Discrete Variation Principle and First Integrals of Lagrange-Maxwell Mechanico-Electrical Dynamical Systems[1]

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Abstract We give discrete variational principle and integral algorithm for the finite dimensional Lagrange-Maxwell mechanico-electrical system, which includes nonconservative force and dissipative function. The discrete variational principle and Euler-Lagrange equation are derived by introducing discrete action of the system. The first integral algorithm is obtained by introducing the infinitesimal transformation with respect to generalized coordinates and electric quantities of the system. This work firstly expands discrete Noether symmetry to mechanico-electrical dynamical systems. A practical example is presented to illustrate these results.

Keywords: discrete variational principle; first integral; mechanico-electrical system

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1. Introduction

Dynamical system with symmetry plays an important role in the mathematical modelling of a variety of physical and mechanical processes. For the study of mechanical systems, it is useful to integrate the equations of motion, and then obtain the invariants of the systems.[1] Now the invariants are used effectively for the numerical integration of various equation in mathematical physics.[2,3] Here the approaches to find first integrals for continuous cases are extended to the discrete mechanical systems.

Based on certain application, discrete variational principles and discrete first integrals of mechanics have been discussed for a long time. The theory of discrete variational mechanics is set up in the 1960s: Jordan and Polak[4] first proposed the discrete variational mechanics in the optimal control literature. Cadzow[5]motivated and discussed discrete calculus of variations, and obtained the discrete Euler-Lagrange equation. In the discrete variational principle and the first integrals of the mechanical systems, some early work was done by some authors. Logan[6]obtained the first integrators in the discrete calculus of variation, and further studied the multi-freedom and higher-order problems. Maeda[7,8] has given the canonical structure and symmetries for discrete systems, and extended Noether's theorem to the discrete case. Lee[9] first studied the version of regarding the time as a discrete dynamical variable. Subsequently, this theory was then pursued to integrable systems by Veselov[10,11], Moser and Veselov[12]. Jaroszkiewiez and Norteo[13-15]furthermore extended to discrete mechanics, which include particle systems, classical field and

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quantum theory. The variational view and numerical implementation of discrete mechanics is further developed by Wendlandt and Marsden[16], Kane, Marsden and Ortiz[17], Bobenko and Suris[18], [19-21]. Afterward, Marsden, Pekarsky and Shkoller[19,20] Bokenko and Suris[21] presented this problems on symmetry reduction of discrete Lagrangian mechanics and discrete Lagrangian reduction etc. Kane and Marsden, Ortiz and West[21]spreaded discrete variational integration algorithm to dissipative mechanical systems. Marsden and West[22] given a comprehensive and unified view of much of these works on both discrete mechanics and integration methods for mechanical systems. Recently, Guo, Wu et al[23-27] given many results on difference discrete variational principles, Euler Lagrange cohomology and sympletic, multisympletic structures and total variation in Hamiltonian Formalism and symletic-energy integrators etc.

It is well known that discrete variational principles and first integrals of mechanical systems has become a field of intensive research activity for a long time. The increasing interest in this subject is mainly due to its dual character. On one hand, discrete variational principles and the first integrators of mechanical systems allow for the construction of integration schemes, that turn out to be numerically competitive in many problems. On the other hand, many of the geometric properties of mechanical systems in the continuous cases admit an appropriate counterpart in the discrete setting, which makes it a rich area to be explored. Both aspects of the discrete model play a key role in the explanation of the good behavior of the integrators deriver from it in a number of situations.

Mechanico-electrical systems are so called for the systems in which the mechanical process and electromagnetic process are affected by each other. In general case, mechanico-electrical systems there are nonlinear term, then that it is very difficulty when we to obtain solution of the systems. From the point of view of applications to problems in mechanical systems case definitely deserves special attention. Moreover, discrete variational principle and integration method of mechanical systems appear as an indispensable tool in modern engineering technology domain. The purpose of this paper is to do first attempt which extend the discrete variational principle and the first integration algorithm of the mechanical systems to the case of mechanico-electrical systems. Firstly, we use discrete Lagrangian to define a discrete action, which includes generalized coordinates and generalized electric quantities. Secondly, we obtain discrete variational principle of Lagrange-maxwell mechanico-electrical systems when take variation to discrete Lagrangian, and further obtain discrete Euler-Lagranian equations. Thirdly, we derive the first integration method of Lagrange-maxwell mechanico-electrical systems when introduce the infinitesimal transformations with respect to generalized coordinates and generalized electric quantities. In a further work, we will test our algorithm in relevant examples.

2. The discrete variational principle and Euler-Lagrange equation of mechanico-electrical systems

These systems are called the mechanico-electrical dynamical systems when the mechanical process and electromagnetic process in the systems are related to each other. The mechanical part of N particles component is described with the model of the mechanics. If the system is subjected to d ideal, bilateral, holonomic constraints, the position in space of the system is determined by $n=3N-d$ generalized coordinates q_s $(s=1,\cdots,n)$. The electrodynamical part of electric circuit and magnetic circuit is described with the model of electricity by *m* generalized electric quantities. Suppose the mechanico-electrical system has m closed circuits, every closed circuit consisted of the line conductor and capacitance, there is not any relation among electric circuits, and the electromagnetic process of closed circuits are not independent. Suppose i_k is the current, V_k is the electric potential, e_k is the electric quantity in the capacitance ($\dot{e}_k = i_k$), R_k is the resistance, C_k is the capacitance, then the Lagrangian of the mechanico-electrical systems is

$$L = T(\boldsymbol{q}, \dot{\boldsymbol{q}}, \boldsymbol{e}, \dot{\boldsymbol{e}}) - V(\boldsymbol{q}) + W_m(\boldsymbol{q}, \dot{\boldsymbol{e}}) - W_e(\boldsymbol{q}, \boldsymbol{e}), \tag{1}$$

where

$$W_e = \frac{1}{2}\frac{e_k^2}{C_k}, \; W_m = \frac{1}{2} L_{kr} i_k i_r \quad (k, r = 1, \cdots, m) \tag{2}$$

are electric field energy and magnetic field energy of the m-th circuit, respectively, in which $C_k = C_k(q_s)$ is capacitance of the k-th circuit, $L_{kr}(k \neq r)$ is mutual inductance between k-th and r-th circuit, $L_{kr} = L_{rk}(q_s)$, and L_{kk} is self inductance of the k-th return circuit.

Equations of motion of the system are

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{e}_k} - \frac{\partial L}{\partial e_k} + \frac{\partial F}{\partial \dot{e}_k} = u_k, \; \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_s} - \frac{\partial L}{\partial q_s} + \frac{\partial F}{\partial \dot{q}_s} = Q_s^{"} \; (k = 1, \cdots, m; s = 1, \cdots, n), \tag{3}$$

Equation (3) is called the Lagrange-Maxwell equations those are two order $n+m$ dimensions normal differential equations of the generalized coordinates and the generalized electric quantities. In Eq. (3)

$$F = F_e(\dot{\boldsymbol{e}}) + F_m(\boldsymbol{q}, \dot{\boldsymbol{q}}), \tag{4}$$

and

$$F_e = \frac{1}{2} R_k i_k^2 = \frac{1}{2} R_k \dot{e}_k{}^2 \quad (k = 1, \cdots, m). \tag{5}$$

where F_m is the dissipative function of the viscous frictional damping force, F_e is the lead-through electric dissipative function, $-\partial F / \partial \dot{q}_s$ and $-\partial F / \partial \dot{e}_k$ are called dissipative forces, and $Q_s^{"}$ is non-potential general force, u_k is general electromotive force of kth circuit.

Given a configuration space Q, a discrete Lagrangian of mechanico-electrical dynamical systems is a map

$$L_d : Q \times Q \to R \tag{6}$$

In practice, L_d is obtained by approximating a given Lagrangian as we shall discuss later, but regard L_d as given for the moment. We regard L_d as a function of 2 nearby points $(q_{k-1}, q_k; e_{l-1}, e_l)$

For the positive integer N, M, the action sum is the map $S_d : Q^{N+M} \to R$ defined by

$$S_d = \sum_{k=1, l=1}^{N, M} L_d(q_{k-1}, q_k, e_{l-1}, e_l) \tag{7}$$

where $q_{k-1}, e_{l-1} \in Q$ and k is a nonnegative integer, and $\Delta q = q_k - q_{k-1}, \Delta e = e_l - e_{l-1}$. The action sum is the discrete analog of the action integral in mechanico-electrical dynamical systems.

The discrete variational principle states that the evolution equations extremize the action sum given fixed end points q_0, q_N, e_0, e_M. Extremizing S_d over $q_1,\cdots,q_{N-1};e_1,\cdots, e_{M-1}$ leads to as follows

$$\begin{aligned}
&\sum_{k=1,l=1}^{N,M}\left[\frac{\partial L_d\left(q_{k-1},q_k,e_{l-1},e_l\right)}{\partial q_{k-1}}\delta q_{k-1}+\frac{\partial L_d\left(q_{k-1},q_k,e_{l-1},e_l\right)}{\partial q_k}\delta q_k\right]-\sum_{k=1,l=1}^{N,M}\frac{\partial F}{\partial \dot q_k}\left(q_{k-1},q_k,e_{l-1},e_l\right)\delta q_k\\
&+\sum_{k=1}^{N}Q_d^{"}\left(q_{k-1},q_k\right)\delta q_k+\sum_{k=1,l=1}^{N,M}\left[\frac{\partial L_d\left(q_{k-1},q_k,e_{l-1},e_l\right)}{\partial e_{l-1}}\delta e_{l-1}+\frac{\partial L_d\left(q_{k-1},q_k,e_{l-1},e_l\right)}{\partial e_l}\delta e_l\right]-\\
&\sum_{k=1,l=1}^{N,M}\frac{\partial F}{\partial \dot e_l}\left(q_{k-1},q_k,e_{l-1},e_l\right)\delta e_l+\sum_{l=1}^{M}V_d\left(e_{l-1},e_l\right)\delta e_l\\
&=\sum_{k=1,l=1}^{N-1,M-1}\left[\frac{\partial L_d\left(q_{k-1},q_k,e_{l-1},e_l\right)}{\partial q_k}+\frac{\partial L_d\left(q_k,q_{k+1},e_{l-1},e_l\right)}{\partial q_k}\right]\delta q_k-\sum_{k=1,l=1}^{N-1,M-1}\frac{\partial F}{\partial \dot q_k}\left(q_{k-1},q_k,e_{l-1},e_l\right)\delta q_k\\
&+\sum_{k=1}^{N-1}Q_d^{"}\left(q_{k-1},q_k\right)\delta q_k+\sum_{k=1,l=1}^{N-1,M-1}\left[\frac{\partial L_d\left(q_{k-1},q_k,e_{l-1},e_l\right)}{\partial e_l}+\frac{\partial L_d\left(q_{k-1},q_k,e_l,e_{l+1}\right)}{\partial e_l}\right]\delta e_l-\\
&\sum_{k=1,l=1}^{N=1,M-1}\frac{\partial F}{\partial \dot e_l}\left(q_{k-1},q_k,e_{l-1},e_l\right)\delta e_l+\sum_{l=1}^{M-1}V_d\left(e_{l-1},e_l\right)\delta e_l=0
\end{aligned}\tag{8}$$

From Eq.(8), we obtain the discrete Euler-Lagrangian equations as

$$\begin{aligned}
&\frac{\partial L_d\left(q_{k-1},q_k,e_{l-1},e_l\right)}{\partial q_k}+\frac{\partial L_d\left(q_k,q_{k+1},e_{l-1},e_l\right)}{\partial q_k}-\frac{\partial F_d}{\partial \dot q_k}\left(q_{k-1},q_k,e_{l-1},e_l\right)+Q_d^{"}\left(q_{k-1},q_k\right)=0,\\
&\frac{\partial L_d\left(q_{k-1},q_k,e_{l-1},e_l\right)}{\partial e_l}+\frac{\partial L_d\left(q_{k-1},q_k,e_l,e_{l+1}\right)}{\partial e_l}-\frac{\partial F}{\partial \dot e_l}\left(q_{k-1},q_k,e_{l-1},e_l\right)+V_d\left(e_{l-1},e_l\right)=0
\end{aligned}\tag{9}$$

For convenience later, equation (4) can been written as the following form

$$\begin{aligned}
&D_2L_d\left(q_{k-1},q_k,e_{l-1},e_l\right)+D_1L_d\left(q_k,q_{k+1},e_{l-1},e_l\right)-\frac{\partial F}{\partial \dot q_k}\left(q_{k-1},q_k,e_{l-1},e_l\right)\\
&+Q_d^{"}\left(q_{k-1},q_k,e_{l-1},e_l\right)=0,\\
&D_4L_d\left(q_{k-1},q_k,e_{l-1},e_l\right)+D_3L_d\left(q_{k-1},q_k,e_{l-1},e_l\right)-\frac{\partial F}{\partial e_l}\left(q_{k-1},q_k,e_{l-1},e_l\right)\\
&+V_d\left(e_{l-1},e_l\right)=0,
\end{aligned}\tag{10}$$

Where

$$\begin{aligned}
&D_2L_d\left(q_{k-1},q_k,e_{l-1},e_l\right)=\frac{\partial L_d\left(q_{k-1},q_k,e_{l-1},e_l\right)}{\partial q_k},\\
&D_1L_d\left(q_k,q_{k+1},e_{l-1},e_l\right)=\frac{\partial L_d\left(q_k,q_{k+1},e_{l-1},e_l\right)}{\partial q_k},\\
&D_4L_d\left(q_{k-1},q_k,e_{l-1},e_l\right)=\frac{\partial L_d\left(q_{k-1},q_k,e_{l-1},e_l\right)}{\partial e_l},\\
&D_3L_d\left(q_{k-1},q_k,e_l,e_{l+1}\right)=\frac{\partial L_d\left(q_{k-1},q_k,e_l,e_{l+1}\right)}{\partial e_l}.
\end{aligned}\tag{11}$$

3. The first integral of discrete nonconservative mechanico-electrical systems

We now derive a discrete version of Noether's theorem. For continuous systems, Noether's theory states that symmetry of Lagrangian leads to a conserved quantity. We now derive a method of the first integral of discrete Lagrange-Maxwell mechanico-electrical dynamical systems via investigate of invariance of discrete Lagrangian of the systems.

For the discrete coordinates and electric quantity, one introduce the following infinitesimal transformations

$$q_k^* = q_k + \varepsilon\xi_k(q_s, e_l),\ e_l^* = e_l + \varepsilon\eta_l(q_k, e_j) \quad (k, s = 1, \cdots, n; l, j = 1, \cdots, m), \tag{12}$$

Where 兆 is a small parameter, and ξ_k and η_l are infinitesimal generators. We first give a definition as follows

Definition If there are function $v(q_{k-1}, q_k, e_{l-1}, e_l)$, for each $k = 1, \cdots, n$ and $l = 1, \cdots, m$, and subject to the nonconservative force Q_d and general electromotive force V_d, then the following equations held

$$\text{Extremize } L_d\left[\varepsilon\xi_k(q_s, e_l), \varepsilon\xi_l(q_k, e_j)\right] = \text{兆}[\ \frac{\partial F}{\partial \dot{q}_k}(q_{k-1}, q_k, e_l, e_{l-1}) - Q_d^{''}(q_{k-1}, q_k)$$

$$+ \Delta v(q_{k-1}, q_k, e_l, e_{l-1}) + \frac{\partial F}{\partial \dot{e}_l}(q_{k-1}, q_k, e_l, e_{l-1}) - V_d(e_{l-1}, e_l)\ ] \tag{13}$$

and Extremize L_d is written as the following form

$$\begin{aligned} &\left[D_2L_d(q_{k-1}, q_k, e_{l-1}, e_l) + D_1L_d(q_k, q_{k+1}, e_{l-1}.e_l)\right]\delta q_k + \Delta\left[-\delta q_{k-1}D_1L_d(q_{k-1}, q_k, e_{l-1}, e_l)\right] \\ &+ \left[D_4L_d(q_{k-1}, q_k, e_{l-1}, e_l) + D_3L_d(q_{k-1}, q_k, e_l, e_{l+1})\right]\delta e_l + \Delta\left[-\delta e_{l-1}D_3L_d(q_{k-1}, q_k, e_{l-1}, e_l)\right], \end{aligned} \tag{14}$$

In which Δ is difference operator, i.e. $\Delta q_k = q_{k+1} - q_k$, then we call the discrete Lagrangian as generalized difference invariant associated with infinitesimal transformation (12).

Based on discrete Lagrange-D'Alembert principle, we can present the following proposition:

Proposition For the infinitesimal transformation (12), the discrete Lagrange-Maxwell mechanico-electrical dynamical systems possess the first integral as

$$\begin{aligned} &\xi(q_{k-1}, e_{l-1}, e_l)D_1L_d(q_{k-1}, q_k, e_{l-1,} e_l) + \eta(q_{k-1}, q_k, e_{l-1})D_3L_d(q_{k-1,} q_k, e_{l-1}, e_l) \\ &+ v((q_{k-1}, q_k, e_{l-1}, e_l)) = \text{const} \end{aligned} \tag{15}$$

if discrete Lagrangian is a generalized difference invariant, and Eq.(10) hold.

Proof: We, from Eqs. (13) and (14), have

$$\begin{aligned} &[D_2L_d(q_{k-1}, q_k, e_{l-1}, e_l) + D_1L_d(q_k, q_{k+1}, e_{l-1}.e_l)]\varepsilon\xi(q_k, e_l) \\ &+ \Delta[-\varepsilon\xi(q_{k-1}, e_{l-1})D_1L_d(q_{k-1}, q_k, e_{l-1}, e_l)] \\ &+ [D_4L_d(q_{k-1}, q_k, e_{l-1}, e_l) + D_3L_d(q_{k-1}, q_k, e_l, e_{l+1})]\varepsilon\eta(q_k, e_l) \\ &+ \Delta\left[-\varepsilon\eta(q_{k-1}, e_{l-1})D_3L_d(q_{k-1}, q_k, e_{l-1}, e_l)\right] \\ &= \varepsilon[(\frac{\partial F}{\partial \dot{q}}(q_k, q_{k+1}, e_{l-1}.e_l) - Q_d^{''}(q_k, q_{k+1}, e_{l-1}.e_l))\xi(q_k, e_l) + \Delta v(q_{k-1}, q_k, e_{l-1}, e_l) \\ &+ (\frac{\partial F}{\partial \dot{e}}(q_k, q_{k+1}, e_{l-1}.e_l) - V_d(q_k, q_{k+1}, e_{l-1}.e_l))\eta(q_k, e_l)] \end{aligned} \tag{16}$$

Using Eq. (9) in Eq. (16), and further to reduce the result, one leads to

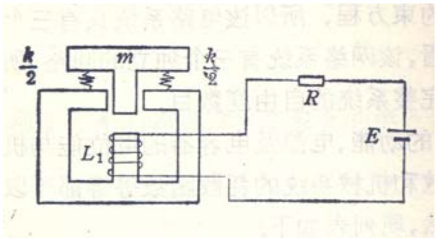
$$\Delta[\xi_k(q_{k-1},e_{l-1})D_1L_d(q_{k-1},q_k,e_{l-1},e_l)+\eta_l(q_{k-1},e_{l-1})D_3L_d(q_{k-1},q_k,e_{l-1},e_l)+v(q_{k-1},q_k,e_{l-1},e_l)]=0 \quad (17)$$

Then Eq. (15) is right. This is the desired result to us.

It is been show that Eq. (15) is a connection of one order difference, which is a first integral of Euler-Lagrange equation (10) for the Lagrange-Maxwell mechanico-electrical systems.

4. Numerical example

Fig 1 expresses a circuit of electromotion sensor to record mechanical vibration. Where m denotes mass of armature, k denotes total hardly coefficient, $L_1=L_1(x)$ denotes self-induction in the winding, and x is plumb displacement which is calculated from the position of originality length of winding in L_1. This circuit composed with winding, battery and resistance. We using E denotes electromotive force of battery, R denotes resistance.



Mechanical part and electrical part there being one freedom in system respectively, we taking armature displacement x and electric quantity q denote generalized coordinates, then kinetic energy and magnetic energy is

$$T=\frac{1}{2}m\dot{x}^2+\frac{1}{2}L_1(x)\dot{q}^2 \quad (18)$$

potential energy is

$$V=\frac{1}{2}kx^2-mgx \quad (19)$$

dissipation function is

$$F=-\frac{1}{2}R\dot{q}^2 \quad (20)$$

We will use a Lagrangian that is of the standard form kinetic energy mins potential energy, namely

$$L=\frac{1}{2}m\dot{x}^2+\frac{1}{2}L_1(x)\dot{q}^2-\frac{1}{2}kx^2+mgx \quad (21)$$

where $x,\dot{x},q$ and $\dot{q}$ are real numbers, with the corresponding discrete Lagrangian given by

$$L_d=\frac{1}{2}m(x_k-x_{k-1})^2+\frac{1}{2}L_1(x_{k-1})(q_l-q_{l-1})^2-\frac{1}{2}kx_{k-1}^2+mgx_{k-1} \quad (22)$$

and discrete dissipation function given by

$$F_d=-\frac{1}{2}R(q_l-q_{l-1})^2 \quad (23)$$

where x_{k-1},x_k,q_{k-1} and q_k are also real numbers. Then the discrete action sum is

$$J_d=\sum_{k=1,l=1}^{N,M}\left[\frac{1}{2}m(x_k-x_{k-1})^2+\frac{1}{2}L_1(x_{k-1})(q_l-q_{l-1})^2-\frac{1}{2}kx_{k-1}^2+mgx_{k-1}\right] \quad (24)$$

Substituting of the Eq. (24) into Eq. (10), one obtains discrete Euler-Lagrange equation which is a second difference equation in the form

$$m\left(k_{k+1}-2x_k+x_{k-1}\right)-\frac{1}{2}L_1^{'}\left(x_k\right)\left(q_l-q_{l-1}\right)^2+k\left(x_x+x_{k-1}\right)+mg=0, \qquad (25)$$
$$L_1\left(x_k\right)\left(q_{l+1}-q_l\right)-L_1\left(x_{k-1}\right)\left(q_l-q_{l-1}\right)+R\left(q_l-q_{l-1}\right)+E_d=0$$

Discrete varitional principle of the mechanico-electrical system shows discrete Lagrangian

$$L_d=\frac{1}{2}m\left(x_k-x_{k-1}\right)^2+\frac{1}{2}L_1\left(x_{k-1}\right)\left(q_l-q_{l-1}\right)^2-\frac{1}{2}kx_{k-1}^2+mgx_{k-1}$$

is a generalized difference invariant with respect to infinitesimal transformations $\xi=1,\eta=1$, i.e.

$$x^*=x+\varepsilon,\ q^*=q+\varepsilon \qquad (26)$$

Therefore, using Eq. (10), we obtain a first integrator

$$I=-m\left(x_k-x_{k-1}\right)+\frac{1}{2}\frac{\partial L_1\left(x_{k-1}\right)}{\partial x_{k-1}}\left(q_l-q_{l-1}\right)-kx_{k-1}+mg-\frac{1}{2}L_1\left(x_{k-1}\right)\left(q_l-q_{l-1}\right) \qquad (27)$$
$$+v\left(x_k,x_{k-1},q_l,q_{l-1}\right)=\text{const}$$

When $L_1(x)$ is known form, we can give the function $v\left(x_k,x_{k-1},q_l,q_{l-1}\right)$. For example, if $L_1(x)$=const, then we can take function v in the form

$$v=-\frac{1}{2}L_1\left(q_k-q_{k-1}\right)+Rq_k+Et \qquad (28)$$

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