CONFORMAL INVARIANTS ASSOCIATED TO A MEASURE: CONFORMALLY COVARIANT OPERATORS

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Abstract. In this paper we study Riemannian manifolds (M^n, g) equipped with a smooth measure m. In particular, we show that the construction of conformally covariant operators due to Graham-Jenne-Mason-Sparling can be adapted to this setting. As a by-product, we define a family of scalar curvatures, one of which corresponds to Perelman's scalar curvature function. We also study the variational problem naturally associated to these curvature/operator pairs.

1. Introduction

This paper draws its inspiration from an observation about the scalar curvature function introduced by Perelman [Per02], with the goal of illustrating the connection between conformally covariant operators and the $\mathcal{W}$-functional of Perelman.

Let (M^n, g) be a Riemannian manifold endowed with a smooth measure m, which we write as

$$dm = e^{-f} dVol(g).$$

The Bakry-Emery Ricci tensor of the Riemannian measure space (M^n, g, m) is

$$Ric^m(g) = Ric^m_\infty(g) = Ric(g) + \nabla^2 f.$$

Although typically attributed to Bakry-Emery [BE85], this tensor was studied much earlier by Lichnerowicz [Lic70]. Perelman [Per02] introduced a notion of scalar curvature in this setting given by

$$R^m(g) = R^m_\infty(g) = R(g) + 2\Delta f - |\nabla f|^2.$$

When the measure m is the canonical Riemannian measure, then $f \equiv 0$ and the generalized curvatures agree with their classical counterparts.

From the perspective of conformal geometry, the scalar curvature is naturally considered in conjunction with the *conformal laplacian*, the linear second-order operator which describes how the scalar curvature transforms under a conformal change of metric. In our conventions, if $\hat{g} = v^{4/(n-2)} g$ then

$$R(\hat{g}) = \frac{4(n-1)}{(n-2)} v^{\frac{n+2}{n-2}} L_g v, \tag{1.1}$$

Research of the first and third authors was partially supported by NSF Grant DMS-0758601.
Research of the second author was partially supported by NSF Grant DMS-0800084.

where

$$L = -\Delta + \frac{(n-2)}{4(n-1)} R(g). \tag{1.2}$$

Moreover, the conformal laplacian is *conformally covariant*: now writing $\hat{g} = e^{2w} g$,

$$L_{\hat{g}} \phi = e^{-\frac{n+2}{2} w} L_g (e^{\frac{n-2}{2} w} \phi). \tag{1.3}$$

The question naturally arises: is there a linear, conformally covariant differential operator associated to Perelman's scalar curvature? What are the corresponding transformation formulas?

The answer to the first question is, somewhat surprisingly, 'yes': The operator is given by

$$\mathcal{L}^m_{2,\infty} = -\Delta + 2\langle \nabla f, \cdot \rangle + \frac{n+2}{4} R^m_\infty(g) \tag{1.4}$$

(see Section 4.1). Moreover, if $\hat{g} = v^{-\frac{4}{n+2}} g$ is a conformal metric, then

$$R^m_\infty(\hat{g}) = \frac{4(n-1)}{(3n-2)} v^{-\frac{n-2}{n+2}} \mathcal{L}^m_{2,\infty} v. \tag{1.5}$$

Writing $\hat{g} = e^{2w} g$, the operator in (1.4) satisfies the covariance property

$$(\mathcal{L}^m_{2,\infty})_{\hat{g}} \phi = e^{\frac{n-2}{2} w} (\mathcal{L}^m_{2,\infty})_g (e^{-\frac{n+2}{2} w} \phi). \tag{1.6}$$

Note the interesting comparison with the "bidegree" of the conformal laplacian in (1.3).

Our first goal in this paper is to put the preceding formulas for R^m_∞ and $\mathcal{L}^m_{2,\infty}$ into a broader context. That is, by adapting the construction of Graham-Jenne-Mason-Sparling [GJMS92] to Riemannian measure spaces we prove the existence of a 1-parameter family of conformally covariant operators, of which $\mathcal{L}^m_{2,\infty}$ is a particular example (i.e., $\alpha = 2$). As a by-product of this construction we define a new family of scalar curvature functions $R^{(m,\alpha)}$ which generalize Perelman's scalar curvature. Thus, for each value of the parameter α, we have a pair $(R^{(m,\alpha)}, \mathcal{L}^m_\alpha)$ consisting of a scalar curvature function and covariant operator. The relationship between curvature and operator are completely analogous to the case of the scalar curvature/conformal laplacian detailed above. We remark that the *conformally invariant curvatures* of [CGY06] figure in this construction in an important way.

The second goal of this paper is to study the variational problem naturally associated to this new family $(R^{(m,\alpha)}, \mathcal{L}^m_\alpha)$. As we shall see in Section 3, the Euler-Lagrange equation can be subcritical, critical (as it is for the usual scalar curvature), or even supercritical, depending on the value of α. In Section 4 we prove existence of extremals for the Lagrangian in the subcritical case. For the remaining cases existence seems to be a difficult issue.

In Section 5 we study another special case ($\alpha = 1$), and formulate a weighted L^2-eigenvalue problem. We then give a characterization of the Yamabe invariant as the solution of a mini-max problem for this eigenvalue. This result is directly inspired by Perelman's work; indeed, the Lagrangian associated to the operator $\mathcal{L}^m_1$ is (up to a constant) Perelman's entropy functional. Why it is that a Lagrangian which comes from a construction in conformal geometry should coincide with Perelman's functional—which characterizes gradient Ricci solitons—is somewhat mysterious. In some sense, Section 5 brings us full circle: what started with an observation about Perelman's scalar curvature brings us via the construction of Graham-Jenne-Mason-Sparling back to Perelman's work.

Some material in this paper was announced in [CGY06].

2. Conformally covariant operators on RM-spaces

In this Section we adapt the construction of Fefferman-Graham [FG85] and Graham-Jenne-Mason-Sparling [GJMS92] to construct families of conformally covariant differential operators associated to an RM-space. As we shall see here and in Section 3, the conformally invariant scalar and Ricci curvatures of [CGY06] arise naturally in these constructions.

Let (M^n, g) be a Riemannian manifold of dimension $n \geq 2$. A metrically defined differential operator $\mathcal{A} = \mathcal{A}_g$ is said to be conformally covariant of bi-degree (a, b) if it obeys the following transformation under a conformal change of metric $\hat{g} = e^{2w} g$:

$$\mathcal{A}_{\hat{g}}(\psi) = e^{-bw}\mathcal{A}_g(e^{aw}\psi) \tag{2.1}$$

for some constants a, b and all $\psi \in C^\infty(M^n)$. For example, when $n = 2$, $\mathcal{A}_g = \Delta_g$ is conformally covariant with $a = 0$ and $b = 2$. More generally, when $n \geq 3$ the conformal laplacian $\mathcal{A}_g = L_g = -\Delta_g + \frac{(n-2)}{4(n-1)}R(g)$ is conformally covariant with $a = (n-2)/2$ and $b = (n+2)/2$.

In [GJMS92], Graham-Jenne-Mason-Sparling constructed conformally covariant operators P_k for all positive integers k when n is odd, and for $1 \leq k \leq n/2$ when n is even, with $a = (n-2k)/2$ and $b = (n+2k)/2$. The principal part of P_k is given by $(-\Delta)^k$; when $k = 1$ then P_1 is just the conformal laplacian. These operators were derived from the ambient metric construction of Fefferman-Graham which is briefly described below. Aside from their intrinsic interest, they have also played a role in the recent work of Fefferman-Graham [FG02], Fefferman-Hirachi [FH03], and Graham-Zworksi [GZ03]. Given an RM-space (M^n, g, m), we can modify the method of [GJMS92] to derive a family of operators $\mathcal{A}_g^m$ satisfying

$$\hat{g} = e^{2w} g \quad \Longrightarrow \quad \mathcal{A}_{\hat{g}}^m(\psi) = e^{-bw}\mathcal{A}_g^m(e^{aw}\psi), \tag{2.2}$$

for some constants a, b and all $\psi \in C^\infty(M^n)$.

Theorem 2.1. *Let (M^n, g, m) be an RM-space with $n \geq 3$. Let k be a positive integer; if n is even we assume in addition that $1 \leq k \leq n/2$. For $\alpha \in \mathbb{R}$, denote $\beta_k(\alpha) = (n\alpha - n + 2k)/2$. Then, given any $\alpha \in \mathbb{R}$ there is an operator $P_{\alpha,k}^m$ satisfying (2.2) with $a = -\beta_k(\alpha)$ and $b = 2k - \beta_k(\alpha)$, the leading term of which is given by*

$$P_{\alpha,k}^m = \big(-\Delta_g + \alpha\langle\nabla f, \nabla\,\cdot\,\rangle\big)^k + \cdots \tag{2.3}$$

When $\alpha = 0$ the operator $P^m_{\alpha,k}$ coincides with P_k. For $k = 1$ we have the formula

$$P^m_{\alpha,k}(\psi) = -\Delta_g \psi + \alpha\langle \nabla f, \nabla \psi\rangle + \frac{n-2-n\alpha}{2(n-2)}\big(\alpha \Delta_g f + \frac{n\alpha + n - 2}{2(n-1)} R(g)\big)\psi. \tag{2.4}$$

As in [GJMS92], our operators are constructed by an inductive algorithm; when k becomes large the formulas become increasingly complicated. In fact, Graham-Jenne-Mason-Sparling presented two (equivalent) ways of deriving their operators. We will briefly describe one of their methods, indicating the modifications necessary to produce the measure-dependent operators $P^m_{\alpha,k}$.

To begin, given a Riemannian manifold (M^n, g), let $\mathcal{G} \subset S^2T^*M^n$ denote the ray bundle consisting of metrics in the conformal class of g. Fixing a representative $g \in [g]$ determines a fiber variable t on $\mathcal{G}$, by writing a general point in $\mathcal{G}$ in the form $(x, t^2 g(x))$. If $\{x^i\}$ are local coordinates on M^n, the coordinate system (t, x^i) on $\mathcal{G}$ extends to a coordinate system (t, x^i, ρ) on $\tilde{\mathcal{G}} = \mathcal{G} \times (-1, 1)$, where ρ is a defining function for $\mathcal{G}$, homogeneous of degree 0 (see [FG85] for details). Using these coordinates we can define the *ambient metric* $\tilde{g}$ on $\tilde{\mathcal{G}}$ by

$$\tilde{g} = 2\rho dt^2 + 2t dt\, d\rho + t^2 g_{ij}(x,\rho) dx^i dx^j, \tag{2.5}$$

where $g_{ij}(x, 0) = g_{ij}(x)$ is the given representative of $[g]$. For $\rho \neq 0$ the Taylor expansion of $g_{ij}(x, \rho)$ is determined by formally solving the Einstein equation

$$Ric(\tilde{g}) = 0. \tag{2.6}$$

We remark that in the construction of [FG85], when n is even, (2.6) determines the Taylor coefficient of g_{ij} up to the $(\rho)^{\frac{n}{2}}$ term; the trace part of $g^{ij}(\partial_\rho)^{\frac{n}{2}} g_{ij}$ is determined at $\rho = 0$ but the trace-free part of $(\partial_\rho)^{\frac{n}{2}} g_{ij}$ is not. When n is odd, (2.6) determines the expansion of all orders. This partially explains the constraint on the order k for the existence part of the GJMS operator when the dimension n is even.

Let $\delta_s : \mathcal{G} \to \mathcal{G}$ denote the dilations $\delta_s(g) = s^2 g$, with $s > 0$. Functions on $\mathcal{G}$ which are homogeneous of degree β with respect to δ_s are known as *conformal densities of weight* β. Given a density ϕ of weight β, consider the problem of extending ϕ to a harmonic function on $\tilde{\mathcal{G}}$ with the same homogeneity. That is, we want to find the formal power series solution of

$$\tilde{\Delta}(t^\beta \phi) = 0. \tag{2.7}$$

The operators of [GJMS92] arise as the obstruction to formally solving (2.7) with $\beta + \frac{1}{2}n = k = 1, 2, 3, \dots$.

Given an RM-space (M^n, g, m) we can also construct the ambient metric $\tilde{g}$, but we need to extend the density function f associated to m as well.

Lemma 2.1. *Let (M^n, g, m) be a Riemannian measure space with $dm = e^{-f} dVol(g)$. Let k be a positive integer; if n is even we assume in addition that $1 \leq k < \frac{n}{2}$. Then there is an extension $\tilde{f} : \tilde{\mathcal{G}} \to \mathbb{R}$ with $\tilde{f}(t, x, \rho) = f(x, \rho) + n \log t$, such that $f(x, 0) = f(x)$ for all $x \in M^n$, and $\tilde{f}$ satisfies*

$$\tilde{\Delta}\tilde{f} = O(\rho^k) \tag{2.8}$$

near $\mathcal{G}$ *on* $\tilde{\mathcal{G}}$.

This Lemma is a special case of Proposition 2.2 in [GJMS92]; see also Lemma 2.1 in [FH03]. In order to make the paper self contained–and to derive specific formulas for $P^m_{\alpha,k}$ in (2.4) for the case $k=1$–we will outline the proof here.

Proof. We will establish (2.8) by induction on k. Given a function ψ defined on the ambient space $\psi = \psi(t,x,\rho)$, denote $\psi' = \frac{\partial}{\partial\rho}\psi$, $\psi'' = \frac{\partial^2}{\partial\rho^2}\psi$. Then

$$\tilde{\Delta}\psi \;=\; t^{-2}\{\Delta_g\psi + (n-2)\psi' - 2\rho\psi'' + 2t\partial_t\psi' + \frac{1}{2}tg^{ij}g'_{ij}\partial_t\psi - \rho(log|g|)'\psi'\}, \tag{2.9}$$

where $g = g_{ij}(x,\rho)dx_i dx_j$. Thus, for a function $\tilde{f}(t,x,\rho) = f(x,\rho) + n\log t$ with $f(x,0) = f(x)$ we have

$$t^2\tilde{\Delta}\tilde{f} \;=\; \Delta_g\tilde{f} + (n-2)f' - 2\rho f'' + 2t\partial_t f' + \frac{n}{2}g^{ij}g'_{ij} - \rho(log|g|)'f'. \tag{2.10}$$

To see that $\tilde{f}$ can be chosen to satisfy the equation (2.8) for $k=1$ and all $n>2$, we use the identities

$$\begin{aligned} g'_{ij}(x,0) = 2P_{ij} &= \frac{2}{n-2}\{R_{ij} - \frac{1}{2(n-1)}Rg_{ij}\} \\ (log|g|)'|_{\rho=0} &= \frac{1}{(n-1)}R, \end{aligned} \tag{2.11}$$

where R_{ij} and R are respectively the Ricci and scalar curvature of the metric g. Substituting these into the formula (2.10), we see that (2.8) for $k=1$ is equivalent to finding $f(x,\rho)$, with

$$f'(x,0) = -\frac{1}{n-2}\Delta_g f(x) - \frac{n}{2(n-1)(n-2)}R, \tag{2.12}$$

which can easily be done.

To see that (2.8) can be solved for all k with $1 \le k < \frac{n}{2}$ if n is even and for all k when n is odd, we apply the same strategy that appears in the construction of the operators in [GJMS92]. That is, we inductively differentiate $\tilde{\Delta}\tilde{f}$ exactly $(k-1)$-times w.r.t. ρ, then evaluate at $\rho=0$. For example, when $k=2$, using the identities in (2.11) and doing some routine calculations we obtain

$$\begin{aligned} t^2(\tilde{\Delta}\tilde{f})'|_{\rho=0} = &-2P^{ij}\nabla_i\nabla_j f - \frac{1}{2(n-1)}\nabla_j R\nabla_j f \\ &+ \Delta_g f' + (n-4)f'' - nP^{ij}P_{ij} - \frac{1}{(n-1)}Rf'. \end{aligned} \tag{2.13}$$

From (2.13), it is clear that to solve (2.8) for $k=2$ and $n \neq 4$ one only needs to choose $f(x,\rho)$ with $f''(x,0)$ satisfying

$$\begin{aligned} (n-4)f''(x,0) = &\, 2P^{ij}\nabla_i\nabla_j f(x) + \frac{1}{2(n-1)}\nabla R\nabla f(x) \\ &- \Delta_g f'(x,0) + nP^{ij}P_{ij}(x) + \frac{1}{(n-1)}Rf'(x,0), \end{aligned} \tag{2.14}$$

with $f'(x,0)$ satisfying equation (2.12). We refer to [GJMS92] for the proof of the general k. $\square$

Proof of Theorem 2.1 To derive the operators $P^m_{\alpha,k}$ we replace (2.7) with

$$-\tilde{\Delta}(t^{\beta}\phi) + \alpha < \tilde{\nabla}\tilde{f}, \tilde{\nabla}\phi >= 0, \tag{2.15}$$

where $\phi = \phi(x,\rho)$ is any extension of a given function ϕ defined on M and where $\tilde{f}$ is an extension of f chosen according to Lemma 2.1. The operators $P^m_{\alpha,k}$ arise as the obstruction to formally solving (2.15) up to order ρ^k independent of the extension $\phi = \phi(x,\rho)$ of ϕ. We then find that a suitable choice of β is $\beta = \beta_k(\alpha) = (n\alpha - n + 2k)/2$ for each $k \geq 1$ when n is odd, and for $1 \leq k < \frac{n}{2}$ when n is even. As the proof is by induction on k and very similar to the proof in [GJMS92] we will only give an outline.

Given a smooth function $\tilde{\phi} = \phi(t,x,\rho)$ defined on the ambient space $\tilde{\mathcal{G}}$, we define the operator

$$\tilde{L}^m_{\alpha,\tilde{g}}(\tilde{\phi}) = -\tilde{\Delta}(\tilde{\phi}) + \alpha < \tilde{\nabla}\tilde{f}, \tilde{\nabla}\tilde{\phi} > . \tag{2.16}$$

Let $\phi, f \in C^\infty(M)$ and suppose $\phi(x,\rho)$ and $f(x,\rho)$ are smooth extensions defined on $\mathcal{G}$; i.e., $\phi(x,0) = \phi(x)$ and $f(x,0) = f(x)$. Given $\beta \in \mathbb{R}$, denote $\tilde{\phi}(t,x,\rho) = t^{\beta}\phi(x,\rho)$ and $\tilde{f}(t,x,\rho) = f(x,\rho) + n\,log\,t$; then

$$\begin{aligned}\tilde{L}^m_{\alpha\tilde{g}}(\tilde{\phi}) = t^{\beta-2}\{&2\rho\phi'' - [2\beta + (n-2) - \frac{1}{n-1}\rho R - n\alpha]\phi' - \Delta_g\phi \\ &- \frac{1}{2(n-1)}\beta R\phi + \alpha\beta\phi f' + \alpha g^{ij}\nabla_i\phi\nabla_j f - 2\rho\alpha\phi' f'\}.\end{aligned} \tag{2.17}$$

Therefore,

$$\begin{aligned}t^{2-\beta}\tilde{L}^m_{\alpha,\tilde{g}}(\tilde{\phi})|_{\rho=0} = &[n\alpha - (n-2) + 2\beta]\,\phi' - \Delta_g\phi \\ &- \frac{1}{2(n-1)}\beta R\phi + \alpha\beta\phi f' + \alpha < \nabla_g\phi, \nabla_g t > .\end{aligned} \tag{2.18}$$

Consequently, if we choose $\beta = \beta_1(\alpha)$ so that $n\alpha - (n-2) - 2\beta = 0$, and choose $\tilde{f}$ to satisfy (2.12) in Lemma 2.1, the operator $P^m_{\alpha,1}$ given by

$$P^m_{\alpha,1}(\phi) = t^{2-\beta}L^m_{\alpha,\tilde{g}}(\tilde{\phi})|\rho = 0 \tag{2.19}$$

is well defined and satisfies covariance property

$$(P^m_{\alpha,1})_{\hat{g}}(\phi) = e^{(\beta-2)w}(P^m_{\alpha,1})_g(e^{-\beta w}\phi) \tag{2.20}$$

for all functions $\phi \in C^\infty(M)$, where $\hat{g} = e^{2w}g$. Note in the formula of $P^m_{\alpha,1_{\hat{g}}}$ we should replace f by $\hat{f} = f + nw$. The explicit formula for $P^m_{\alpha,1}$ for the choice of f' in (2.12) is given by (2.4).

As before, for general k the idea of the proof is to differentiate the term $\tilde{\mathcal{L}}^m_{\alpha,\tilde{g}}(\tilde{\phi})$ exactly $(k-1)$-times w.r.t. ρ and inductively define the operators $P^m_{\alpha,k}$ in a similar fashion. We refer to [GJMS92] for details.

Remarks.
1. The conformally invariant curvatures of [CGY06] can also be defined in terms of the extension $\tilde{f}$. For example, $R_n^m(g)$ is given by

$$R_n^m(g) = -\frac{(n-1)(n-2)}{n^2}|\tilde{\nabla}\tilde{f}|^2\Big|_{M^n}. \tag{2.21}$$

2. When n is even, the operators of [GJMS92] exist up to $k \leq \frac{n}{2}$, but our construction above only gives the existence of operators for $k < \frac{n}{2}$ due to the choice of the extension $\tilde{f}$ in Lemma 2.1. However, when $k = 1$ the preceding Remark indicates a way of modifying our construction, as follows. First, note that one can add a multiple of $R_n^m(g)$ to the operator $P_{\alpha,1}^m$ and obtain an operator with the same conformal covariance property. For example, if one adds the term $CR_n^m(g)$, with $C = C(\alpha, n) = \frac{n^2}{4(n-1)(n-2)}\alpha\beta_1(\alpha)$, then the operator defined by

$$\begin{aligned}\tilde{L}_{\alpha,1}^m(\phi) &= P_{\alpha,1}^m(\phi) + C(\alpha,n)R_n^m(g)\phi \\ &= -\Delta_g\phi + \alpha < \nabla_g f, \nabla_g\phi > \\ &- \beta_1[(1-\frac{n\alpha}{2})\frac{1}{2(n-1)}R_g - \frac{\alpha}{2}\Delta_g f + \frac{\alpha}{4}|\nabla_g f|^2]\phi\end{aligned} \tag{2.22}$$

satisfies the conformal covariance property (2.2), with $a = -\beta_1(\alpha)$ and $b = 2-\beta_1(\alpha)$. It has the additional advantage that it exists for all $n \geq 2$, including $n = 2$. When $k \geq 2$, it is not yet clear how to modify the operator $P_{\alpha,k}^m$. On the other hand, the existence of m-conformally covariant operators for all k when n is even and for $1 \leq k \leq \frac{n}{2}$ (when n is odd) follows from an observation of R. Graham. The details are given in the next Remark.

3. R. Graham pointed out to us another possible construction of conformally covariant operators on RM-spaces, by using the operators P_k of [GJMS92]. Letting

$$G_{\alpha,k}^m(\phi) = e^{\frac{\alpha f}{2}}P_k(e^{-\frac{\alpha f}{2}}\phi), \tag{2.23}$$

it is easy to see that these operators satisfy the same conformal covariance as the operators $P_{\alpha,k}^m$ in Theorem 2.1. Interestingly, in general $G_{\alpha,k}^m$ and $P_{\alpha,k}^m$ do not agree. For example, when $k = 1$ they again differ by a multiple $C_{\alpha,n}$ of $R_n^m(g)$.

3. Properties of the operators

In this section we will discuss some properties of the operators constructed in Section 2. To simplify the presentation, we will restrict ourselves to a discussion of the case $k = 1$.

As before, (M^n, g, dm) will be an RM-space, and $dm = e^{-f} dv_g$ defines the density function f. Let us denote

$$\begin{aligned} \mathcal{L}_\alpha^m \psi &= P^m_{\alpha,k=1}\psi \\ &= -\Delta_g \psi + \alpha\langle \nabla f, \nabla \psi\rangle + \frac{n-2-n\alpha}{2(n-2)}\big(\alpha \Delta_g f + \frac{n\alpha+n-2}{2(n-1)} R(g)\big)\psi. \end{aligned} \tag{3.1}$$

We begin by summarizing some elementary properties of the operators $\mathcal{L}_\alpha^m$.

Proposition 3.1. *For each $\alpha \in \mathbb{R}$,*

(i) $\mathcal{L}_\alpha^m$ is self-adjoint with respect to the measure

$$dm_\alpha = e^{-\alpha f} dVol(g). \tag{3.2}$$

(ii) Suppose $\hat{g} = e^{2w} g$ is a conformal metric, then

$$(\mathcal{L}_\alpha^m)_{\widehat{g}}(\phi) = e^{(\beta(\alpha)-2)w} \mathcal{L}_\alpha^m (e^{-\beta(\alpha)w}\phi), \tag{3.3}$$

for all $\phi \in C^\infty(M)$, where

$$\beta(\alpha) = \frac{n\alpha - n + 2}{2}. \tag{3.4}$$

(iii) Denote $v = v_\alpha = e^{-\beta(\alpha)w}$. Then

$$(\mathcal{L}_\alpha^m)_{\widehat{g}}(1) = v^{-\gamma_\alpha} (\mathcal{L}_\alpha^m)_g(v), \tag{3.5}$$

where

$$\gamma_\alpha = \frac{n+2-n\alpha}{n-2-n\alpha}, \quad \alpha \neq \frac{n-2}{n}. \tag{3.6}$$

Proof. The properties $(i) - (iii)$ follow from the properties of the operators $P^m_{\alpha,k=1}$ described in Section 2. □

Remarks.

1. The properties of $\mathcal{L}_\alpha^m$ listed in Proposition 3.1 are shared by any operator which differs from $\mathcal{L}_\alpha^m(g)$ by a constant multiple of $R_n^m(g)$. In particular, the operators $\mathcal{G}_\alpha^m$ satisfy the same properties.

2. One can interpret equation (3.5) as defining a a scalar curvature associated to the triple (g, m, α). Let

$$R^{(m,\alpha)} = R^{(m,\alpha)}(g) = \frac{n-2-n\alpha}{n-2}\big(R(g) + \frac{2\alpha(n-1)}{n-2+n\alpha}\Delta_g f\big). \tag{3.7}$$

We will refer to $R^{(m,\alpha)}$ as the (g, m, α)-*scalar curvature*, or just the α-*scalar curvature* if the context is clear. Note we can also write

$$R^{(m,\alpha)}(g) = \frac{4(n-1)}{(n-2+n\alpha)}\mathcal{L}_\alpha^m(1). \tag{3.8}$$

By (3.5) and (3.8), given a conformal metric

$$\hat{g} = e^{2w} g = v^{\frac{4}{n-\alpha n-2}} g, \tag{3.9}$$

the α-scalar curvature of $\hat{g}$ is given by

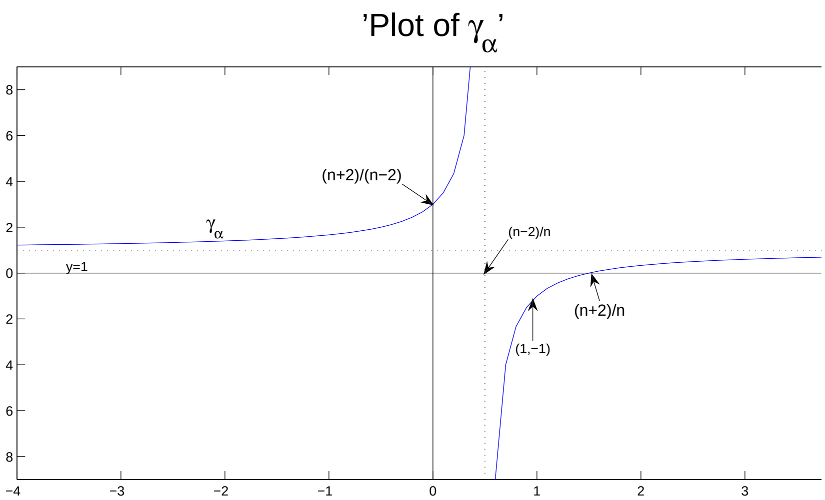
$$R^{(m,\alpha)}(\hat{g}) = \frac{4(n-1)}{(n-2+n\alpha)} v^{-\gamma_\alpha} (\mathcal{L}_\alpha^m)_g(v). \tag{3.10}$$

These formulas define a pair $(R^{(m,\alpha)}, \mathcal{L}_\alpha^m)$ generalizing the well known example of the scalar curvature/conformal laplacian (R, L). Indeed, the pair (R, L) is just $(R^{(m,0)}, \mathcal{L}_0^m)$, i.e., the case $\alpha = 0$.

3. It is interesting to note that the semilinear equation (3.10) associated to the α-scalar curvature can be sub-critical, critical, or super-critical with respect to the Sobolev imbedding, depending on α. To see this, we note the following apparent properties of the exponent γ_α:

$$\begin{aligned}
&(a.)\ \gamma_0 = \frac{n+2}{n-2}, \gamma_1 = -1, \gamma_{(n+2)/n} = 0. \\
&(b.)\ \lim_{\alpha \to \pm\infty} \gamma_\alpha = 1. \\
&(c.)\ \frac{d}{d\alpha}\gamma_\alpha = \frac{4n}{(n-2-n\alpha)^2}, \quad \alpha \neq \frac{n-2}{n}. \\
&(d.)\ \lim_{\alpha \to (\frac{n-2}{n})^-} \gamma_\alpha = +\infty. \\
&(e.)\ \lim_{\alpha \to (\frac{n-2}{n})^+} \gamma_\alpha = -\infty.
\end{aligned} \tag{3.11}$$

The figure below shows γ_α as a function of α.



4. When $\alpha = (n-2)/n$ one needs to modify the definition of the α-scalar curvature, since the definition (3.7) gives zero. In addition, one sees from the figure above that the exponent in equation (3.10) becomes infinite. Using an ansatz due to Branson known as *continuation in the dimension*, we can construct an operator T^m to supplant

$$\mathcal{L}^m_{(n-2)/n} = -\Delta + \frac{n-2}{n}\langle \cdot, \nabla f\rangle, \tag{3.12}$$

and this permits us to define a scalar curvature $K^m(g)$ corresponding to the case $\alpha = (n-2)/n$. Indeed, denote $\bar{\alpha} = (n-2)/n$, and define

$$\begin{aligned} T^m_g \phi &= \lim_{\alpha\to\bar{\alpha}} \frac{1}{\beta(\alpha)}\big[\mathcal{L}^m_\alpha(e^{\beta(\alpha)\phi}) - \mathcal{L}^m_\alpha(1)\big] \\ &= \lim_{\alpha\to\bar{\alpha}} \frac{1}{\beta(\alpha)}\big[-\Delta(e^{\beta(\alpha)\phi}-1) + \alpha\langle \nabla((e^{\beta(\alpha)\phi}-1), \nabla f\rangle \\ &\qquad - \frac{\beta}{(n-2)}\big(\alpha\Delta f + \frac{n\alpha+n-2}{2(n-1)}R\big)(e^{\beta(\alpha)\phi}-1)\big] \\ &= -\Delta w + \frac{n-2}{n}\langle \nabla w, \nabla f\rangle. \end{aligned}$$

We also define

$$K^m(g) = \lim_{\alpha\to\bar\alpha} \frac{1}{\beta(\alpha)}\mathcal{L}^m_\alpha(1)$$
$$= -\frac{1}{(n-1)}\big(R(g) + \frac{(n-1)}{n}\Delta f\big).$$

If $\hat{g} = e^{2w}g$, then

$$T^m_{\hat{g}} = e^{-2w}T^m_g, \tag{3.13}$$

in analogy with the laplacian on surfaces. Also, the behavior of K^m under a conformal change is given by

$$T^m_g w + c_n K^m(\hat{g})e^{2w} = c_n K^m(g), \tag{3.14}$$

where

$$c_n = \frac{(n-2)}{(n-1)}.$$

Note the obvious parallel with the prescribed Gauss curvature equation.

As we observed in Remark 3. above, equation (3.10) can be sub-critical, critical, or supercritical depending on the value of α. In the next Section we will study the existence of conformal metrics with constant α-scalar curvature for the sub-critical case; i.e., $-\infty < \alpha < 0$ and $\alpha > 1$.

4. $-\infty < \alpha < 0$ and $\alpha > 1$: the sub-critical cases

To introduce the variational problems associated to the operators defined in Section 3 we define the functionals

$$\begin{aligned} E^m_\alpha[v] &= \int v\ \mathcal{L}^m_\alpha v\ dm_\alpha(g) \\ &= \big\langle v, \mathcal{L}^m_\alpha v\big\rangle_{L^2(dm_\alpha)}, \end{aligned} \tag{4.1}$$

where

$$dm_\alpha(g) = e^{-\alpha f}dVol(g).$$

We also define the constraint set

$$\mathcal{C}_\alpha = \big\{v \in W^{1,2}(M)\ \big|\ v \geq 0, \int v^{\gamma_\alpha+1}\ dm_\alpha(g) = 1\big\}. \tag{4.2}$$

Note that

$$1 + \gamma_\alpha = \frac{2n(\alpha-1)}{n(\alpha-1)+2}, \tag{4.3}$$

which is positive when $-\infty < \alpha < 0$ or $\alpha > 1$.

Consider the variational problem

$$\inf_{v\in\mathcal{C}_\alpha} E^m_\alpha[v]. \tag{4.4}$$

By the identity (3.10) this is equivalent to the following geometric variational problem: define

$$\mathcal{R}^{(m,\alpha)} : g \mapsto \int R^{(m,\alpha)}(g)\ dm_\alpha(g), \tag{4.5}$$

$$\mathcal{C}_\alpha([g]) = \left\{ \hat{g} = v^{\frac{4}{n-\alpha n-2}} g \ \middle|\ v \in C^\infty(M), v > 0, v \in \mathcal{C}_\alpha \right\}. \tag{4.6}$$

Then

$$\mathcal{R}^{(m,\alpha)}[\hat{g}] = \frac{4(n-1)}{(n-2+n\alpha)} E^m_\alpha[v], \tag{4.7}$$

where $\hat{g} = v^{\frac{4}{n-\alpha n-2}} g$. Consequently,

$$\inf_{\hat{g}\in\mathcal{C}_\alpha([g])} \mathcal{R}^{(m,\alpha)}[\hat{g}] = \frac{4(n-1)}{(n-2+n\alpha)} \inf_{v\in\mathcal{C}_\alpha} E^m_\alpha[v], \quad \text{if } \alpha < \frac{n-2}{n}, \tag{4.8}$$

and

$$\sup_{\hat{g}\in\mathcal{C}_\alpha([g])} \mathcal{R}^{(m,\alpha)}[\hat{g}] = \frac{4(n-1)}{(n-2+n\alpha)} \inf_{v\in\mathcal{C}_\alpha} E^m_\alpha[v], \quad \text{if } \alpha > \frac{n-2}{n}. \tag{4.9}$$

Again, when $\alpha = 0$ we recover the familiar relation between the total scalar curvature and the Yamabe quotient. Moreover, when $\alpha < 0$ or $\alpha > 1$, the exponent in the definition of the constraint set $\mathcal{C}_\alpha$ is subcritical for the Sobolev embedding.

Theorem 4.1. (*i*) *Suppose $\alpha \leq 0$ or $\alpha > 1$. Then*

$$\inf_{v\in\mathcal{C}_\alpha} E^m_\alpha[v] > -\infty. \tag{4.10}$$

(ii) If $\alpha < 0$ or $\alpha > 1$, then the infimum in (4.10) is attained by a positive function $v = v_\alpha \in C^\infty(M)$ satisfying

$$\mathcal{L}^m_\alpha v_\alpha = c v^{\gamma_\alpha} \tag{4.11}$$

for some constant c.

Equivalently, if $\alpha < 0$ there is a conformal metric $\hat{g} = v_\alpha^{\frac{4}{n-\alpha n-2}} g \in \mathcal{C}_\alpha([g])$ which attains the infimum of $\mathcal{R}^{m,\alpha}$; if $\alpha > 1$ there is a conformal metric $\hat{g} = v_\alpha^{\frac{4}{n-\alpha n-2}} g \in \mathcal{C}_\alpha([g])$ which attains the supremum of $\mathcal{R}^{m,\alpha}$. In both cases, the α-scalar curvature of $\hat{g}$ satisfies

$$R^{(m,\alpha)}(\hat{g}) = c.$$

Proof. To verify (4.10), let $v \in \mathcal{C}_\alpha$; then

$$E^m_\alpha[v] \geq \int |\nabla v|^2\ dVol(g) - C(g,f) \int v^2\ dVol(g). \tag{4.12}$$

Therefore, by the Sobolev imbedding theorem,

$$
\begin{aligned}
\Big(\int v^{\frac{2n}{n-2}}\ dVol(g)\Big)^{(n-2)/n} &\le C\|v\|_{W^{1,2}} \\
&\le C\big(E_\alpha^m[v] + \int v^2\ dVol(g)\big).
\end{aligned} \tag{4.13}
$$

When $\alpha \le 0$, then $1+\gamma_\alpha$ satisfies

$$
2 < 1+\gamma_\alpha \le \frac{2n}{n-2}, \tag{4.14}
$$

and by Hölder's inequality

$$
\int v^2\ dV \le \Big(\int v^{1+\gamma_\alpha}\ dV\Big)^{\frac{2}{1+\gamma_\alpha}} \le C. \tag{4.15}
$$

It follows from (4.13) that

$$
\begin{aligned}
E_\alpha^m[v] &\ge C\|v\|^2_{\frac{2n}{n-2}} - C \\
&\ge -C.
\end{aligned} \tag{4.16}
$$

If $\alpha > 1$, then by Hölder's inequality

$$
\int v^2\ dV \le \Big(\int v^{\frac{2n}{n-2}}\ dV\Big)^{\theta}\Big(\int v^{1+\gamma_\alpha}\ dV\Big)^{1-\theta}, \tag{4.17}
$$

where

$$
\theta = \frac{n-2}{\alpha n} < \frac{n-2}{n}. \tag{4.18}
$$

Substituting this into (4.13) and using the constraint one verifies that (4.16) also holds for $\alpha > 1$.

For existence, we now suppose $\alpha < 0$ or $\alpha > 1$, and let $\{v_k\}$ be a minimizing sequence for E_α^m with $v_k \in \mathcal{C}_\alpha$. We may assume

$$
E_\alpha^m[v_k] \le \inf_{\mathcal{C}_\alpha} E_\alpha^m + 1.
$$

By (4.16) we see that

$$
\|v_k\|_{2n/(n-2)} \le C,
$$

and from (4.12) we conclude that $\{v_k\}$ is bounded in $W^{1,2}$. Since

$$
1+\gamma_\alpha < \frac{2n}{n-2}
$$

when $\alpha < 0$ or $\alpha > 1$, the embedding

$$
W^{1,2} \hookrightarrow L^{1+\gamma_\alpha}
$$

is compact. Therefore, a subsequence of $\{v_k\}$ will converge weakly in $W^{1,2}$, but strongly in $L^{1+\gamma_\alpha}$, to a minimizer $v \in \mathcal{C}_\alpha$. Using the fact that $\mathcal{L}_\alpha^m$ is self-adjoint it is easy to check that a $W^{1,2}$-critical point of E_α^m subject to the constraint in (4.2) will satisfy (4.11) weakly. Elliptic regularity implies $v \in C^\infty$. $\square$

4.1. $\alpha = 2$**: Perelman's scalar curvature.** The case $\alpha = 2$ is of particular interest. Note that

$$R^{(m,2)}(g) = -(\frac{n+2}{n-2})\big(R(g) + \frac{4(n-1)}{(3n-2)}\Delta f\big).$$

Recall the definition of the conformally invariant scalar curvature in [CGY06]:

$$R_n^m(g) = R(g) + \frac{2(n-1)}{n}\Delta f - \frac{(n-1)(n-2)}{n^2}|\nabla f|^2.$$

When $n = \infty$, this corresponds formally to the scalar curvature introduced by Perelman [Per02]:

$$R_\infty^m(g) = R(g) + 2\Delta f - |\nabla f|^2.$$

Comparing these formulas we see that

$$R^{(m,2)}(g) = \frac{(n+2)(n-1)}{(3n-2)}R_\infty^m(g) - \frac{n^2(n+2)}{(3n-2)(n-2)}R_n^m(g). \tag{4.19}$$

In particular, if we define the operator

$$\mathcal{L}_{2,\infty}^m = \mathcal{L}_2^m + \frac{n^2(n+2)}{4(n-1)(n-2)}R_n^m(g), \tag{4.20}$$

then by Remark 1 following Proposition 3.1, $\mathcal{L}_{2,\infty}^m$ enjoys the same conformal covariance properties as $\mathcal{L}_2^m$. One can check that

$$\mathcal{L}_{2,\infty}^m = -\Delta + 2\langle\nabla\cdot, \nabla f\rangle + \frac{n+2}{4}R_\infty^m(g), \tag{4.21}$$

so that the "scalar curvature" associated to $\mathcal{L}_{2,\infty}^m$ is a multiple of Perelman's scalar curvature. This leads to the following corollary of Theorem 4.1:

Corollary 4.1. *Given an RM-space (M, g, m), there is a conformal metric $\hat{g} = v^{-\frac{4}{n+2}}g$ with*

$$R_\infty^m(\hat{g}) = const. \tag{4.22}$$

Moreover, v can be realized as the infimum of the functional

$$E_{2,\infty}^m[\phi] = \int \langle\phi, \mathcal{L}_{2,\infty}^m\phi\rangle\ dm_2(g) \tag{4.23}$$

subject to the constraint $\int \phi^{\frac{2n}{n+2}}\ dm_2 = 1$.

5. The case $\alpha = 1$: Perelman's Entropy functional

For the borderline case $\alpha = 1$, the parameter $\gamma_{-1} = -1$, and the measure $m_{-1} = m$. Also, the 1-scalar curvature is given by

$$R^{(m,1)}(g) = 2\mathcal{L}_1^m(1) = -\frac{2}{n-2}(\Delta f + R(g)). \tag{5.1}$$

It follows that the functional $\mathcal{R}^{(m,1)}$ defined in (4.5) is

$$\mathcal{R}^{(m,1)}[g] = \int R^{(m,1)}(g)\ dm = -\frac{2}{n-2}\int (R(g) + \Delta f)\ dm = -\frac{2}{n-2}\int (R(g) + |\nabla f|^2)\ dm. \tag{5.2}$$

Up to a constant, this is precisely the entropy functional defined by Perelman in §1 of [Per02]. The difficulty in studying the corresponding variational problem (4.4) is that the constraint set $\mathcal{C}_\alpha$ is not well defined when $\alpha = 1$, since then $\gamma_1 = -1$ (or, to be more precise, it does not impose any constraint). In this Section we study a related eigenvalue problem inspired by Perelman's work and point out an interesting connection to the Yamabe invariant.

To begin, let us introduce the modified constraint set

$$\mathcal{D}^m(g) = \left\{ v \in W^{1,2}(M) \;\middle|\; v \geq 0,\ \int v^2 e^{-\frac{2}{n}f}\ dm = 1 \right\}. \tag{5.3}$$

In a slight abuse of notation we will write $\hat{g} = v^{-2}g \in \mathcal{D}^m(g)$ whenever $v > 0, v \in C^\infty(M^n)$, and $v \in \mathcal{D}^m(g)$.

A key property used in our analysis is that the functional E_1^m enjoys a certain conformal covariance when restricted to $\mathcal{D}^m$. To explain this, let us modify our notation slightly to emphasize the dependence of E_1^m on the choice of metric, and write

$$E_1^m(g)[v] = E_1^m[v] = \left\langle v, (\mathcal{L}_1^m)_g v \right\rangle_{L^2(dm)}. \tag{5.4}$$

Lemma 5.1. *For all smooth functions $\rho > 0$, $v \in W^{1,2}(M^n)$, we have*

$$E_1^m(g)[v] = E_1^m(\rho^2 g)[\rho v], \tag{5.5}$$

$$v \in \mathcal{D}^m(g) \Leftrightarrow \rho v \in \mathcal{D}^m(\rho^2 g). \tag{5.6}$$

Proof. To prove (5.5), we use the covariance of $\mathcal{L}_1^m$ given in Proposition 3.1, (ii): If $\hat{g} = e^{2w}g$, then

$$(\mathcal{L}_1^m)_{e^{2w}g}\phi = e^{-w}(\mathcal{L}_1^m)_g(e^{-w}\phi).$$

Taking $e^w = \rho$, this implies

$$\begin{aligned} E_1^m(\rho^2 g)[\rho v] &= \int \langle \rho v, (\mathcal{L}_1^m)_{\rho^2 g}(\rho v)\rangle\ dm \\ &= \int \langle \rho v, \rho^{-1}(\mathcal{L}_1^m)_g(\rho^{-1}\rho v)\rangle\ dm \\ &= \int \langle v, (\mathcal{L}_1^m)_g v\rangle\ dm \\ &= E_1^m(g)[v]. \end{aligned}$$

To prove (5.6), suppose $\rho > 0$ is smooth and write

$$dm = e^{-f_\rho} dVol(\rho^2 g),$$

where $f_\rho = f + n\log\rho$. Therefore,

$$\begin{aligned}\int (\rho v)^2 e^{-\frac{2}{n}f}\ dm &= \int (\rho v)^2 e^{-\frac{2}{n}(f+n\log\rho)}\ dm \\ &= \int (\rho v)^2 e^{-\frac{2}{n}f_\rho}\ dm.\end{aligned}$$

If follows that $v \in \mathcal{D}^m(g)$ if and only if $(\rho v) \in \mathcal{D}^m(\rho^2 g)$, as claimed. $\square$

For simplicity we now adopt Perelman's notation and write

$$\begin{aligned}\mathcal{F}^m[g] &= -\frac{(n-2)}{2}\mathcal{R}^{(m,1)}[g] \\ &= \int (R(g) + \Delta f)\ dm \\ &= \int (R(g) + |\nabla f|^2)\ dm.\end{aligned} \tag{5.7}$$

It will be convenient if we normalize the measure m to have total mass one; let $\mathcal{P}$ denote the set of all such smooth probability measures on M^n.

Theorem 5.1. *Let (M^n, g) be a Riemannian manifold.*

(i) For each $m \in \mathcal{P}$,

$$\lambda(m,[g]) = \sup_{\hat{g}\in\mathcal{D}^m(g)} \mathcal{F}^m[\hat{g}] \tag{5.8}$$

is attained by some metric $\sigma_m \in [g]$ satisfying

$$R(\sigma_m) + \Delta_{\sigma_m} f_m = \lambda(m,[g]) e^{-\frac{2}{n}f_m}, \tag{5.9}$$

where f_m is the density function of m relative to σ_m.

(ii) Let $Y(M^n,[g])$ denote the Yamabe invariant of $[g]$. Then

$$\lambda_*([g]) = \inf_{m\in\mathcal{P}} \lambda(m,[g]) = Y(M^n,[g]), \tag{5.10}$$

and the infimum is attained by all Yamabe measures, *i.e., measures $m \in \mathcal{P}$ such that*

$$dm = e^{-f_Y} dVol(g), \tag{5.11}$$

with $g_Y = e^{-\frac{2}{n}f_Y} g$ a Yamabe metric.

Proof. (i). First, by (4.7) we have

$$\begin{aligned}\mathcal{F}^m(\hat{g}) &= -(\frac{n-2}{2})\mathcal{R}^{(m,1)}(\hat{g}) \\ &= -(n-2)E_1^m(g)(v),\end{aligned}$$

where $\hat{g} = v^{-2}g$. Therefore, the variational problem in (5.8) is equivalent to a weighted L^2-eigenvalue problem for the operator $\mathcal{L}_1^m$. It follows that there is a function $v \in C^\infty(M^n) \cap \mathcal{D}^m(g), v > 0$ satisfying the Euler-Lagrange equation

$$\mathcal{L}_1^m v = \mu v e^{-\frac{2}{n}f}, \tag{5.12}$$

where

$$\begin{aligned} \mu &= \inf_{v \in \mathcal{D}^m(g)} E_1^m(g)[v] \\ &= -\frac{1}{(n-2)} \sup_{\hat{g} \in \mathcal{D}^m(g)} \mathcal{F}^m[\hat{g}] \\ &= -\frac{1}{(n-2)} \lambda(m, [g]). \end{aligned}$$

Using (3.10), equation (5.12) implies the metric $\sigma_m = v^{-2}g$ satisfies

$$R^{(m,1)}(\sigma_m) = -\frac{2}{(n-2)} \lambda(m, [g]) v^2 e^{-\frac{2}{n}f}. \tag{5.13}$$

Since $dVol(\sigma_m) = v^{-n} dVol(g)$, it follows that $f_m = f - n \log v$, hence

$$v^2 = e^{-\frac{2}{n}f_m} e^{\frac{2}{n}f}.$$

Substituting this into (5.13) and using the definition in (5.1) we find

$$\begin{aligned} R(\sigma_m) + \Delta_{\sigma_m} f_m &= -\frac{n-2}{2} R^{(m,1)}(\sigma_m) \\ &= \lambda(m, [g]) e^{-\frac{2}{n}f_m}, \end{aligned}$$

as claimed.

(ii) We will prove part (ii) through a series of claims.

Claim 5.1. *For each $m \in \mathcal{P}$,*

$$\lambda_*([g]) \leq Y(M^n, [g]). \tag{5.14}$$

Proof. Let $g_Y = \rho_0^2 g$ denote a Yamabe metric in $[g]$ and $m_Y = dVol(g_Y)$ denote the Yamabe measure associated to g_Y. We will assume that g_Y has been normalized to have unit volume, so that dm_Y is a probability measure and

$$R(g_Y) = Y(M^n, [g]). \tag{5.15}$$

By the definitions above,

$$\begin{aligned} \lambda(m_Y, [g]) &= \sup_{v^{-2}g \in \mathcal{D}^{m_Y}(g)} \mathcal{F}^{m_Y}[v^{-2}g] \\ &= \sup_{v \in \mathcal{D}^{m_Y}(g)} -(n-2) E_1^{m_Y}(g)[v]. \end{aligned}$$

By Lemma 5.1,

$$\begin{aligned} E_1^{m_Y}(g)[v] &= E_1^{m_Y}(\rho_0^2 g)[\rho_0 v] \\ &= E_1^{m_Y}(g_Y)[\rho_0 v], \end{aligned}$$

and $v \in \mathcal{D}^{m_Y}(g) \Leftrightarrow w = \rho_0 v \in \mathcal{D}^{m_Y}(g_Y)$. Thus,

$$\lambda(m_Y, [g]) = \sup_{w \in \mathcal{D}^{m_Y}(g_Y)} -(n-2)E_1^{m_Y}(g_Y)[w].$$

Now,

$$-(n-2)E_1^{m_Y}(g_Y)[w] = \int \big[-(n-2)|\nabla w|^2 + (R(g_Y) + \Delta f_Y) w^2 \big] \, dm.$$

Since $m_Y = dVol(g_Y)$, the density function $f_Y \equiv 0$. Therefore,

$$\begin{aligned} \lambda(m_Y, [g]) &= \sup_{w \in \mathcal{D}^{m_Y}(g_Y)} -(n-2)E_1^{m_Y}(g_Y)[w] \\ &= \sup_{w \in \mathcal{D}^{m_Y}(g_Y)} \int \big[-(n-2)|\nabla w|^2 + R(g_Y) w^2 \big] \, dm \\ &\leq \sup_{w \in \mathcal{D}^{m_Y}(g_Y)} R(g_Y) \int w^2 \, dm_Y \\ &= R(g_Y) \\ &= Y(M^n, [g]). \end{aligned}$$

□

Claim 5.2. *As in* [CGY06], *define the conformally invariant functional*

$$\mathcal{S}^m[g] = \int R_n^m(g) \, e^{\frac{2}{n} f} \, dm. \tag{5.16}$$

Then for each $m \in \mathcal{P}$,

$$\lambda(m, [g]) \geq \mathcal{S}^m[\sigma_m]. \tag{5.17}$$

Proof. Recall from above the definition of $R_n^m(g)$:

$$R_n^m(g) = R(g) + \frac{2(n-1)}{n}\Delta f - \frac{(n-1)(n-2)}{n^2}|\nabla f|^2. \tag{5.18}$$

Taking $g = \sigma_m$, and using equation (5.9), we have

$$R_n^m(\sigma_m) = \frac{(n-2)}{n}\Delta f_m - \frac{(n-1)(n-2)}{n^2}|\nabla f_m|^2 + \lambda(m, [g]) e^{-\frac{2}{n}}.$$

Therefore,

$$\begin{aligned} \mathcal{S}^m[\sigma_m] &= \int \Big(\frac{(n-2)}{n}\Delta f_m - \frac{(n-1)(n-2)}{n^2}|\nabla f_m|^2 + \lambda(m, [g]) e^{-\frac{2}{n}} \Big) e^{\frac{2}{n} f_m} \, dm \\ &= \lambda(m, [g]) - \frac{(n-2)}{n^2} \int |\nabla f_m|^2 e^{\frac{2}{n} f_m} \, dm, \end{aligned}$$

which implies (5.17). □

Claim 5.3. *We have*

$$\inf_{m\in\mathcal{P}} \mathcal{S}^m[g] = Y(M^n,[g]), \tag{5.19}$$

and the infimum is achieved by a measure m_Y if and only if m_Y is a Yamabe measure.

Proof. Let $m\in\mathcal{P}$ with density function f. By (5.16) and (5.18),

$$\begin{aligned}\mathcal{S}^m[g] &= \int \big[R(g) + \frac{2(n-1)}{n}\Delta f - \frac{(n-1)(n-2)}{n^2}|\nabla f|^2\big] e^{\frac{2}{n}f}\, dm \\ &= \int \big[R(g) + \frac{(n-1)(n-2)}{n^2}|\nabla f|^2\big] e^{-\frac{n-2}{n}f}\, dVol(g).\end{aligned} \tag{5.20}$$

Let $g_m = e^{-\frac{2}{n}f}g$; then (5.20) implies that

$$\mathcal{S}^m[g] = \int R(g_m)\, dVol(g_m).$$

Since m is a probability measure, g_m has unit volume, and it follows that

$$\begin{aligned}\inf_{m\in\mathcal{P}} \mathcal{S}^m[g] &= \inf_{g_m = e^{-\frac{2}{n}f}g} \int R(g_m)\, dVol(g_m) \\ &= Y(M^n,[g]).\end{aligned}$$

□

Combining (5.17) and (5.19), we see that for any $m\in\mathcal{P}$,

$$\begin{aligned}\lambda(m,[g]) &\geq \mathcal{S}^m[\sigma_m] \\ &\geq \inf_{m\in\mathcal{P}} \mathcal{S}^m[g] \\ &= Y(M^n,[g]).\end{aligned}$$

Therefore,

$$\lambda_*([g]) \geq Y(M^n,[g]).$$

Combining this with (5.14), we arrive at (5.10).

Moreover, it is clear from the proofs of the Claims that any Yamabe measure attains $\lambda_*([g])$.

□

References

[BE85] E. Bakry and Michel Émery, *Diffusions hypercontractives*, Sminaire de probabilits, XIX, 1983/84, Lecture Notes in Math., vol. 1123, Springer, Berlin, 1985, pp. 177–206.

[CGY06] Sun-Yung Alice Chang, Matthew J. Gursky, and Paul Yang, *Conformal invariants associated to a measure*, Proc. Nat. Acad. Sci. **103** (2006), 2535–2540.

[FG85] Charles Fefferman and C. Robin Graham, *Conformal invariants*, The mathematical heritage of Élie Cartan (Lyon, 1984), Astrisque, vol. Numero Hors Serie, 1985, pp. 95–116.

[FG02] C. Fefferman and C.R. Graham, *Q-curvature and Poincaré metrics*, Math. Res. Lett. **9** (2002), 139–152.

[FH03] Charles Fefferman and Kengo Hirachi, *Ambient metric construction of Q-curvature in conformal and CR geometries.*, Math. Res. Lett. **10** (2003), 819–831.

[GJMS92] C. Robin Graham, Ralph Jenne, Lionel J. Mason, and George A.J. Sparling, *Conformally invariant powers of the Laplacian, I. Existence.*, J. London Math. Soc. **46** (1992), 557–565.

[GZ03] C.R. Graham and Maciej Zworski, *Scattering matrix in conformal geometry*, Invent. Math. **152** (2003), 89–118.

[Lic70] Andre Lichnerowicz, *Varietes Riemanniennes a tenseur C non negatif*, C. R. Acad. Sci. Paris Sr. A-B **271** (1970), A650–A653.

[Per02] Grisha Perelman, *The entropy formula for the Ricci flow amd its geometric applications*, preprint, 2002.

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