrix} 1 \\ m+2 \end{bmatrix} = \Psi_{m+1}(S_n), \quad D_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \Psi_m(\Delta^{q+1} S_{n+1}), \\
&D_2 = \Psi_{m+1}(\Delta^{i+1} S_n), \quad D_2 \begin{bmatrix} 1 & 2 \\ 1 & m+3 \end{bmatrix} = \Psi_{m+1}(\Delta^{i-1} S_{n+1}), \\
&D_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \Phi_{m+2}(\Delta^{i-1} S_{n+1}), \quad D_2 \begin{bmatrix} 2 \\ m+3 \end{bmatrix} = \Psi_{m+1}(\Delta^i S_n), \\
&D_2 \begin{bmatrix} 1 \\ m+3 \end{bmatrix} = \Phi_{m+2}(\Delta^{i-1} S_n), \quad D_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \Psi_{m+1}(\Delta^i S_{n+1}),
\end{aligned}$$

then we get (2.9)(2.10) immediately.

Next, we consider equations (2.5) and (2.6), which are equivalent to

$$\begin{aligned}
&\Psi_{m+1}(\Delta^i S_n)\Phi_{m+1}(\Delta^{i-1} S_{n+p+1}) - \Psi_m(\Delta^i S_{n+p+1})\Phi_{m+2}(\Delta^{i-1} S_n) \\
&= \Psi_m(\Delta^{i+1} S_{n+p})\Psi_{m+1}(\Delta^{i-1} S_{n+1}), \quad i = 2, \ldots, q, \qquad (2.13) \\
&\Psi_m(S_{n+p+1})\Psi_m(\Delta^{q+1} S_n) - \Psi_{m+1}(S_n)\Psi_{m-1}(\Delta^{q+1} S_{n+p+1}) \\
&= \Psi_m(\Delta S_{n+p})\Psi_m(\Delta^q S_{n+1}), \qquad (2.14) \\
&\Psi_m(\Delta S_{n+p+1})\Psi_m(\Delta^{q+2} S_n) - \Psi_{m+1}(\Delta S_n)\Psi_{m-1}(\Delta^{q+2} S_{n+p+1}) \\
&= \Psi_m(\Delta^2 S_{n+p})\Psi_m(\Delta^{q+1} S_{n+1}). \qquad (2.15)
\end{aligned}$$

We use Jacobi identity (2.12) to show the validity of (2.13) and (2.14). Set

$$D_3 = \begin{vmatrix} 1 & 1 & \cdots & 1 & 0 \\ n+mp & n+mp+1 & \cdots & n+mp+m & 0 \\ \Delta^i S_{n+(m-1)p} & \Delta^i S_{n+(m-1)p+1} & \cdots & \Delta^i S_{n+(m-1)p+m} & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ \Delta^{i+(m-2)q} S_{n+p} & \Delta^{i+(m-2)q} S_{n+p+1} & \cdots & \Delta^{i+(m-2)q} S_{n+p+m} & 0 \\ \Delta^{i+(m-1)q} S_n & \Delta^{i+(m-1)q} S_{n+1} & \cdots & \Delta^{i+(m-1)q} S_{n+m} & 1 \end{vmatrix},$$

$$D_4 = \begin{vmatrix} 1 & 1 & \cdots & 1 & 0 \\ S_{n+mp} & S_{n+mp+1} & \cdots & S_{n+mp+m} & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ \Delta^{(m-1)q} S_{n+p} & \Delta^{(m-1)q} S_{n+p+1} & \cdots & \Delta^{(m-1)q} S_{n+p+m} & 0 \\ \Delta^{mq} S_n & \Delta^{mq} S_{n+1} & \cdots & \Delta^{mq} S_{n+m} & 1 \end{vmatrix},$$

we have the following relations

$$\begin{aligned}
&D_3 = \Psi_{m-1}(\Delta^{i+2} S_{n+p}), \quad D_3 \begin{bmatrix} 1 & 2 \\ 1 & m+2 \end{bmatrix} = \Psi_m(\Delta^i S_{n+1}), \\
&D_3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \Phi_m(\Delta^i S_{n+p+1}), \quad D_3 \begin{bmatrix} 2 \\ m+2 \end{bmatrix} = \Psi_m(\Delta^{i+1} S_n), \\
&D_3 \begin{bmatrix} 1 \\ m+2 \end{bmatrix} = \Phi_{m+1}(\Delta^i S_n), \quad D_3 \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \Psi_{m-1}(\Delta^{i+1} S_{n+p+1}), \\
&D_4 = \Psi_m(\Delta S_{n+p}), \quad D_4 \begin{bmatrix} 1 & 2 \\ 1 & m+2 \end{bmatrix} = \Psi_m(\Delta^q S_{n+1}), \\
&D_4 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \Psi_m(S_{n+p+1}), \quad D_4 \begin{bmatrix} 2 \\ m+2 \end{bmatrix} = \Psi_m(\Delta^{q+1} S_n), \\
&D_4 \begin{bmatrix} 1 \\ m+2 \end{bmatrix} = \Psi_{m+1}(S_n), \quad D_4 \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \Psi_{m-1}(\Delta^{q+1} S_{n+p+1}).
\end{aligned}$$

Then equations (2.13) and (2.14) are obtained by applying Jacobi identity (2.12) to D_3 and D_4 (with $i_1 = 1, i_2 = 2\ ; j_1 = 1, j_2 = m+2$), respectively.

The proof of (2.15) is nearly the same as that of (2.14), thus we omit it.

Consequently, equations (2.3)–(2.6) hold, which complete the proof. □

From Theorem 1 and the dependent variable transformation (2.1), we obtain the molecule solution of (1.5) immediately.

3. A new sequence transformation and the corresponding recursive algorithm. In this section, we construct a new sequence transformation related to the molecule solution given by Theorem 1, and derive a convergence acceleration algorithm for its implementation.

Let us consider a new sequence transformation defined by

$$T_k^{(p,q)}(S_n) = \frac{\begin{vmatrix} S_{n+kp} & S_{n+kp+1} & \cdots & S_{n+k(p+1)} \\ \Delta^q S_{n+(k-1)p} & \Delta^q S_{n+(k-1)p+1} & \cdots & \Delta^q S_{n+(k-1)p+k} \\ \vdots & \vdots & & \vdots \\ \Delta^{kq} S_n & \Delta^{kq} S_{n+1} & \cdots & \Delta^{kq} S_{n+k} \end{vmatrix}}{\begin{vmatrix} 1 & 1 & \cdots & 1 \\ \Delta^q S_{n+(k-1)p} & \Delta^q S_{n+(k-1)p+1} & \cdots & \Delta^q S_{n+(k-1)p+k} \\ \vdots & \vdots & & \vdots \\ \Delta^{kq} S_n & \Delta^{kq} S_{n+1} & \cdots & \Delta^{kq} S_{n+k} \end{vmatrix}}, \quad (3.1)$$

where p and q are nonnegative integers satisfying $p \leq q$, $k = 0, 1, \ldots$. It is obvious that when $k = 1, p = 1$ and $q = 2$, (3.1) is nothing but the rational transformation $C_n(3, 2)$ given by (1.2), and when $p = 0$, it is equivalent to the multistep Shanks' transformation proposed in [12]. Thus, (3.1) is an extension of the both. We mention that $T_k^{(p,q)}(S_n)$ can also be expressed as

$$T_k^{(p,q)}(S_n) = \frac{\Psi_{k+1}^{(p,q)}(S_n)}{\Psi_k^{(p,q)}(\Delta^{q+1} S_n)} = \frac{g_{kT}^{(n)}}{f_{kT}^{(n)}},$$

which motivate us to implement the dependent variable transformation

$$u_k^{(n)} = \frac{g_k^{(n)}}{f_k^{(n)}} \quad (3.2)$$

to bilinear equations (2.3)-(2.6) to see whether there exists a recursive relation satisfied by $u_k^{(n)}$. In fact, we have the following theorem.

Theorem 2 If $g_k^{(n)}$ and $f_k^{(n)}$ satisfy bilinear equations (2.3)–(2.6), then $u_k^{(n)}$ defined by (3.2) can be computed recursively:

$$u_{k+1}^{(n)} = u_{k-q}^{(n+p+1)} - \frac{(u_{k-q-1}^{(n+p+1)} - u_k^{(n)})(u_{k-q}^{(n+p+1)} - u_{k-q}^{(n+p)})}{u_k^{(n+1)} - u_k^{(n)}}, \; k = 1, 2, \ldots, \quad (3.3)$$

with the initial values

$$u_{-q}^{(n)} = 0, u_{-q+1}^{(n)} = \cdots = u_{-1}^{(n)} = n, u_0^{(n)} = S_n, u_1^{(n)} = \frac{1}{\Delta S_n}. \quad (3.4)$$

Proof: It is obvious that the initial conditions (3.4) can be obtained directly from the dependent variable transformation (3.2) and Theorem 1. Thus, we only need to prove equation (3.3), which is equivalent to the following identity

$$(u_{k+q}^{(n+1)} - u_{k+q}^{(n)})(u_k^{(n+p+1)} - u_{k+q+1}^{(n+1)}) = (u_{k-1}^{(n+p+1)} - u_{k+q}^{(n)})(u_k^{(n+p+1)} - u_k^{(n+p)}) \quad (3.5)$$

From the dependent variable transformation (3.2) and equations (2.3)–(2.6), we obtain

$$u_k^{(n+1)} - u_k^{(n)} = \frac{f_{k-1}^{(n+1)} f_{k+1}^{(n)}}{f_k^{(n+1)} f_k^{(n)}}, \quad (3.6)$$

$$u_k^{(n+p+1)} - u_{k+q+1}^{(n)} = \frac{f_{k+1}^{(n+p)} f_{k+q}^{(n+1)}}{f_k^{(n+p+1)} f_{k+q+1}^{(n)}}, \quad (3.7)$$

where $k \neq mT+1$, and when $k = mT+1$, it only needs to change the sign of the right hand side of (3.6) and (3.7).

Consider the case when $k = mT+i, i = 3, \ldots, q+1$, in (3.5). We have

$$\begin{aligned}
&(u_{k+q}^{(n+1)} - u_{k+q}^{(n)})(u_k^{(n+p+1)} - u_{k+q+1}^{(n+1)}) \\
&= \frac{f_{k+q-1}^{(n+1)} f_{k+q+1}^{(n)}}{f_{k+q}^{(n+1)} f_{k+q}^{(n)}} \cdot \frac{f_{k+1}^{(n+p)} f_{k+q}^{(n+1)}}{f_k^{(n+p+1)} f_{k+q+1}^{(n)}} = \frac{f_{k+q-1}^{(n+1)} f_{k+1}^{(n+p)}}{f_k^{(n+p+1)} f_{k+q}^{(n)}}, \\
&(u_{k-1}^{(n+p+1)} - u_{k+q}^{(n)})(u_k^{(n+p+1)} - u_k^{(n+p)}) \\
&= \frac{f_k^{(n+p)} f_{k+q-1}^{(n+1)}}{f_{k-1}^{(n+p+1)} f_{k+q}^{(n)}} \cdot \frac{f_{k-1}^{(n+p+1)} f_{k+1}^{(n+p)}}{f_k^{(n+p+1)} f_k^{(n+p)}} = \frac{f_{k+q-1}^{(n+1)} f_{k+1}^{(n+p)}}{f_k^{(n+p+1)} f_{k+q}^{(n)}},
\end{aligned}$$

which shows that (3.5) holds when $k = mT+i, i = 3, \ldots, q+1$. The proofs of this identity when $k = mT+1, mT+2$ are nearly the same, and thus be omitted.

Consequently, (3.5) holds for all $k \in \mathbb{N}$, which leads to the validity of (3.3). Then we complete the proof. □

It is obvious that $u_{kT}^{(n)}$ is nothing but the new sequence transformation (3.1). Thus, according to Theorem 2, $T_k^{(p,q)} : (S_n) \to (u_{kT}^{(n)})$ can be implemented via (3.3) with initial values (3.4). In other words, (3.3) together with (3.4) can be viewed as a convergence acceleration algorithm corresponding to sequence transformation $T_k^{(p,q)}$.

Since transformation (3.1) can be regarded as an extension of the multistep Shanks' transformation, it is natural to investigate the relationship between their corresponding recursive algorithms. In fact, the following corollary shows that the multistep ε-algorithm [12] is just a special case of our new algorithm.

Corollary 3 If we set $p = 0$ in the new algorithm (3.3), then it can be reduced to the multistep ε-algorithm.

Proof: In this case, (3.3) is written as

$$\left(u_{k+1}^{(n)} - u_{k-q}^{(n+1)}\right)\left(u_k^{(n+1)} - u_k^{(n)}\right) = \left(u_k^{(n)} - u_{k-q-1}^{(n+1)}\right)\left(u_{k-q}^{(n+1)} - u_{k-q}^{(n)}\right).$$

Multiplying both sides of the above equation by $\prod_{i=1}^{q-1}\left(u_{k-i}^{(n+1)} - u_{k-i}^{(n)}\right)$, we obtain

$$\left(u_{k+1}^{(n)} - u_{k-q}^{(n+1)}\right)\prod_{i=0}^{q-1}\left(u_{k-i}^{(n+1)} - u_{k-i}^{(n)}\right) = \left(u_k^{(n)} - u_{k-q-1}^{(n+1)}\right)\prod_{i=1}^{q}\left(u_{k-i}^{(n+1)} - u_{k-i}^{(n)}\right),$$

which can be simplified further yielding

$$u_{k+1}^{(n)} = u_{k-q}^{(n+1)} + \frac{1}{\prod_{i=0}^{q-1}\left(u_{k-i}^{(n+1)} - u_{k-i}^{(n)}\right)}.$$

This formula is nothing but the multistep ε-algorithm, corresponding to $m = q$. Thus completing the proof. □

As the end of this section, we give the kernel of the new sequence transformation, that is the set of sequences which would be transformed into a constant.

Theorem 4 A necessary and sufficient condition for all $n > N$, $T_k^{(p,q)}(S_n) = S$ is that for $\forall n > N$,

$$S_{n+kp} = S + a_1\Delta^q S_{n+(k-1)p} + \cdots + a_k\Delta^{kq} S_n,$$

where $a_1, \ldots, a_k$ are constants independent of n and $a_k \neq 0$, $S = \lim_{n\to\infty} S_n$.

4. Convergence and stability analysis. The analysis of convergence and stability is an important topic in the theory of convergence acceleration. In this section, we consider the convergence and stability of the sequence transformation (3.1) as it is applied to the following three different sequences.

I: Logarithmic sequences for which

$$S_n \sim S + \sum_{i=0}^{\infty} \alpha_i n^{\gamma-i} \quad as \quad n \to \infty;\ \alpha_0 \neq 0, \gamma \neq 0, 1, 2 \ldots \quad . \tag{4.1}$$

II: Linear sequences for which

$$S_n \sim S + \xi^n \sum_{i=0}^{\infty} \alpha_i n^{\gamma-i} \quad as \quad n \to \infty;\ \alpha_0 \neq 0,\ \xi \neq 1. \tag{4.2}$$

III: Factorial sequences for which

$$S_n \sim S + (n!)^{-r}\xi^n \sum_{i=0}^{\infty} \alpha_i n^{\gamma-i} \quad as \quad n \to \infty; \alpha_0 \neq 0, r = 1, 2 \ldots \quad . \tag{4.3}$$

4.1. Convergence analysis. We give the following two lemmas, which are useful in the subsequent proofs. In fact, the first lemma is an obvious result of asymptotic expansions.

Lemma 5 Given (A_n), assume $A_n \sim \sum_{i=0}^{\infty} a_i n^{\gamma-i},\ as\ n \to \infty;\ a_0 \neq 0$, then

(i) if $\gamma \neq 0, \Delta A_n \sim \sum_{i=0}^{\infty} \widehat{a}_i n^{\gamma-i-1},\ as\ n \to \infty;\ \widehat{a}_0 = \gamma a_0 \neq 0$;

(ii) if $\gamma = 0, \Delta A_n \sim \sum_{i=\mu}^{\infty} \widehat{a}_i n^{-i-1},\ as\ n \to \infty;\ \widehat{a}_\mu = -\mu a_\mu \neq 0$ (a_μ is the first nonzero a_i with $i \geq 1$);

(iii) if $\xi \neq 1, \Delta^k(\xi^n A_n) \sim \xi^n \sum_{i=0}^{\infty} \widehat{a}_i n^{\gamma-i},\ as\ n \to \infty;\ \widehat{a}_0 = (\xi-1)^k a_0 \neq 0$;

(iv) if $r = 1, 2 \cdots, \Delta(\frac{\xi^n}{(n!)^r} A_n) \sim \frac{\xi^n}{(n!)^r} \sum_{i=0}^{\infty} \widehat{a}_i n^{\gamma-i},\ as\ n \to \infty;\ \widehat{a}_0 = -a_0 \neq 0$;

(v) if $r = 1, 2 \cdots, \Delta(\frac{(n!)^r}{\xi^n} A_n) \sim \frac{(n!)^r}{\xi^n} \sum_{i=0}^{\infty} \widehat{a}_i n^{\gamma+r-i},\ as\ n \to \infty;\ \widehat{a}_0 = \frac{1}{\xi} a_0 \neq 0$.

Lemma 6 If $u_k^{(n)}$ is computed by algorithm (3.3), then $u_{(k+1)T}^{(n)}$ can be expressed as

$$\begin{aligned} u_{(k+1)T}^{(n)} = \frac{1}{\prod_{i=1}^{q} \Delta u_{kT+i}^{(n)}} \Bigg\{ & u_{kT}^{(n+p+1)} \Delta u_{kT+1}^{(n)} \prod_{i=2}^{q} \Delta u_{kT+i}^{(n)} \\ & + u_{kT+1}^{(n)} \Delta u_{kT}^{(n+p)} \prod_{i=2}^{q} \Delta u_{(k-1)T+i}^{(n+p)} \\ & - \Delta u_{kT}^{(n+p)} u_{kT-q}^{(n+p+1)} \prod_{i=2}^{q} \Delta u_{(k-1)T+i}^{(n+p)} \Bigg\}, \end{aligned} \tag{4.4}$$

where the forward difference operator Δ is applied to superscripts.

Proof: Equation (3.3) can be rewritten as

$$u_{m+1}^{(n)} - u_{m-q}^{(n+p+1)} = \frac{\Delta u_{m-q}^{(n+p)}}{\Delta u_m^{(n)}}(u_m^{(n)} - u_{m-q-1}^{(n+p+1)}). \tag{4.5}$$

Multiplying together these equations for $m = (k+1)T-1, (k+1)T-2, \cdots kT+1$, we will obtain

$$u_{(k+1)T}^{(n)} - u_{kT}^{(n+p+1)} = \frac{\prod_{i=2}^{T} \Delta u_{(k-1)T+i}^{(n+p)}}{\prod_{i=1}^{q} \Delta u_{kT+i}^{(n)}}(u_{kT+1}^{(n)} - u_{kT-q}^{(n+p+1)}), \tag{4.6}$$

which is equivalent to the expression (4.4). □

Then we have the following convergence theorem.

Theorem 7 Applying the new algorithm (3.3) with (3.4) to sequence (S_n), then for any nonnegative integer k, we have:

(i) If (S_n) behaves like (4.1), then

$$u_{kT}^{(n)} - S \sim (-1)^k \alpha_0 \frac{q^k \cdot k!}{(\gamma-q)(\gamma-2q)\cdots(\gamma-kq)} n^{\gamma} \quad as \quad n \to \infty. \tag{4.7}$$

(ii) If (S_n) behaves like (4.2), then

$$u_{kT}^{(n)} - S \sim \xi^n \sum_{i=0}^{\infty} \alpha_{k,i}^{(0)} n^{\gamma_k - i} \ as \ \ n \to \infty, \alpha_{k,0}^{(0)} \neq 0, \tag{4.8}$$

$$u_{kT+1}^{(n)} \sim \xi^{-n} \sum_{i=0}^{\infty} \alpha_{k,i}^{(1)} n^{-\gamma_k - i} \ as \ \ n \to \infty, \alpha_{k,0}^{(1)} = \frac{1}{\alpha_{k,0}^{(0)}(\xi-1)} \neq 0, \tag{4.9}$$

$$u_{kT+j}^{(n)} \sim n + \sum_{i=0}^{\infty} \alpha_{k,i}^{(j)} n^{-i} \ as \ \ n \to \infty,$$
$$\alpha_{k,0}^{(j)} = (k+1)(p+1+\frac{\xi}{1-\xi}), \ j = 2, 3 \ldots q, \tag{4.10}$$

with $\gamma_0 = \gamma, \gamma_k = \gamma_{k-1} - 2 - \mu_k, k = 1, 2 \ldots$, where μ_k are some nonnegative integers.

(iii) If (S_n) behaves like (4.3), then

$$u_{kT}^{(n)} - S \sim \frac{\xi^n}{(n!)^r} \sum_{i=0}^{\infty} \alpha_{k,i}^{(0)} n^{\gamma_k - i} \ as \ \ n \to \infty, \alpha_{k,0}^{(0)} \neq 0, \tag{4.11}$$

$$u_{kT+1}^{(n)} \sim \frac{(n!)^r}{\xi^n} \sum_{i=0}^{\infty} \alpha_{k,i}^{(1)} n^{-\gamma_k - i} \ as \ \ n \to \infty, \alpha_{k,0}^{(1)} = -\frac{1}{\alpha_{k,0}^{(0)}} \neq 0, \tag{4.12}$$

$$u_{kT+j}^{(n)} \sim n + p + 1 + \sum_{i=0}^{\infty} \alpha_{k,i}^{(j)} n^{-r-i} as \ \ n \to \infty,$$
$$\alpha_{k,0}^{(j)} = (k+1)\xi, j = 2, 3 \ldots q, \tag{4.13}$$

with $\gamma_0 = \gamma, \gamma_k = \gamma_{k-1} - r - 2 - \mu_k$ when $p = 0$ and $\gamma_k = \gamma_{k-1} - (p+1)r - 1 - \mu_k$ when $p \neq 0$, where μ_k are some nonnegative integers.

Proof : (i) For logarithmic sequences in (4.1), the convergence results can be obtained by following the similar steps given by Garibotti-Grinstein [17] in the proof of ε-algorithm. Here we omit the details.

(ii) Now we prove the convergence results for linear sequences in (4.2). We proceed by induction on k.

Base step. Consider the case when $k = 0$. On the one hand, since $u_0^{(n)} = S_n$ and $u_1^{(n)} = 1/\Delta S_n$, it is obvious that (4.8) and (4.9) hold for $k = 0$ with $\gamma_0 = \gamma$, $\alpha_{k,i}^{(0)} = \alpha_0$. On the other hand, according to the recursive relation (3.3) and initial values (3.4), expression (4.10) for $k = 0$ can be easily obtained with the help of Lemma 6.

Inductive step. Assume that expressions (4.8)–(4.10) hold for $k = 1, 2, \ldots, m$, where m is a positive integer. Next, we will prove that they also hold for $k = m + 1$.

Firstly, consider the proof of (4.8) when $k = m + 1$. Subtracting S form both sides of equation (4.4) in Lemma 6, we get

$$\begin{aligned} u_{(k+1)T}^{(n)} - S = \frac{1}{\prod_{i=1}^{q} \Delta u_{kT+i}^{(n)}} \Bigg\{ & \left(u_{kT}^{(n+p+1)} - S\right) \Delta u_{kT+1}^{(n)} \prod_{i=2}^{q} \Delta u_{kT+i}^{(n)} \\ & + u_{kT+1}^{(n)} \Delta \left(u_{kT}^{(n+p)} - S\right) \prod_{i=2}^{q} \Delta u_{(k-1)T+i}^{(n+p)} \\ & - \Delta \left(u_{kT}^{(n+p)} - S\right) u_{kT-q}^{(n+p+1)} \prod_{i=2}^{q} \Delta u_{(k-1)T+i}^{(n+p)} \Bigg\}. \end{aligned} \tag{4.14}$$

For simplicity, set

$$A_k^{(n)} = \left(u_{kT}^{(n+p+1)} - S\right) \Delta u_{kT+1}^{(n)} \prod_{i=2}^{q} \Delta u_{kT+i}^{(n)} + u_{kT+1}^{(n)} \Delta \left(u_{kT}^{(n+p)} - S\right) \prod_{i=2}^{q} \Delta u_{(k-1)T+i}^{(n+p)},$$

$$B_k^{(n)} = \Delta \left(u_{kT}^{(n+p)} - S\right) u_{kT-q}^{(n+p+1)} \prod_{i=2}^{q} \Delta u_{(k-1)T+i}^{(n+p)},$$

$$C_k^{(n)} = \prod_{i=1}^{q} \Delta u_{kT+i}^{(n)},$$

then (4.14) can be written as

$$u_{(k+1)T}^{(n)} - S = \frac{A_k^{(n)} - B_k^{(n)}}{C_k^{(n)}}. \tag{4.15}$$

Thus, in order to prove (4.8) for $k = m+1$, we only need to analyze the asymptotic behaviours of $A_m^{(n)}$, $B_m^{(n)}$ and $C_m^{(n)}$ as $n \to \infty$, respectively.

In fact, according to the inductive hypothesis and Lemma 5, we have

$$\Delta(u_{kT}^{(n)} - S) \sim \xi^n \sum_{i=0}^{\infty} \hat{\alpha}_{k,i}^{(0)} n^{\gamma_k - i},\ as\ \ n \to \infty,\ \ \hat{\alpha}_{k,0}^{(0)} = (\xi - 1)\alpha_{k,0}^{(0)}, \tag{4.16}$$

$$\Delta u_{kT+1}^{(n)} \sim \xi^{-n} \sum_{i=0}^{\infty} \hat{\alpha}_{k,i}^{(1)} n^{-\gamma_k - i},\ as\ \ n \to \infty,\ \ \hat{\alpha}_{k,0}^{(1)} = (\frac{1}{\xi} - 1)\alpha_{k,0}^{(1)}, \tag{4.17}$$

$$\Delta u_{kT+j}^{(n)} \sim 1 + \sum_{i=2}^{\infty} \hat{\alpha}_{k,i}^{(j)} n^{-i},\ as\ \ n \to \infty,\ \ j = 2, 3 \ldots q. \tag{4.18}$$

Furthermore,

$$(u_{kT}^{(n+p)} - S) u_{kT+1}^{(n)} \sim \sum_{i=0}^{\infty} b_{k,i} n^{-i}, as\ \ n \to \infty, \tag{4.19}$$

which leads to

$$\Delta[(u_{kT}^{(n+p)} - S) u_{kT+1}^{(n)}] \sim \sum_{i=2}^{\infty} \hat{b}_{k,i} n^{-i},\ as\ \ n \to \infty, \tag{4.20}$$

where $k = 1, \ldots, m$.

With the help of the above relations, we obtain

$$\begin{aligned}
A_m^{(n)} \sim & \left(u_{mT}^{(n+p+1)} - S\right) \Delta u_{mT+1}^{(n)} \prod_{j=2}^{q} \left(1 + \sum_{i=2}^{\infty} \hat{\alpha}_{m,i}^{(j)} n^{-i}\right) \\
& + u_{mT+1}^{(n)} \Delta \left(u_{mT}^{(n+p)} - S\right) \prod_{j=2}^{q} \left(1 + \sum_{i=2}^{\infty} \hat{\alpha}_{m-1,i}^{(j)} (n+p)^{-i}\right) \\
\sim & \left(u_{mT}^{(n+p+1)} - S\right) \Delta u_{mT+1}^{(n)} \left(1 + O(n^{-2})\right) + u_{mT+1}^{(n)} \Delta \left(u_{mT}^{(n+p)} - S\right) \left(1 + O(n^{-2})\right) \\
\sim & \Delta \left[\left(u_{mT}^{(n+p)} - S\right) u_{mT+1}^{(n)}\right] \left(1 + O(n^{-2})\right) \\
\sim & \sum_{i=2}^{\infty} \beta_{m,i} n^{-i},\ as\ \ n \to \infty,
\end{aligned} \tag{4.21}$$

$$\begin{aligned}
B_m^{(n)} \sim & \left[\xi^{-n-p-1} \sum_{i=0}^{\infty} \alpha_{m-1,i}^{(1)} (n+p+1)^{-\gamma_{m-1} - i}\right] \\
& \cdot \left[\xi^{n+p} \sum_{i=0}^{\infty} \hat{\alpha}_{m,i}^{(0)} (n+p)^{\gamma_m - i}\right] \prod_{j=2}^{q} \left(1 + \sum_{i=2}^{\infty} \hat{\alpha}_{m-1,i}^{(j)} (n+p)^{-i}\right) \\
\sim & \left[\xi^{-n-p-1} \sum_{i=0}^{\infty} \alpha_{m-1,i}^{(1)} (n+p+1)^{-\gamma_{m-1} - i}\right] \cdot \left[\xi^{n+p} \sum_{i=0}^{\infty} \hat{\alpha}_{m,i}^{(0)} (n+p)^{\gamma_m - i}\right] \left(1 + O(n^{-2})\right) \\
\sim & \sum_{i=0}^{\infty} \theta_{m,i} n^{\gamma_m - \gamma_{m-1} - i},\ as\ \ n \to \infty,
\end{aligned} \tag{4.22}$$

$$C_m^{(n)} \sim \left[\xi^{-n}\sum_{i=0}^{\infty}\hat{\alpha}_{m,i}^{(1)}n^{-\gamma_m-i}\right]\prod_{j=2}^{q}\left(1+\sum_{i=2}^{\infty}\hat{\alpha}_{m,i}^{(j)}n^{-i}\right)$$
$$\sim \left[\xi^{-n}\sum_{i=0}^{\infty}\hat{\alpha}_{m,i}^{(1)}n^{-\gamma_m-i}\right]\left(1+O(n^{-2})\right)$$
$$\sim \xi^{-n}\sum_{i=0}^{\infty}\tau_{m,i}n^{-\gamma_m-i},\ as\ \ n\to\infty. \tag{4.23}$$

Substituting expressions (4.21)–(4.23) into (4.15), from the fact that $\gamma_m-\gamma_{m-1}\leq -2$, we finally get the following result

$$u_{(m+1)T}^{(n)} - S \sim \xi^n\sum_{i=0}^{\infty}\rho_{m,i}^{(0)}n^{\gamma_m-2-i}$$
$$= \xi^n\sum_{i=0}^{\infty}\alpha_{m+1,i}^{(0)}n^{\gamma_{m+1}-i}, as\ n\to\infty,\ \alpha_{m+1,0}^{(0)}\neq 0, \tag{4.24}$$

with $\gamma_{m+1}=\gamma_m-2-\mu_{m+1}$ for some nonnegative integer μ_{m+1}, which implies that (4.8) holds for $k=m+1$.

Secondly, we prove (4.9) for $k=m+1$. Replacing k by $(m+1)T$, equation (3.3) can be written as

$$u_{(m+1)T+1}^{(n)} = u_{mT+1}^{(n+p+1)} + \frac{\Delta u_{mT+1}^{(n+p)}}{\Delta u_{(m+1)T}^{(n)}}(u_{(m+1)T}^{(n)} - u_{mT}^{(n+p+1)}). \tag{4.25}$$

Using the hypothesis and expression (4.24) we have just proved, we obtain

$$u_{(m+1)T+1}^{(n)} \sim \xi^{-n}\sum_{i=0}^{\infty}\alpha_{m+1,i}^{(1)}n^{-\gamma_{m+1}-i}\ as\ \ n\to\infty, \tag{4.26}$$

where $\alpha_{m+1,0}^{(1)} = \frac{1}{\alpha_{m,0}^{(0)}(\xi-1)}$, which can be derived by the relation $\alpha_{m+1,0}^{(1)}\alpha_{m+1,0}^{(0)} = \alpha_{m,0}^{(1)}\alpha_{m,0}^{(0)}$.

Finally, we investigate the asymptotic behaviours of $u_{(m+1)T+j}^{(n)}$ with $j=2,\ldots,q$, as $n\to\infty$. In fact, similar to the analysis of $u_{(m+1)T+1}^{(n)}$ given above, the asymptotic behaviour of $u_{(m+1)T+j}^{(n)}$ can be easily derived from that of $u_{(m+1)T+j-1}^{(n)}$ and the inductive hypothesis, that is

$$u_{(m+1)T+j}^{(n)} \sim n+\sum_{i=0}^{\infty}\alpha_{m+1,i}^{(j)}n^{-i} as\ \ n\to\infty, j=2,3\ldots q, \tag{4.27}$$

with $\alpha_{m+1,0}^{(j)} = (m+2)(p+1+\frac{\xi}{1-\xi})$, which can be obtained from $\alpha_{m+1,0}^{(j+1)} - \alpha_{m,0}^{(j+1)} = \alpha_{m+1,0}^{(j)} - \alpha_{m,0}^{(j)} = p+1+\frac{\xi}{1-\xi}, j=2,3\ldots q-1$.

Consequently, expressions (4.8)–(4.10) hold for $k=m+1$, which complete the proof of (ii) by inductive principle.

(iii) The proof of (4.11)–(4.13) to factorial sequences in (4.3) can be achieved in a similar way as we did in the case of linear sequences in (ii).

Thus proving the theorem. □

Theorem 7 indicates that our new method accelerate the convergence of both linear sequences (4.2) and factorial sequences (4.3), but fails in logarithmic sequences (4.1).

4.2. Stability. We now turn to the investigation of stability. From equation (3.3), we obtain

$$u_{(k+1)T}^{(n)} = \lambda_k^{(n)} u_{kT}^{(n+p+1)} + \mu_k^{(n)} u_{kT}^{(n+p)}, \quad \lambda_k^{(n)} + \mu_k^{(n)} = 1, \tag{4.28}$$

where

$$\lambda_k^{(n)} = \frac{u_{kT+q}^{(n+1)} - u_{(k-1)T+q}^{(n+p+1)}}{\Delta u_{kT+q}^{(n)}} \quad and \quad \mu_k^{(n)} = -\frac{u_{kT+q}^{(n)} - u_{(k-1)T+q}^{(n+p+1)}}{\Delta u_{kT+q}^{(n)}}. \tag{4.29}$$

Using mathematical induction on k and noticing that $u_0^{(n)} = S_n$, we have

$$u_{kT}^{(n)} = \sum_{i=0}^{k} \gamma_{k,i}^{(n)} S_{n+kp+i}, \quad \sum_{i=0}^{k} \gamma_{k,i}^{(n)} = 1. \tag{4.30}$$

From the context of other extrapolation methods [36, 38, 39], the quantities of relevance to stability are

$$\Gamma_k^{(n)} = \sum_{i=0}^{k} |\gamma_{k,i}^{(n)}|. \tag{4.31}$$

In fact, if $\tilde{S_n} = S_n + \epsilon_n$ are the initial values with small perturbations, then the calculated values $\tilde{u}_{kT}^{(n)}$ are given by

$$\tilde{u}_{kT}^{(n)} \approx \sum_{i=0}^{k} \gamma_{k,i}^{(n)} \tilde{S}_{n+kp+i} = u_{kT}^{(n)} + \sum_{i=0}^{k} \gamma_{k,i}^{(n)} \epsilon_{n+kp+i}. \tag{4.32}$$

Therefore,

$$|\tilde{u}_{kT}^{(n)} - u_{kT}^{(n)}| \approx |\sum_{i=0}^{k} \gamma_{k,i}^{(n)} \epsilon_{n+kp+i}| \leq \Gamma_k^{(n)} \epsilon, \text{ with } \epsilon = \max_{0 \leq i \leq k} |\epsilon_{n+kp+i}|, \tag{4.33}$$

which implies that $\Gamma_k^{(n)}$ control the propagation of errors in computing process. When $sup_n \Gamma_k^{(n)} = \infty$, the sequence $\left(u_{kT}^{(n)}\right)_{n=0}^{\infty}$ is unstable, and when $sup_n \Gamma_k^{(n)} < \infty$, it is stable. Since $\sum_{i=0}^{k} \gamma_{k,i}^{(n)} = 1$, we hope these $\Gamma_k^{(n)}$ are as close to 1 as possible to get good numerical stability. Next, we will consider the asymptotic behaviour of $\Gamma_k^{(n)}$, as $n \to \infty$.

As the following lemma can be proved in a way similar to [40], we simply list it without proof.

Lemma 8 Let $P_k^{(n)}(z) = \sum_{i=0}^{k} \gamma_{k,i}^{(n)} z^i$, then

$$P_{k+1}^{(n)}(z) = \lambda_k^{(n)} z P_k^{(n+p+1)}(z) + \mu_k^{(n)} P_k^{(n+p)}(z). \tag{4.34}$$

Lemma 9 For any nonnegative k,
(i) if (S_n) behaves like (4.2), then

$$\lambda_k^{(n)} \sim \frac{1}{1-\xi} \quad and \quad \mu_k^{(n)} \sim \frac{-\xi}{1-\xi}, \quad as \quad n \to \infty. \tag{4.35}$$

(ii)if (S_n) behaves like (4.3), then

$$\lambda_k^{(n)} \sim 1 \quad and \quad \mu_k^{(n)} \sim 0, \quad as \quad n \to \infty, \quad if \ q = 1, \tag{4.36}$$
$$\lambda_k^{(n)} \sim -p \quad and \quad \mu_k^{(n)} \sim p+1, \quad as \quad n \to \infty, \quad if \ q > 1. \tag{4.37}$$

Proof: The proof can be easily obtained by using the expressions (4.29) and the results of Theorem 7. □

Finally, we give the main stability results.

Theorem 10 (i) If (S_n) behaves like (4.2), then

$$P_k^{(n)}(z) \sim \left(\frac{\xi - z}{\xi - 1}\right)^k \quad and \quad \Gamma_k^{(n)} \sim \left(\frac{|\xi|+1}{|\xi-1|}\right)^k, \quad as \quad n \to \infty. \tag{4.38}$$

(ii) If (S_n) behaves like (4.3), then

$$P_k^{(n)}(z) \sim z^k \quad and \quad \Gamma_k^{(n)} \sim 1, \quad as \ n \to \infty, \quad if \ q = 1, \tag{4.39}$$
$$P_k^{(n)}(z) \sim (p+1-pz)^k \quad and \quad \Gamma_k^{(n)} \sim (2p+1)^k \quad as \ n \to \infty, \quad if \ q > 1. \tag{4.40}$$

Proof: Combining Lemma 8 and Lemma 9, expressions (4.38)–(4.40) hold immediately by induction on k. □

Remark. Since our new method is ineffective on logarithmic sequences (4.1), we only consider the stability corresponding to linear sequences (4.2) and factorial sequences (4.3) in the above theorem.

We close this section by concluding that our new sequence transformation is stable for both linear sequences (4.2) and factorial sequences (4.3). Concretely, for linear sequences, the stability is better when ξ is a real negative number, while becomes weak when ξ approaches 1, since $\Gamma_k^{(n)} \to \infty$ as $\xi \to 1$. Noticing that ξ^l with some positive integer $l \geq 2$ is farther away from 1, we propose to apply the method to the subsequences (S_{ln}) for better numerical stability. This strategy is APS [41], which was first proposed by Sidi [37]. As for factorial sequences, our new sequence transformation (3.1) with $q = 1$ has better stability than that with $q > 1$. In addition, for a fixed $q > 1$, the sequence transformation becomes more and more stable as p shrinks to 0.

5. Numerical examples. In this section, we give some numerical examples, which illustrate the performance of algorithm (3.3)-(3.4) numerically.

Example 1. Consider the following alternating series

$$S_n = \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k},$$

TABLE 5.1
Numerical results of example 1.

n	$\lvert S_n - S\rvert$	$\lvert T_{k_1}^{(1,1)}(S_{n-2k_1})\rvert$	$\lvert T_{k_2}^{(1,2)}(S_{n-3k_2})\rvert$	$\lvert T_{k_3}^{(2,2)}(S_{n-3k_3})\rvert$
5	0.0902	1.8615×10^{-4}	9.4288×10^{-4}	0.0069
10	0.0475	3.8072×10^{-8}	6.0792×10^{-7}	2.7318×10^{-5}
15	0.0322	3.7446×10^{-12}	4.0163×10^{-10}	2.4731×10^{-8}
20	0.0244	5.5511×10^{-16}	3.6915×10^{-13}	3.2922×10^{-11}

TABLE 5.2
Numerical results of example 2.

n	$\lvert S_n - S\rvert$	$\lvert T_{k_1}^{(1,1)}(S_{n-2k_1})\rvert$	$\lvert T_{k_2}^{(1,2)}(S_{n-3k_2})\rvert$	$\lvert T_{k_3}^{(2,2)}(S_{n-3k_3})\rvert$
10	0.0306	1.7538×10^{-4}	0.0079	0.0061
20	0.0019	1.3000×10^{-8}	3.3291×10^{-5}	7.8235×10^{-6}
30	1.4341×10^{-4}	1.8611×10^{-10}	3.0440×10^{-7}	1.6116×10^{-7}
40	1.1916×10^{-5}	3.1393×10^{-12}	2.0094×10^{-8}	7.4446×10^{-10}

with $\lim_{n\to\infty} S_n = S = \ln 2 = 0.693147181\cdots$. According to the *Euler-Maclaurin summation formula* (see [6] [30] for details), we have

$$S_n - \ln 2 \sim (-1)^{n-1}\left(\frac{1}{2n} - \sum_{j=1}^{\infty}\frac{B_{2j}(2^{2j}-1)}{(2j)n^{2j}}\right), \quad \text{as } n \to \infty,$$

where B_{2j} are *Bernoulli numbers*. This asymptotic expansion is a special case of (4.2), with $\xi = -1, \gamma = -1$ and $\alpha_0 = -\frac{1}{2} \neq 0$. The numerical results corresponding to S_n, $T_{k_1}^{(1,1)}(S_{n-2k_1})$, $T_{k_2}^{(1,2)}(S_{n-3k_2})$ and $T_{k_3}^{(2,2)}(S_{n-3k_3})$ are presented in Table 5.1, where $k_1 = [(n-1)/2], k_2 = [(n-1)/3]$ and $k_3 = [(n-2)/3]$.

Example 2. Consider the linearly convergent series

$$S_n = \sum_{k=1}^{n}\frac{(0.8)^k}{k},$$

which converges to $S = \ln 5 = 1.60943791\cdots$ as $n \to \infty$. As shown in [41, p.84], S_n has the following asymptotic expansion

$$S_n - \ln 5 \sim \frac{(0.8)^n}{n}\left(-4 + O(n^{-1})\right), \quad \text{as } n \to \infty,$$

which is a special case of (4.2) with $\xi = 0.8$. The corresponding numerical results are presented in Table 5.2.

Example 3. Consider the logarithmically convergent series

$$S_n = \sum_{k=1}^{n}\frac{1}{k^2},$$

which converges to $S = \pi^2/6$ as $n \to \infty$. Also from Euler-Maclaurin summation formula, we have

$$S_n - \frac{\pi^2}{6} \sim n^{-1}\left(-1 + \frac{1}{2}n^{-1} - \sum_{j=1}^{\infty}B_{2j}n^{-2j}\right), \quad \text{as } n \to \infty.$$

TABLE 5.3
Numerical results of example 3.

n	$\lvert S_n - S\rvert$	$\lvert T_{k_1}^{(1,1)}(S_{n-2k_1})\rvert$	$\lvert T_{k_2}^{(1,2)}(S_{n-3k_2})\rvert$	$\lvert T_{k_3}^{(2,2)}(S_{n-3k_3})\rvert$
10	0.0951 6634	0.0304 5977	0.0786 4240	0.0707 6553
20	0.0487 7082	0.0086 4618	0.0307 2823	0.0251 5819
30	0.0327 8394	0.0053 7525	0.0187 2470	0.0138 3597
40	0.0246 9010	0.0042 1038	0.0142 6067	0.0109 0439

It is obvious that (S_n) is a logarithmic sequence with the asymptotic expansion (4.1), and the corresponding numerical results are shown in Table 5.3.

The above numerical examples indicate that our new algorithm indeed accelerates the linear sequences having asymptotic expansion (4.2) (Example 1 and Example 2), while fails in the logarithmic sequences in (4.1) (Example 3). That is to say, the numerical results presented here coincide with the theoretical results given in section 4.

6. Conclusion and discussions. In this article, we construct the molecule solution of an extended discrete Lotka-Volterra equation by Hirota's bilinear method, from which a new sequence transformation is derived. From the bilinear form of this extended discrete Lotka-Volterra equation, a two dimensional difference equation which can be used as a convergence acceleration algorithm to implement the new sequence transformation is generated. In addition, our new transformation is nothing but an extension of the multistep Shanks' transformation, and the multistep ε-algorithm is just a special case of our new algorithm. Then we present a rigorous convergence and stability analysis, which implies that our new method accelerates both linear sequences (4.2) and factorial sequences (4.3) with good numerical stability while fails in logarithmic sequences (4.1). Finally, we give numerical examples to demonstrate some of the preceding theoretical results.

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