QUANTUM FEYNMAN–KAC PERTURBATIONS

ALEXANDER C. R. BELTON, J. MARTIN LINDSAY AND ADAM G. SKALSKI

Dedicated to the memory of Bill Arveson

Abstract. We develop fully noncommutative Feynman–Kac formulae by employing quantum stochastic processes. To this end we establish some theory for perturbing quantum stochastic flows on von Neumann algebras by multiplier cocycles. The multiplier cocycles are governed by quantum stochastic differential equations whose coefficients are driven by the unperturbed flow; under certain regularity conditions, we show that every multiplier cocycle is so described. Our results generalise those obtained using classical Brownian motion on the one hand, and results for unitarily implemented free flows on the other.

Introduction

Feynman–Kac formulae for vector field-type perturbations of a class of noncommutative elliptic operators were developed in [LiS] and extended in [BaP]. In those papers classical Brownian motion is employed, so the noncommutativity is confined to the operator algebra which replaces a function space in the classical Feynman–Kac formula. Moreover the unperturbed semigroup in those papers is a Gaussian average of a unitarily implemented automorphism group. In this paper we obtain fully noncommutative Feynman–Kac formulae by employing *quantum* stochastic processes ([Par, Mey, L_1, SiG]). To this end we develop the theory of perturbing quantum stochastic flows by multiplier cocycles, in particular those governed by quantum stochastic differential equations (*cf.* [EvH, DaS, GLW]). We also show that every sufficiently regular multiplier cocycle is governed by a quantum stochastic differential equation, whose coefficients are driven by the free flow; this extends results of [Bra] and [LW_1].

Another route to the Feynman–Kac formulae obtained here is sketched in [BLS]. This starts from the observation that Bahn and Park's ideas in [BaP] may be expressed naturally in vacuum-adapted quantum stochastic calculus ([B]).

Vector field-type perturbations of the Laplacian on $\mathbb{R}^n$ were realised with classical Brownian motion by Parthasarathy and Sinha ([PS_1]). Early work on noncommutative Feynman–Kac formulae was done by Accardi, Frigerio and Lewis ([Acc, AcF, AFL]) and by Hudson, Ion and Parthasarathy ([HIP]); see also [Hud] and [Fa_1]. The operator-algebraic structure of the classical Feynman–Kac formula was elucidated by Arveson ([Ar_1]).

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The plan of the paper is as follows. In Section 1 we give the necessary background material from quantum stochastic analysis, including some recent results, such as Theorem 1.10 and the converse part of Theorem 1.8, to be applied in the later sections. This section also serves to introduce notation and terminology. In Section 2 we introduce adapted multiplier cocycles for a fixed quantum stochastic flow, referred to as the *free flow*. These are shown to give rise to perturbations of the free flow; they are themselves completely bounded quantum stochastic cocycles. The relevant existence and uniqueness results for quantum stochastic differential equations with time-dependent coefficients are recalled in Section 3, where Hölder properties of their solutions are also discussed. With these, perturbation processes are constructed in Section 4 and their contractivity, isometry and coisometry properties are characterised in terms of the driving coefficients of the quantum stochastic differential equation. In Section 5 bounded perturbation processes having locally uniform bounds are shown to be multiplier cocycles for the free flow. In Section 6 the class of multiplier cocycles constructed in Section 5 is characterised, under the assumption that the free flow is Markov regular. In Section 7 the perturbation theorem for quantum stochastic flows is deduced from results of Section 6, by application of the quantum Itô product formula. In the final Section we obtain quantum Feynman–Kac formulae for quantum Markov semigroups via their quantum stochastic dilation.

General notation. All Hilbert spaces here are complex, with inner products linear in their second argument. We usually abbreviate the simple tensor $u \otimes \xi$ of Hilbert-space vectors $u \in \mathsf{h}$ and $\xi \in \mathsf{H}$ to $u\xi$. We write $|\mathsf{h}\rangle$ and $\langle\mathsf{h}|$ for the column and row spaces $B(\mathbb{C};\mathsf{h})$ and $B(\mathsf{h};\mathbb{C})$ respectively, with $|u\rangle$ and $\langle u|$ denoting the operator $\mathbb{C} \to \mathsf{h}$, $\lambda \mapsto \lambda u$ and the linear functional $\mathsf{h} \to \mathbb{C}$, $v \mapsto \langle u, v\rangle$ respectively. We also use the notation

$$E_u := I_{\mathsf{h}'} \otimes |u\rangle \otimes I_{\mathsf{h}} \quad \text{and} \quad E^u := I_{\mathsf{h}'} \otimes \langle u| \otimes I_{\mathsf{h}}, \tag{0.1}$$

where context dictates the choice of Hilbert spaces h' and h. Algebraic tensor products are denoted $\underline{\otimes}$ and ultraweak tensor products $\overline{\otimes}$. For a Hilbert space h, ι_{h} denotes the ampliation $B(\mathsf{H}) \to B(\mathsf{H} \otimes \mathsf{h})$, $T \mapsto T \otimes I_{\mathsf{h}}$ or $B(\mathsf{H}) \to B(\mathsf{h} \otimes \mathsf{H})$, $T \mapsto T \otimes I_{\mathsf{h}}$, where the order and Hilbert space H depend on context, and, for a von Neumann allgebra A, $\iota_{\mathsf{h}}^{\mathsf{A}}$ denotes the induced ampliation $\mathsf{A} \to \mathsf{A} \overline{\otimes} B(\mathsf{h})$, or $\mathsf{A} \to B(\mathsf{h}) \overline{\otimes} \mathsf{A}$.

For $u, v \in \mathsf{h}$, $\omega_{u,v}$ denotes the vector functional $x \mapsto \langle u, xv\rangle$ on $B(\mathsf{h})$ and its restrictions to von Neumann subalgebras. Finally, for a vector-valued function $f : \mathbb{R}_+ \to V$ and a subinterval J of $\mathbb{R}_+$, the function which agrees with f on J and is zero on $\mathbb{R}_+ \setminus J$ is denoted f_J, and, for $c \in V$, the function which equals c on J and the zero vector on $\mathbb{R}_+ \setminus J$ is denoted c_J.

1. Quantum stochastic analysis

Fix now, and for the rest of the paper, a von Neumann algebra A acting faithfully on a Hilbert space $\mathfrak{h}$ and a further Hilbert space k, and set $\widehat{\mathsf{k}} = \mathbb{C} \oplus \mathsf{k}$. Under the natural identification $\widehat{\mathsf{k}} \otimes \mathfrak{h} = \mathfrak{h} \oplus (\mathsf{k} \otimes \mathfrak{h})$, elements of

$B(\widehat{\mathsf{k}}) \overline{\otimes} \mathsf{A}$ take the block-matrix form $\begin{bmatrix} k & m \\ l & n \end{bmatrix}$, where $k \in \mathsf{A}$, $l \in |\mathsf{k}\rangle \overline{\otimes} \mathsf{A}$, $m \in \langle\mathsf{k}| \overline{\otimes} \mathsf{A}$ and $n \in B(\mathsf{k}) \overline{\otimes} \mathsf{A}$.

In this section we provide the necessary background material in quantum stochastic (QS) analysis. Further detail may be found in [L$_1$], but some more recent results are also included below.

Definition 1.1. A *Markov semigroup* on A is a pointwise ultraweakly continuous semigroup of normal positive unital maps on A.

Remark. If unitality is relaxed to contractivity then the semigroup is called *sub-Markov*. All the sub-Markov semigroups appearing here will be completely positive. Completely positive sub-Markov semigroups are also known as *quantum dynamical semigroups*.

For a subinterval J of $\mathbb{R}_+$, let $\mathcal{F}_J$ denote the symmetric Fock space over $L^2(J;\mathsf{k})$ and set $\mathsf{N}_J = B(\mathcal{F}_J)$, shortened to $\mathcal{F}$ and N when $J = \mathbb{R}_+$; write I_J for the identity operator on $\mathcal{F}_J$. Thus, for $0 \leqslant r < t \leqslant \infty$,

$$\mathcal{F} = \mathcal{F}_{[0,r[} \otimes \mathcal{F}_{[r,t[} \otimes \mathcal{F}_{[t,\infty[} \quad \text{and} \quad \mathsf{N} = \mathsf{N}_{[0,r[} \overline{\otimes} \mathsf{N}_{[r,t[} \overline{\otimes} \mathsf{N}_{[t,\infty[}. \tag{1.1}$$

The subspace of $L^2(\mathbb{R}_+;\mathsf{k})$ consisting of step functions is denoted $\mathbb{S}$; for elements of $\mathbb{S}$ we always take their right-continuous version. We use normalised exponential vectors,

$$\varpi(f) := e^{-\frac{1}{2}\|f\|^2}\varepsilon(f) = e^{-\frac{1}{2}\|f\|^2}\big((n!)^{-1/2} f^{\otimes n}\big)_{n\geq 0} \quad (f \in \mathbb{S}), \tag{1.2}$$

and let $\mathcal{E}$ denote their linear span. We refer to the map

$$\mathbb{E} := \mathrm{id}_{\mathsf{A}} \overline{\otimes} \omega_{\varpi(0),\varpi(0)} : \mathsf{A} \overline{\otimes} \mathsf{N} \to \mathsf{A}, \ T \mapsto E^{\varpi(0)} T E_{\varpi(0)} \tag{1.3}$$

as the *vacuum-expectation map*.

In terms of the unitary operator

$$T_r : \mathcal{F} \to \mathcal{F}_{[r,\infty[}, \ \varpi(f) \mapsto \varpi(f_r) \quad (f \in \mathbb{S}),$$

where $r \in \mathbb{R}_+$ and $f_r(s) := f(s-r)$ for all $s \in [r,\infty[$, the *ampliated CCR flow on* $\mathsf{A} \overline{\otimes} \mathsf{N}$ is the semigroup of normal unital *-homomorphisms $(\widetilde{\sigma}_r)_{r\geqslant 0}$ determined by the identity

$$\widetilde{\sigma}_r(a \otimes z) = a \otimes I_{[0,r[} \otimes T_r z T_r^* \quad (a \in \mathsf{A}, \ z \in \mathsf{N}). \tag{1.4}$$

We also use the normal *-isomorphisms $\sigma_r : \mathsf{A} \overline{\otimes} \mathsf{N} \to \mathsf{A} \overline{\otimes} \mathsf{N}_{[r,\infty[}$ $(r \geqslant 0)$ determined by the identity

$$\sigma_r(a \otimes z) = a \otimes T_r z T_r^* \quad (a \in \mathsf{A}, \ z \in \mathsf{N}).$$

Definition 1.2. By an *operator process on* $\mathfrak{h}$ we mean a family of (possibly unbounded) operators $X = (X_t)_{t\geqslant 0}$ on $\mathfrak{h} \otimes \mathcal{F}$, with common domain $\mathfrak{h} \underline{\otimes} \mathcal{E}$, which is adapted, in the sense that

$$E^{\varpi(f)} X_t E_{\varpi(g)} = \langle \varpi(f_{[t,\infty[}), \varpi(g_{[t,\infty[})\rangle E^{\varpi(f_{[0,t[})} X_t E_{\varpi(g_{[0,t[})}$$

$(f, g \in \mathbb{S}, \ t \in \mathbb{R}_+)$, and weakly measurable, in that

$$t \mapsto \langle \zeta', X_t \zeta\rangle \text{ is measurable} \quad (\zeta' \in \mathfrak{h} \otimes \mathcal{F}, \ \zeta \in \mathfrak{h} \underline{\otimes} \mathcal{E}).$$

The process X is *bounded* if each X_t is a bounded operator, and X is a *process in* A if the corresponding matrix elements lie in A: that is, $E^{\varpi(f)} X_t E_{\varpi(g)} \in \mathsf{A}$ for all $f, g \in \mathbb{S}$ and $t \in \mathbb{R}_+$. Finally, we call the process X *measurable* or *continuous* if, for all $\zeta \in \mathfrak{h} \underline{\otimes} \mathcal{E}$, the map $t \mapsto X_t\zeta$ is strongly measurable or continuous, respectively.

Remarks. In the literature it is more common to allow the common domain for a QS process to be of the form $\mathcal{D} \underline{\otimes} \mathcal{E}$ where $\mathcal{D}$ is a dense subspace of $\mathfrak{h}$.

When $\mathfrak{h}$ and k are both separable, measurability is automatic, by Pettis' Theorem.

The following notation and terminology are convenient for expressing the basic estimates and formulae of QS calculus. For an operator process X on $\mathfrak{h}$ we set

$$\widetilde{X}_t := I_{\widehat{\mathsf{k}}} \underline{\otimes} X_t \quad (t \in \mathbb{R}_+). \tag{1.5}$$

For all $s \in \mathbb{R}_+$, the (unclosable) operator $\widehat{\nabla}_s$ on $\mathfrak{h} \otimes \mathcal{F}$ with domain $\mathfrak{h} \underline{\otimes} \mathcal{E}$ is given by linear extension of the prescription

$$\widehat{\nabla}_s u\varpi(f) = \widehat{f}(s)u\varpi(f) \quad (u \in \mathfrak{h},\ f \in \mathbb{S});$$

this is well defined for all $s \in \mathbb{R}_+$, since f is right continuous. The *quantum Itô projection* is the operator

$$\Delta := \begin{bmatrix} 0 & 0 \\ 0 & I_{\mathsf{k}} \end{bmatrix} \in B(\widehat{\mathsf{k}}) = \mathbb{C} \oplus B(\mathsf{k}),$$

which will appear in ampliated form below, with the same notation. For $f \in \mathbb{S}$ and any subinterval J of $\mathbb{R}_+$, the relevant constants for the estimates (1.11) and (1.11) below are

$$C(f) := 1 + \|f\| \quad \text{and} \quad C(J,f) := \sqrt{|J| + C(f_J)^2}, \tag{1.6}$$

where $|J|$ is the length of J; see [L$_3$, Theorem 3.4].

A family of (possibly unbounded) operators $(G_t)_{t \geqslant 0}$ on $\widehat{\mathsf{k}} \otimes \mathfrak{h} \otimes \mathcal{F}$ is a *QS-integrable process* if it satisfies the domain and integrability conditions

(i) $\widehat{\nabla}_t\zeta \in \operatorname{Dom} G_t$,
(ii) $s \mapsto \Delta^{\perp} G_s \widehat{\nabla}_s \zeta$ is locally integrable and
(iii) $s \mapsto \Delta G_s \widehat{\nabla}_s \zeta$ is locally square-integrable,

for all $\zeta \in \mathfrak{h} \underline{\otimes} \mathcal{E}$ and $t \in \mathbb{R}_+$, and also the adaptedness condition

(iv) for all $f, g \in \mathbb{S}$ and $t \in \mathbb{R}_+$,

$$E^{\varpi(f)} G_t \widehat{\nabla}_t E_{\varpi(g)} = \langle \varpi(f_{[t,\infty[}), \varpi(g_{[t,\infty[}) \rangle E^{\widehat{f}(t)\varpi(f_{[0,t[})} G_t E_{\widehat{g}(t)\varpi(g_{[0,t[})}. \tag{1.7}$$

Remarks. These conditions imply that (1.10) below is finite.

Like the collection of processes on $\mathfrak{h}$, the collection of QS-integrable processes forms a linear space.

The following conditions on an adapted family of operators $(G_t)_{t \geqslant 0}$ on $\widehat{\mathsf{k}} \otimes \mathfrak{h} \otimes \mathcal{F}$ are sufficient for QS integrability:

(a) $\operatorname{Dom} G_t = \widehat{\mathsf{k}} \underline{\otimes} \mathfrak{h} \underline{\otimes} \mathcal{E}$ for all $t \in \mathbb{R}_+$,
(b) $s \mapsto G_s\xi$ is continuous for all $\xi \in \widehat{\mathsf{k}} \underline{\otimes} \mathfrak{h} \underline{\otimes} \mathcal{E}$.

In particular, for a locally integrable process X on $\mathfrak{h}$, the adapted family of operators (1.5) is a QS-integrable process.

Let $X = \big(\int_0^t G_s\,d\Lambda_s\big)_{t\geqslant 0}$ for a QS-integrable process G. Then X is a continuous operator process on $\mathfrak{h}$ such that, for all $0 \leqslant r \leqslant t$ and $\zeta \in \mathfrak{h} \underline{\otimes} \mathcal{E}$, the *First Fundamental Formula* holds:

$$\langle \zeta, (X_t - X_r)\zeta\rangle = \int_r^t \langle \widehat{\nabla}_s\zeta, G_s\widehat{\nabla}_s\zeta\rangle\,ds. \tag{1.8}$$

Furthermore, the following hold: the *Second Fundamental Formula*,

$$\|(X_t - X_r)\zeta\|^2 = \int_r^t \big(2\,\mathrm{Re}\langle \widetilde{X}_s\widehat{\nabla}_s\zeta, G_s\widehat{\nabla}_s\zeta\rangle + \|\Delta G_s\widehat{\nabla}_s\zeta\|^2\big)\,ds, \tag{1.9}$$

the *Fundamental Estimate*,

$$\begin{aligned}
&\|(X_t - X_r)u\varpi(f)\| \\
&\leqslant \int_r^t \|\Delta^\perp G_s\widehat{f}(s)u\varpi(f)\|\,ds + C\big(f_{[r,t[}\big)\Big\{\int_r^t \|\Delta G_s\widehat{f}(s)u\varpi(f)\|^2\,ds\Big\}^{1/2} \\
&\leqslant C([r,t[,f)\Big\{\int_r^t \|G_s\widehat{f}(s)u\varpi(f)\|^2\,ds\Big\}^{1/2} \quad (u\in\mathfrak{h},\ f\in\mathbb{S})
\end{aligned} \tag{1.10}$$

and the *Fundamental Hölder Estimate*,

$$(t-r)^{-1/2}\|(X_t - X_r)E_{\varpi(f)}\| \leqslant C([r,t[,f)\sup_{s\in[r,t[}\|G_sE_{\widehat{f}(s)}E_{\varpi(f)}\| \quad (f\in\mathbb{S}). \tag{1.11}$$

Let $H = \big(\widetilde{X}_t^*G_t + G_t^*\widetilde{X}_t + G_t^*\Delta G_t\big)_{t\geqslant 0}$. If $\mathrm{Dom}\,H_t\widehat{\nabla}_t \supset \mathfrak{h}\underline{\otimes}\mathcal{E}$ for each $t\in\mathbb{R}_+$ (for example, if X and G are both bounded) then the identity (1.9) may be re-expressed as follows:

$$\|(X_t - X_r)\zeta\|^2 = \int_r^t \langle \widehat{\nabla}_s\zeta, H_s\widehat{\nabla}_s\zeta\rangle\,ds. \tag{1.12}$$

Thus if H is a QS-integrable process then (1.12) and (1.8) combine to yield the *Quantum Itô Product Formula*,

$$X_t^*X_t = \int_0^t H_s\,d\Lambda_s. \tag{1.13}$$

The formulae (1.8), (1.9), (1.12) and (1.13) may all be polarised; moreover, if G^* is QS integrable then

$$\int_0^t G_s^*\,d\Lambda_s \subset X_t^*.$$

We shall need the following extension of the First Fundamental Formula, which is valid under boundedness assumptions that hold in cases of interest.

Lemma 1.3. *Let G be a QS-integrable process on $\mathfrak{h}$, let f, $g \in \mathbb{S}$ and $0 \leqslant r < t$, and suppose that*

(a) $E^{\varpi(f)}\big(\int_r^t G_s\,d\Lambda_s\big)E_{\varpi(g)}$ *is bounded;*

(b) $E^{\widehat{f}(s)\varpi(f)}G_sE_{\widehat{g}(s)\varpi(g)}$ *is bounded, uniformly for $s\in[r,t[$.*

Then, for all $\omega \in B(\mathfrak{h})_*$,

$$\omega\Big(E^{\varpi(f)}\int_r^t G_s\,d\Lambda_s\ E_{\varpi(g)}\Big) = \int_r^t \omega\big(E^{\widehat{f}(s)\varpi(f)}G_sE_{\widehat{g}(s)\varpi(g)}\big)\,ds.$$

Proof. When ω is a vector functional $\omega_{u,v}$ the identity is equivalent to the First Fundamental Formula (1.8). For a general normal linear functional the result follows from Lebesgue's Dominated Convergence Theorem and the norm totality of vector functionals in $B(\mathfrak{h})_*$. □

Definition 1.4. A *mapping process on* A is a family $k = (k_t)_{t \geqslant 0}$ of linear maps with common domain A, such that $\big(k_t(a)\big)_{t \geqslant 0}$ is an operator process in A for all $a \in \mathsf{A}$.

A mapping process k on A is *bounded*, *completely bounded*, *completely positive* or *normal* if k_t has that property for each $t \in \mathbb{R}_+$, and is *continuous* if the operator process $\big(k_t(a)\big)_{t \geqslant 0}$ is continuous for each $a \in \mathsf{A}$. We call a bounded process k on A *ultraweakly continuous* if it is pointwise ultraweakly continuous: $t \mapsto k_t(a)$ is ultraweakly continuous for each $a \in \mathsf{A}$.

For quantum stochastic differential equations and quantum stochastic cocycles, to be defined next, various forms of ampliation are needed.

Let k be a normal completely bounded process on A. Then adaptedness implies that, for all $r \geqslant 0$ and all $a \in \mathsf{A}$,

$$k_r(a) = k_{r)}(a) \otimes I_{[r,\infty[},$$

where $k_{r)}(a)$ is an operator in $\mathsf{A} \,\overline{\otimes}\, \mathsf{N}_{[0,r[}$. The map

$$k_{r)} : \mathsf{A} \to \mathsf{A} \,\overline{\otimes}\, \mathsf{N}_{[0,r[}$$

is normal and completely bounded, so we may define

$$\widehat{k}_r := k_{r)} \,\overline{\otimes}\, \mathrm{id}_{\mathsf{N}_{[r,\infty[}} : \mathsf{A} \,\overline{\otimes}\, \mathsf{N}_{[r,\infty[} \to \mathsf{A} \,\overline{\otimes}\, \mathsf{N}_{[0,r[} \,\overline{\otimes}\, \mathsf{N}_{[r,\infty[} = \mathsf{A} \,\overline{\otimes}\, \mathsf{N}.$$

Also define a normal completely bounded process $\widetilde{k}$ on $B(\widehat{\mathsf{k}}) \,\overline{\otimes}\, \mathsf{A}$ by setting

$$\widetilde{k}_t := \mathrm{id}_{B(\widehat{\mathsf{k}})} \,\overline{\otimes}\, k_t \quad (t \in \mathbb{R}_+).$$

Definition 1.5. A normal completely bounded process k on A is a *quantum stochastic cocycle* if

$$k_0 = \iota_{\mathcal{F}}^{\mathsf{A}} \quad \text{and} \quad k_{r+t} = \widehat{k}_r \circ \sigma_r \circ k_t \quad (r, t \in \mathbb{R}_+); \tag{1.14}$$

it is a *quantum stochastic flow* if furthermore k is ultraweakly continuous, *-homomorphic and unital.

More generally, a process k on A for which the maps

$$\kappa_t^{f,g} := E^{\varpi(f_{[0,t[})} k_t(\cdot) E_{\varpi(g_{[0,t[})} \quad (f, g \in \mathbb{S},\, t \in \mathbb{R}_+)$$

are bounded is a quantum stochastic cocycle if it satisfies the *weak cocycle relations*

$$\kappa_0^{f,g} = \mathrm{id}_{\mathsf{A}} \quad \text{and} \quad \kappa_{r+t}^{f,g} = \kappa_r^{f,g} \circ \kappa_t^{S_r^* f, S_r^* g} \quad (f, g \in \mathbb{S},\, r, t \in \mathbb{R}_+), \tag{1.15}$$

where S_r is the right-shift isometry on $L^2(\mathbb{R}_+; \mathsf{k})$ such that

$$(S_r f)(u) = 1_{[r,\infty[}(u) f(u - r) \quad (u \in \mathbb{R}_+).$$

The weak cocycle relation is equivalent to (1.14) when k is completely bounded and normal.

Such processes arise naturally; see Theorem 1.8.

Lemma 1.6 (*Cf.* [LW$_1$, Proposition 4.3])**.** *Let k be a normal completely bounded process on A and consider the family $K := \big(\widehat{k_t} \circ \sigma_t\big)_{t \geqslant 0}$ of normal completely bounded maps on $\mathsf{A} \overline{\otimes} \mathsf{N}$.*

(a) *For each $t \in \mathbb{R}_+$, $\|K_t\|_{\mathrm{cb}} = \|k_t\|_{\mathrm{cb}}$. Moreover, K is unital or *-homomorphic if and only if k has the same property.*

(b) *The following are equivalent:*
   (i) *k is a QS cocycle;*
   (ii) *K is a semigroup.*

Proof. (a) Let $t \in \mathbb{R}_+$. Since $\iota_{\mathcal{F}}^{\mathsf{A}}$ and σ_t are both unital and *-homomorphic, and thus completely isometric, the claim follows from the definition of K_t and the identity

$$k_t = K_t \circ \iota_{\mathcal{F}}^{\mathsf{A}}. \tag{1.16}$$

(b) Let $s, t \in \mathbb{R}_+$. For a simple tensor $A = a \otimes z$, the identity

$$K_s\big(AX\big) = K_s(A)\widetilde{\sigma}_s(X) \quad (A \in \mathsf{A} \overline{\otimes} \mathsf{N},\, X \in 1_{\mathsf{A}} \otimes \mathsf{N})$$

holds, since both sides equal $k_s(a)\widetilde{\sigma}_s(ZX)$, where $Z = 1_{\mathsf{A}} \otimes z$; it therefore holds in general, by normality. In particular, if (i) holds then, for $a \in \mathsf{A}$ and $z \in \mathsf{N}$,

$$\begin{aligned}(K_s \circ K_t)(a \otimes z) &= K_s\big(k_t(a)\widetilde{\sigma}_t(1_{\mathsf{A}} \otimes z)\big) \\ &= K_s\big(k_t(a)\big)\widetilde{\sigma}_{s+t}(1_{\mathsf{A}} \otimes z) \\ &= k_{s+t}(a)\widetilde{\sigma}_{s+t}(1_{\mathsf{A}} \otimes z) = K_{s+t}(a \otimes z),\end{aligned}$$

so (ii) holds by normality. The converse follows from (1.16). □

Remark. When k is *-homomorphic and ultraweakly continuous we refer to K as the *corresponding E-semigroup on* $\mathsf{A} \overline{\otimes} \mathsf{N}$ or, if k is also unital, and thus a QS flow, the *corresponding E_0-semigroup on* $\mathsf{A} \overline{\otimes} \mathsf{N}$.

Definition 1.7. If k is a normal completely bounded QS cocycle on A then, for each c, $d \in \mathsf{k}$, setting

$$\mathcal{Q}_t^{c,d}(a) := E^{\varpi(c_{[0,t[})} k_t(a) E_{\varpi(d_{[0,t[})} \quad (a \in \mathsf{A},\, t \in \mathbb{R}_+) \tag{1.17}$$

defines a normal completely bounded semigroup $\mathcal{Q}^{c,d}$ on A. The cocycle is called *Markov regular* if each of these *associated semigroups* is norm continuous, and $\mathcal{Q}^{0,0} = \big(\mathbb{E} \circ k_t\big)_{t \geqslant 0}$ is called the *expectation semigroup* of k, or its *Markov semigroup* when k is positive and unital.

Remark. For a completely bounded cocycle whose cb norm is locally uniformly bounded, Markov regularity is equivalent to norm continuity of the expectation semigroup.

The following result is an amalgamation of the basic existence theorem for quantum stochastic differential equations with bounded constant coefficients with a recent characterisation theorem for quantum stochastic cocycles. In order to state it we introduce some more notation. For a mapping process k on A such that $\operatorname{Dom} k_t(a^*)^* \supset \mathfrak{h} \underline{\otimes} \mathcal{E}$ for all $a \in \mathsf{A}$ and $t \in \mathbb{R}_+$, setting

$$k_t^{\dagger}(a) := k_t(a^*)^*\big|_{\mathfrak{h} \underline{\otimes} \mathcal{E}} \quad (a \in \mathsf{A},\, t \in \mathbb{R}_+)$$

defines a mapping process $k^\dagger$ on A. In this case we say that k is *adjointable*. The dagger notation is also used as follows: for Hilbert spaces h and h', and a map $\psi : \mathsf{A} \to B(\mathsf{h}; \mathsf{h}')$, $\psi^\dagger$ is the map

$$\mathsf{A} \to B(\mathsf{h}'; \mathsf{h}),\ a \mapsto \psi(a^*)^*.$$

Also, for all $\varepsilon \in \mathcal{E}$ set $\iota^{\mathsf{A}}_{\mathcal{F},\varepsilon} := \iota^{\mathsf{A}}_{\mathcal{F}}(\cdot)E_\varepsilon$ and $k_{t,\varepsilon} := k_t(\cdot)E_\varepsilon$.

Theorem 1.8 ([LW$_2$, L$_2$])**.** *Let $\phi \in CB(\mathsf{A}; B(\widehat{\mathsf{k}}) \overline{\otimes} \mathsf{A})$. The QS differential equation*

$$k_0 = \iota^{\mathsf{A}}_{\mathcal{F}}, \quad dk_t = \widetilde{k}_t \circ \phi \, d\Lambda_t \tag{1.18}$$

has a unique weakly regular, weak solution k^ϕ. Moreover, k^ϕ is a QS cocycle, it is adjointable with $(k^\phi)^\dagger = k^{\phi^\dagger}$, and it satisfies

$$\limsup_{t\to 0+} t^{-1/2}\big\{\|k_{t,\varepsilon} - \iota^{\mathsf{A}}_{\mathcal{F},\varepsilon}\|_{\mathrm{cb}} + \|k^\dagger_{t,\varepsilon} - \iota^{\mathsf{A}}_{\mathcal{F},\varepsilon}\|_{\mathrm{cb}}\big\} < \infty \quad (\varepsilon \in \mathcal{E}). \tag{1.19}$$

In particular, k^ϕ is continuous and therefore a strong solution of (1.18)*. In the converse direction, if k is a completely bounded QS cocycle on A with locally bounded cb norm that satisfies* (1.19) *then $k = k^\phi$ for a unique map $\phi \in CB(\mathsf{A}; B(\widehat{\mathsf{k}}) \overline{\otimes} \mathsf{A})$.*

Remarks. The process $k = k^\phi$ given by Theorem 1.8 need not be bounded; *weak regularity* means that $k^{f,g}_t$ is bounded with norm locally bounded in t, for all f, $g \in \mathbb{S}$; in particular, the terms in (1.15) are well defined. To be a *weak solution* means that

$$\langle \zeta, (k_t - k_0)(a)\zeta\rangle = \int_0^t \langle \widehat{\nabla}_s\zeta, \widetilde{k}_s\big(\phi(a)\big)\widehat{\nabla}_s\zeta\rangle \, ds \quad (\zeta \in \mathfrak{h} \underline{\otimes} \mathcal{E},\ a \in \mathsf{A},\ t \in \mathbb{R}_+),$$

and to be a *strong solution* means furthermore that $\big(\widetilde{k}_t(\phi(a))\big)_{t\geqslant 0}$ is QS integrable for all $a \in \mathsf{A}$. If k is completely bounded with locally bounded cb norm, then k is ultraweakly continuous; in this case, the stochastic generator ϕ is normal if and only if the cocycle k is normal.

The condition (1.19) is stronger than Markov regularity in general, but they are equivalent when k is a completely positive QS cocycle which is *completely quasicontractive*, that is, there exists $\beta \in \mathbb{R}$ such that $\|e^{-\beta t}k_t\|_{\mathrm{cb}} \leqslant 1$ for all $t \in \mathbb{R}_+$.

Necessary and sufficient conditions on the stochastic generator ϕ for the cocycle k^ϕ to be *-homomorphic are given next, in terms of a *Lindbladian*

$$\mathcal{L}_{l,\pi,h} : \mathsf{A} \to \mathsf{A},\ a \mapsto l^*\pi(a)l - \tfrac{1}{2}(l^*la + al^*l) + i(ah - ha)$$

and an inner π-derivation

$$\delta_{l,\pi} : \mathsf{A} \to |\mathsf{H}\rangle \overline{\otimes} \mathsf{A},\ a \mapsto \pi(a)l - la,$$

where $l \in |\mathsf{H}\rangle \overline{\otimes} \mathsf{A}$, $h = h^* \in \mathsf{A}$ and $\pi : \mathsf{A} \to B(\mathsf{H} \otimes \mathfrak{h})$ is a representation of A.

Theorem 1.9. *Let $j = k^\theta$, where $\theta \in CB(\mathsf{A}; B(\widehat{\mathsf{k}}) \overline{\otimes} \mathsf{A})$. The following are equivalent:*

(i) *j is normal and *-homomorphic;*
(ii) *$\theta = \left[\begin{smallmatrix} \tau & \delta^\dagger \\ \delta & \pi - \iota \end{smallmatrix}\right]$, where $\iota := \iota^{\mathsf{A}}_{\mathsf{k}}$, π is a normal *-homomorphism, $\delta = \delta_{l,\pi}$ and $\tau = \mathcal{L}_{l,\pi,h}$ for some $h = h^* \in \mathsf{A}$ and $l \in |\mathsf{k}\rangle \overline{\otimes} \mathsf{A}$.*

In this case j is ultraweakly continuous; moreover, j is unital if and only if π is unital.

Proof. Suppose that (i) holds. The quantum Itô product formula (1.13) implies that θ has the above block-matrix form, where π is a *-homomorphism, δ is a π-derivation and $\tau(a^*b) - \tau(a)^*b - a^*\tau(b) = \delta(a)^*\delta(b)$ for all $a \in \mathsf{A}$. Since $\operatorname{Ran}\delta \subset |\mathsf{k}\rangle \,\overline{\otimes}\, \mathsf{A}$, (ii) follows by applying twice the Christensen–Evans characterisation ([ChE, Theorem 2.1]) of a ρ-derivation $\gamma : \mathsf{A} \to B(\mathfrak{h}; \mathsf{K})$ satisfying $\gamma(a)^*\gamma(a) \in \mathsf{A}$ for all $a \in \mathsf{A}$, where $\rho : \mathsf{A} \to B(\mathsf{K})$ is a representation; see [L_1, Theorem 6.9].

For the reverse implication see [GoS, Theorem 3.3.6] and [LW_3, Corollary 4.2]. □

Remark. In [L_1, Theorem 6.8], the hypothesis $\gamma(\mathsf{A})^*\gamma(\mathsf{A}) \subset \mathsf{A}$ is missing but should be included.

We next describe the situation for QS cocycles which are not necessarily Markov regular. To any ultraweakly continuous normal completely positive QS cocycle k on A we may associate the maps

$$\phi_t : \mathsf{A} \to B(\widehat{\mathsf{k}}) \,\overline{\otimes}\, \mathsf{A},\ a \mapsto \begin{bmatrix} t^{-1/2}E^{\varpi(0)} \\ V_t^* \otimes I_{\mathfrak{h}} \end{bmatrix} \big(k_t(a) - \iota_{\mathcal{F}}^{\mathsf{A}}(a)\big) \begin{bmatrix} t^{-1/2}E_{\varpi(0)} & V_t \otimes I_{\mathfrak{h}} \end{bmatrix}$$

where $t > 0$ and V_t denotes the isometry

$$\mathsf{k} \to \mathcal{F},\ c \mapsto (0, t^{-1/2}c_{[0,t[}, 0, 0, \cdots),$$

and also the operator ϕ from A to $B(\widehat{\mathsf{k}}) \,\overline{\otimes}\, \mathsf{A}$, given by

$$\phi(x) = \operatorname*{uw\,lim}_{t \to 0+} \phi_t(x),$$

with domain equal to the set of elements $x \in \mathsf{A}$ for which the ultraweak limit exists. Thus, letting $\mathcal{D}^{c,d}$ denote the domain of the ultraweak generator $\tau_{c,d}$ of the (c,d)-associated semigroup $\mathcal{Q}^{c,d}$ of k, $\operatorname{Dom}\phi$ is a *-invariant subspace of A contained in $\mathcal{D}^{0,0}$. Set

$$\mathcal{D}_2^{0,0} := \big\{x \in \mathcal{D}^{0,0} : x^*x,\ xx^* \in \mathcal{D}^{0,0}\big\}. \tag{1.20}$$

Theorem 1.10 ([L_2])**.** *Let k be an ultraweakly continuous normal completely positive QS cocycle on A which is completely quasicontractive.*

(a) *The cocycle k strongly satisfies the quantum stochastic differential equation*

$$k_0 = \iota_{\mathcal{F}}^{\mathsf{A}}, \quad dk_t = \widetilde{k}_t \circ \phi \, d\Lambda_t \quad \text{on } \operatorname{Dom}\phi. \tag{1.21}$$

(b) *The following relations hold:*

$$\mathcal{D}_2^{0,0} \subset \operatorname{Dom}\phi = \big\{x \in \mathsf{A} : \limsup_{t \to 0+} \|\phi_t(x)\| < \infty\big\}$$
$$\subset \big\{x \in \mathsf{A} : \liminf_{t \to 0+} \|\phi_t(x)\| < \infty\big\} \subset \bigcap_{c,d \in \mathsf{k}} \mathcal{D}^{c,d},$$

with the latter two being equalities when k has finite dimension.

(c) *For all $x \in \operatorname{Dom}\phi$,*

$$\tau_{c,d}(x) = E^{\widehat{c}}\phi(x)E_{\widehat{d}} - \chi(c,d)x,$$

where $\chi(c,d) := \frac{1}{2}\big(\|c\|^2 + \|d\|^2\big) - \langle c, d\rangle$.

(d) *If* $\mathrm{Dom}\,\phi$ *is an ultraweak core for* $\tau_{0,0}$ *then* (1.21) *has no other ultraweakly continuous normal weak solution which is completely quasicontractive.*

We refer to ϕ as the *ultraweak stochastic derivative* of k.

Remark. With the aid of the quantum martingale representation theorem ([PS$_3$]), Accardi and Mohari effectively obtained part (a) of Theorem 1.10, on the domain $\mathcal{D}_2^{0,0}$, assuming that $\mathfrak{h}$ and k are separable ([AcM]). There are cases of interest for which the latter domain is *not* ultraweakly dense ([Fa$_2$, Ar$_2$]).

We end this section with an easily verified lemma.

Lemma 1.11. *Let* j *be a *-homomorphic QS process on* A. *If* j *is ultraweakly continuous then* $t \mapsto j_t(a)$ *is strongly continuous for all* $a \in \mathsf{A}$*; if* j *is normal then* $a \mapsto j_t(a)$ *is strongly continuous on bounded sets for all* $t \in \mathbb{R}_+$.

2. Multiplier cocycles for a quantum stochastic flow

Fix now, and for the rest of the paper, a QS flow j on A, which we refer to as the *free flow* (i.e. unperturbed flow), and let J denote the corresponding E_0 semigroup, according to Lemma 1.6. Thus J is the ultraweakly continuous normal unital *-homomorphic semigroup on $\mathsf{A} \overline{\otimes} \mathsf{N}$ given by $(\widehat{\jmath}_t \circ \sigma_t)_{t \geqslant 0}$.

Definition 2.1. A *multiplier cocycle for* j, or *adapted right* J*-cocycle*, is a bounded operator process Y in A such that

$$Y_0 = I_{\mathfrak{h}\otimes\mathcal{F}} \quad \text{and} \quad Y_{s+t} = J_s(Y_t)Y_s \quad (s, t \in \mathbb{R}_+).$$

Remarks. A multiplier cocycle for the trivial flow $j \equiv \iota_{\mathcal{F}}^{\mathsf{A}}$ is precisely a bounded right QS operator cocycle,

$$X_0 = I_{\mathfrak{h}\otimes\mathcal{F}} \quad \text{and} \quad X_{s+t} = \widetilde{\sigma}_s(X_t)X_s \quad (s, t \in \mathbb{R}_+), \tag{2.1}$$

which is a QS process in A ([LW$_4$]). If a multiplier cocycle for j is strongly continuous on $\mathfrak{h} \otimes \mathcal{F}$ then, by the Banach–Steinhaus Theorem, it is necessarily locally uniformly bounded.

A bounded process Y is an adapted right J-cocycle if and only if the adjoint process $Z = (Y_t^*)_{t\geqslant 0}$ is an adapted left J-cocycle, that is, Z satisfies $Z_0 = I_{\mathfrak{h}\otimes\mathcal{F}}$ and $Z_{s+t} = Z_s J_s(Z_t)$ for all s, $t \in \mathbb{R}_+$.

In this paper we work exclusively with right cocycles.

Lemma 2.2. *Let* Y *be a multiplier cocycle for* j *and suppose that* Y *is locally uniformly bounded. Then* Y *is* exponentially bounded*: there exist* $M \geqslant 1$ *and* $\beta \in \mathbb{R}$ *such that* $\|Y_t\| \leqslant Me^{\beta t}$ *for all* $t \in \mathbb{R}_+$.

Proof. In view of the inequality $\|Y_{s+t}\| \leqslant \|Y_s\|\,\|Y_t\|$, this follows from a standard semigroup argument; see, for example, [Dav, Proposition 1.18]. □

Remarks. Similarly, a QS cocycle k on A is exponentially completely bounded (in the obvious sense) if its cb norm is locally uniformly bounded, since $\|k_{s+t}\|_{\mathrm{cb}} \leqslant \|k_s\|_{\mathrm{cb}}\|k_t\|_{\mathrm{cb}}$.

A multiplier cocycle Y for j is said to be *quasicontractive* if there exists $\beta \in \mathbb{R}$ such that $\|e^{-\beta t}Y_t\| \leqslant 1$ for all $t \in \mathbb{R}_+$.

Proposition 2.3. *Let Y and Z be multiplier cocycles for j. Setting*

$$k_t(a) := Y_t^* j_t(a) Z_t \quad (a \in \mathsf{A},\, t \in \mathbb{R}_+)$$

defines a normal completely bounded QS cocycle k on A, provided only that it is pointwise weakly measurable. If Y and Z are strongly continuous on $\mathfrak{h} \otimes \mathcal{F}$ then k is ultraweakly continuous and exponentially completely bounded.

Proof. The last part follows from the fact that Y and Z are locally uniformly bounded when they are strongly continuous, together with Lemma 2.2 and the ultraweak continuity of j. By Lemma 1.6, it suffices therefore to show that the normal completely bounded maps $\big(K_t := \widehat{k}_t \circ \sigma_t\big)_{t \geqslant 0}$ form a semigroup on $\mathsf{A} \overline{\otimes} \mathsf{N}$. The identity

$$K_s(T) = Y_s^* J_s(T) Z_s \quad (s \in \mathbb{R}_+,\, T \in \mathsf{A} \overline{\otimes} \mathsf{N}),$$

where $J_s = \widehat{\jmath}_s \circ \sigma_s$, is easily verified for a simple tensor T and so holds in general by normality. Therefore, as J is a *-homomorphic semigroup,

$$\begin{aligned} K_{s+t}(T) &= Y_s^* J_s(Y_t^*) J_s\big(J_t(T)\big) J_s(Z_t) Z_s \\ &= Y_s^* J_s\big(Y_t^* J_t(T) Z_t\big) Z_s = K_s\big(K_t(T)\big) \quad (s, t \in \mathbb{R}_+,\, T \in \mathsf{A} \overline{\otimes} \mathsf{N}), \end{aligned}$$

as required. □

Remarks. Thus a pair of multiplier cocycles perturbs the free flow to give a normal completely bounded quantum stochastic cocycle on A. When $Z = Y$ the QS cocycle $k = \big(Y_t^* j_t(\cdot) Y_t\big)_{t \geqslant 0}$ is completely positive; if, also, Y is coisometric then k is *-homomorphic, and then k is unital, and thus a QS flow, if and only if Y is furthermore isometric and so unitary.

When k is ultraweakly continuous, the weak* generator of its expectation semigroup is a perturbation of the generator of the Markov semigroup of the free flow j. In Section 8 we describe this perturbation in terms of the stochastic derivative of j, when the multiplier cocycles Y and Z are governed by QS differential equations.

Definition 2.4. A *bounded pure-noise QS cocycle in* A is a bounded operator cocycle in A which is $1_{\mathsf{A}} \otimes \mathsf{N}$-valued.

We end this section by noting that bounded pure-noise QS operator cocycles act on multiplier cocycles.

Proposition 2.5. *Let Y be a multiplier cocycle for j and let Z be a bounded pure-noise QS operator cocycle in A. Then YZ is also a multiplier cocycle for j, provided only that it is weakly measurable.*

Proof. Let s, $t \in \mathbb{R}_+$ and set $X = YZ$. Then

$$\widetilde{\sigma}_s(Z_t) \in 1_{\mathsf{A}} \otimes I_{[0,s[} \otimes \mathsf{N}_{[s,\infty[} \subset \big(\mathsf{A} \overline{\otimes} \mathsf{N}_{[0,s[} \otimes I_{[s,\infty[}\big)',$$

so $J_s(Z_t) = \widetilde{\sigma}_s(Z_t)$ and

$$X_{s+t} = J_s(Y_t) Y_s \widetilde{\sigma}_s(Z_t) Z_s = J_s(Y_t) J_s(Z_t) Y_s Z_s = J_s(X_t) X_s.$$

Since $X_0 = I$, the result follows. □

A useful example of a bounded pure-noise QS operator cocycle is the *vacuum-projection cocycle* Z. This is given by

$$Z_t = 1_{\mathsf{A}} \otimes |\Omega_t\rangle\langle\Omega_t| \otimes I_{[t,\infty[} \quad (t \in \mathbb{R}_+), \tag{2.2}$$

where $\Omega_t := \varpi(0) \in \mathcal{F}_{[0,t[}$. Direct verification confirms that Z strongly satisfies the quantum stochastic differential equation

$$Z_0 = I_{\mathfrak{h}\otimes\mathcal{F}}, \quad dZ_t = -\Delta \widetilde{Z}_t d\Lambda_t, \tag{2.3}$$

where Δ is in ampliated form. This is used in Proposition 4.7.

3. Time-dependent QS differential equations

In this short section we detail the basic existence theorem required for the construction of perturbation processes in Section 4, namely the coordinate-free and dimension-independent counterpart to Proposition 3.1 of [GLW]. We also highlight the Hölder-continuity properties of solutions of such equations.

Theorem 3.1. *Let G be a bounded measurable process in $B(\widehat{\mathsf{k}}) \overline{\otimes} \mathsf{A}$ with locally uniform bounds. The quantum stochastic differential equation*

$$Y_0 = I_{\mathfrak{h}\otimes\mathcal{F}}, \quad dY_t = G_t \widetilde{Y}_t \, d\Lambda_t \tag{3.1}$$

has a unique strong solution. This solution is such that $Y_s E_\varepsilon \in \mathsf{A} \overline{\otimes} |\mathcal{F}\rangle$ for all $s \in \mathbb{R}_+$ and $\varepsilon \in \mathcal{E}$, and

$$\sup_{0\leqslant r<t\leqslant T} (t-r)^{-1/2} \|(Y_t - Y_r)E_\varepsilon\| < \infty \quad (T > 0, \, \varepsilon \in \mathcal{E}). \tag{3.2}$$

Proof. This follows from [L_3, Theorems 10.3 and 10.4]. □

Remark. The proof of existence consists of a verification that the natural Picard iteration scheme is well defined and converges, using the Fundamental Estimate (1.10); uniqueness and the Hölder estimate also follow from the Fundamental Estimate.

Proposition 3.2. *Let G be as in Theorem 3.1 and suppose that the unique solution Y of* (3.1) *is bounded. Then the following are equivalent:*

(i) *Y has locally uniform bounds;*
(ii) *Y is strongly continuous on $\mathfrak{h} \otimes \mathcal{F}$.*

Proof. That (ii) implies (i) is an immediate consequence of the Banach–Steinhaus Theorem; the converse follows from the continuity of Y and the density of $\mathfrak{h} \underline{\otimes} \mathcal{E}$. □

Often the adjoint process satisfies the local Hölder-continuity condition too.

Proposition 3.3. *Let G be as in Theorem 3.1 and suppose that the unique solution Y of* (3.1) *is bounded with locally uniform bounds. Then*

$$\sup_{0\leqslant r<t\leqslant T} (t-r)^{-1/2} \|(Y_t^* - Y_r^*)E_\varepsilon\| < \infty \quad (\varepsilon \in \mathcal{E}, \, T > 0), \tag{3.3}$$

provided only that the process $\widetilde{Y}^* G^*$ is measurable.

Proof. Under these hypotheses, $\widetilde{Y}^*G^*$ is a bounded measurable process on $\widehat{\mathsf{k}} \otimes \mathfrak{h}$ with locally uniform bounds. Hence $\widetilde{Y}^*G^*$ is QS integrable and such that

$$\int_r^t \widetilde{Y}_s^* G_s^* \, d\Lambda_s E_{\varpi(f)} = (Y_t^* - Y_r^*) E_{\varpi(f)} \quad (f \in \mathbb{S}, \, 0 \leqslant r \leqslant t).$$

The Fundamental Hölder Estimate (1.11) now gives (3.3). □

Remark. Assuming that Y is bounded with locally uniform bounds, the process $\widetilde{Y}^*G^*$ is measurable under either of the following conditions:

(a) $\mathfrak{h}$ and k are separable;
(b) Y is coisometric and G^* is strongly continuous.

In the former case this is because of the equivalence of strong and weak measurability for functions with separable range; in the latter it follows from the equivalence of weak and strong continuity for isometry-valued functions.

4. Perturbation processes

Our aim now is to construct multiplier cocycles for the free flow j using quantum stochastic calculus. If $F \in B(\widehat{\mathsf{k}}) \overline{\otimes} \mathsf{A}$ then Lemma 1.11 implies that $G = \big(\widetilde{\jmath}_t(F)\big)_{t \geqslant 0}$ is a measurable process, to which we may apply both Theorem 3.1 and Proposition 3.3. This gives the following result.

Theorem 4.1. *For all $F \in B(\widehat{\mathsf{k}}) \overline{\otimes} \mathsf{A}$ the quantum stochastic differential equation*

$$Y_0 = I_{\mathfrak{h} \otimes \mathcal{F}}, \quad dY_t = \widetilde{\jmath}_t(F) \widetilde{Y}_t \, d\Lambda_t \tag{4.1}$$

has a unique strong solution. This solution is such that $Y_s E_\varepsilon \in \mathsf{A} \overline{\otimes} |\mathcal{F}\rangle$ for all $s \in \mathbb{R}_+$ and

$$\sup_{0 \leqslant r < t \leqslant T} (t - r)^{-1/2} \|(Y_t - Y_r) E_\varepsilon\| < \infty \quad (\varepsilon \in \mathcal{E}, \, T > 0).$$

If either *Y is bounded with locally uniform bounds and $\mathfrak{h}$ and k are separable,* or *Y is coisometric, it also satisfies the condition*

$$\sup_{0 \leqslant r < t \leqslant T} (t - r)^{-1/2} \|\big((Y_t^* - Y_r^*\big) E_\varepsilon\| < \infty \quad (\varepsilon \in \mathcal{E}, \, T > 0). \tag{4.2}$$

The unique solution will be denoted $Y^{j,F}$. The Hölder conditions play a rôle in the characterisation of multiplier cocycles in Section 6.

Proposition 4.2. *The map $F \mapsto Y^{j,F}$ is injective.*

Proof. Let F, $G \in B(\widehat{\mathsf{k}}) \overline{\otimes} \mathsf{A}$ be such that $Y^{j,F} = Y^{j,G}$. Then, setting $H = F - G$ and $Y = Y^{j,F}$,

$$\begin{aligned} 0 &= t^{-1} \langle u\varepsilon(c_{[0,1[}), (Y_t^{j,F} - Y_t^{j,G}) v\varepsilon(d_{[0,1[}) \rangle \\ &= t^{-1} \int_0^t \langle u\varepsilon(c_{[0,1[}), j_s(E^{\widehat{c}} H E_{\widehat{d}}) Y_s v\varepsilon(d_{[0,1[}) \rangle \, ds \\ &\to e^{\langle c,d \rangle} \langle \widehat{c}u, H \widehat{d}v \rangle \quad \text{as } t \to 0+ \quad (u, v \in \mathfrak{h}, \, c, d \in \mathsf{k}), \end{aligned}$$

by the continuity of Y and the strong continuity of $s \mapsto j_s(E^{\widehat{c}} H E_{\widehat{d}})$. It follows that $H = 0$, as required. □

We next investigate the relationship between properties of F and those of the process $Y^{j,F}$ that it generates; for this the algebraic structure of the generator needs clarification.

For all $F \in B(\widehat{\mathsf{k}}) \overline{\otimes} \mathsf{A} = \mathsf{A} \oplus \big(B(\mathsf{k}) \overline{\otimes} \mathsf{A}\big)$ we set $q(F) := F^* + F + F^*\Delta F$, where we ampliate Δ without changing notation. If F has the block-matrix form $\left[\begin{smallmatrix} k & m \\ l & w-1 \end{smallmatrix}\right]$ then

$$q(F) = \begin{bmatrix} k^* + k + l^*l & l^*w + m \\ m^* + w^*l & w^*w - 1 \end{bmatrix} \quad \text{and} \quad q(F^*) = \begin{bmatrix} k^* + k + mm^* & mw^* + l^* \\ l + wm^* & ww^* - 1 \end{bmatrix}.$$

Proposition 4.3. *Let $F \in B(\widehat{\mathsf{k}}) \overline{\otimes} \mathsf{A}$ with the block-matrix form $\left[\begin{smallmatrix} k & m \\ l & w-1 \end{smallmatrix}\right]$, and let $\beta \in \mathbb{R}$. Then the following equivalences hold.*

(a) (i) *$q(F) \leqslant \beta\Delta^\perp$;*

(ii) *w is a contraction, $b_1 := \beta 1 - (k^* + k + l^*l) \geqslant 0$ and there is a contraction $v_1 \in \langle \mathsf{k}| \overline{\otimes} \mathsf{A}$ such that*

$$m = -l^*w + b_1^{1/2} v_1 (1 - w^*w)^{1/2};$$

(iii) *$q(F^*) \leqslant \beta\Delta^\perp$;*

(iv) *w is a contraction, $b_2 := \beta 1 - (k + k^* + mm^*) \geqslant 0$ and there is a contraction $v_2 \in |\mathsf{k}\rangle \overline{\otimes} \mathsf{A}$ such that*

$$l = -wm^* + (1 - ww^*)^{1/2} v_2 b_2.$$

(b) (i) *$q(F) = 0$;*

(ii) *w is an isometry, $k^* + k + l^*l = 0$ and $m = -l^*w$.*

(c) (i) *$q(F^*) = 0$;*

(ii) *w is a coisometry, $k + k^* + mm^* = 0$ and $l = -wm^*$.*

Proof. (a) The equivalence of (i) and (iii) is contained in [L_3, Theorem A.1]. The other equivalences in (a) follow from the classical result that an operator block matrix $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ is nonnegative if and only if $c = b^*$, $a \geqslant 0$, $d \geqslant 0$ and $b = a^{1/2} v d^{1/2}$ for a contraction v; see [FoF, p. 547] or [GLSW, Lemma 2.1]. Parts (b) and (c) are immediate consequences of the definition of q. □

Set $QC(\mathsf{k}, \mathsf{A}) := \bigcup_{\beta \in \mathbb{R}} QC_\beta(\mathsf{k}, \mathsf{A})$, where

$$QC_\beta(\mathsf{k}, \mathsf{A}) := \big\{F \in B(\widehat{\mathsf{k}}) \overline{\otimes} \mathsf{A} : q(F) \leqslant \beta\Delta^\perp\big\};$$

we abbreviate to QC and QC_β respectively. For $F \in B(\widehat{\mathsf{k}}) \overline{\otimes} \mathsf{A}$ and $\beta \in \mathbb{R}$, the identity

$$q(F - \tfrac{1}{2}\beta\Delta^\perp) = q(F) - \beta\Delta^\perp$$

implies that $F \in QC_\beta$ if and only if $F - \frac{\beta}{2}\Delta^\perp \in QC_0$.

Corollary 4.4. *Let $F \in B(\widehat{\mathsf{k}}) \overline{\otimes} \mathsf{A}$ with the block-matrix form $\left[\begin{smallmatrix} k & m \\ l & w-1 \end{smallmatrix}\right]$.*

(a) *The following are equivalent:*

(i) *$F \in QC$;*

(ii) *w is a contraction and $m = -l^*w + r_1(1 - w^*w)^{1/2}$ for some operator $r_1 \in \langle \mathsf{k}| \overline{\otimes} \mathsf{A}$;*

(iii) *w is a contraction and $l = -wm^* + (1 - ww^*)^{1/2} r_2$ for some operator $r_2 \in |\mathsf{k}\rangle \overline{\otimes} \mathsf{A}$.*

(b) *If $m = -l^*w$ or $l = -wm^*$ then the following are equivalent:*

(i) *$F \in QC$;*
(ii) *w is a contraction.*
(c) *If $w = 0$ then $F \in QC$. More specifically, if $w = 0$ then, for all $\beta \in \mathbb{R}$ the following are equivalent:*
(i) *$F \in QC_\beta$;*
(ii) *$\beta 1 \geqslant k^* + k + l^* l + mm^*$.*

Proof. (a) follows from part (a) of Proposition 4.3 and (b) follows from (a).
(c) If $w = 0$ and $\beta \in \mathbb{R}$ then

$$\beta\Delta^\perp - q(F) = \begin{bmatrix} \beta 1 - (k + k^* + l^* l) & -m \\ -m^* & 1 \end{bmatrix},$$

which is nonnegative if and only if $\beta 1 - (k^* + k + l^* l) \geqslant mm^*$. □

The relevance of the above observations is seen in the next theorem.

Theorem 4.5. *Let $Y = Y^{j,F}$, where $F \in B(\widehat{\mathsf{k}}) \overline{\otimes} \mathsf{A}$, and let $\beta \in \mathbb{R}$.*
(a) *The following are equivalent:*
(i) *$\big(e^{-\beta t/2} Y_t\big)_{t \geqslant 0}$ is contractive;*
(ii) *$F \in QC_\beta$.*
Thus Y is quasicontractive if and only if $F \in QC$.
(b) *The following are equivalent:*
(i) *Y is isometric;*
(ii) *$q(F) = 0$.*
(c) *If Y is coisometric then $q(F^*) = 0$.*

Proof. Recalling our convention that $\mathbb{S}$ consists of right-continuous step functions, we use the unbounded (and unclosable) operator R from $\mathfrak{h} \otimes \mathcal{F}$ to $\widehat{\mathsf{k}} \otimes \mathfrak{h}$ determined by the conditions

$$\operatorname{Dom} R = \mathfrak{h} \underline{\otimes} \mathcal{E} \quad \text{and} \quad Ru\varpi(f) = \widehat{f}(0)u.$$

(a), (b) In view of the remarks above and the fact that $\big(e^{-\beta t/2} Y_t^{j,F}\big)_{t \geqslant 0}$ equals $Y^{j,G}$, where $G = F - \frac{1}{2}\beta\Delta^\perp$, to prove (a) it suffices to assume that $\beta = 0$. Let $\zeta \in \mathfrak{h} \underline{\otimes} \mathcal{E}$ and define the function

$$\varphi : \mathbb{R}_+ \to \mathbb{C}, \ s \mapsto \langle \widetilde{Y}_s \widehat{\nabla}_s \zeta, \widetilde{\jmath}_s\big(q(F)\big) \widetilde{Y}_s \widehat{\nabla}_s \zeta \rangle.$$

Then φ is right continuous and, by the Second Fundamental Formula (1.9),

$$\|Y_t \zeta\|^2 - \|\zeta\|^2 = \int_0^t \varphi(s)\, ds \quad (t \in \mathbb{R}_+).$$

Thus $q(F) \leqslant 0$ implies that Y is contractive and $q(F) = 0$ implies that Y is isometric. For the converse, note that

$$t^{-1}\big(\|Y_t \zeta\|^2 - \|\zeta\|^2\big) \to \varphi(0+) = \langle R\zeta, q(F) R\zeta \rangle$$

as $t \to 0+$ and R has dense range $\widehat{\mathsf{k}} \underline{\otimes} \mathfrak{h}$. This proves (a) and (b).

(c) If Y is coisometric then Y^* is weakly continuous and isometric, thus is a continuous process, and $Y_t^* = I_{\mathfrak{h} \otimes \mathcal{F}} + \int_0^t \widetilde{Y}_s^* \widetilde{\jmath}_s(F^*)\, d\Lambda_s$ for all $t \in \mathbb{R}_+$. Therefore, arguing as above,

$$\|Y_t^* \zeta\|^2 - \|\zeta\|^2 = \int_0^t \psi(s)\, ds \quad (t \in \mathbb{R}_+, \, \zeta \in \mathfrak{h} \underline{\otimes} \mathcal{E}),$$

where

$$\psi : \mathbb{R}_+ \to \mathbb{C},\ s \mapsto \langle \widehat{\nabla}_s \zeta, \widetilde{\jmath}_s\big(q(F^*)\big)\widehat{\nabla}_s\zeta\rangle.$$

Hence

$$0 = t^{-1}\big(\|Y_t^*\zeta\|^2 - \|\zeta\|^2\big) \to \psi(0+) = \langle R\zeta, q(F^*)R\zeta\rangle$$

as $t \to 0+$. Using the density of the range of R once more, it follows that $q(F^*) = 0$. □

Remark. Theorem 4.5 contains a coordinate-free and dimension-independent extension of Propositions 3.3 and 3.4 of [GLW]; it will be applied in the proof of Theorem 8.1.

When the free flow j is Markov regular, the converse of part (c) of Theorem 4.5 holds. We prove this next; the argument is a little more involved than that for the converse of part (b), requiring more than a simple application of the quantum Itô product formula.

Proposition 4.6. *Let $Y = Y^{j,F}$, where $F \in B(\widehat{\mathsf{k}}) \mathbin{\overline{\otimes}} \mathsf{A}$, and suppose that j is Markov regular. The following are equivalent:*

(i) *Y is coisometric;*
(ii) *$q(F^*) = 0$.*

Proof. Because of Theorem 4.5, it remains only to prove that (ii) implies (i). Suppose therefore that (ii) holds. Following [GLW, Proposition 3.6], define processes

$$Z := (Y_tY_t^* - I)_{t\geqslant 0}, \quad k := (j_t(\cdot)Y_t)_{t\geqslant 0} \quad \text{and} \quad l := \big(Z_t j_t(\cdot)\big)_{t\geqslant 0}.$$

Part (a) of Theorem 4.5 implies that Y is contractive; hence Z is too and k and l are completely contractive processes. We now show that l is the zero process and so Y is coisometric by the unitality of j. By Theorem 1.8, there exists a map $\theta \in CB(\mathsf{A}; B(\widehat{\mathsf{k}}) \mathbin{\overline{\otimes}} \mathsf{A})$ such that $j = k^\theta$, and the quantum Itô product formula (1.13) implies that $k = k^\psi$ for the completely bounded map

$$\psi : a \mapsto \theta(a) + \iota(a)F + \theta(a)\Delta F, \quad \text{where} \quad \iota = \iota^{\mathsf{A}}_{\widehat{\mathsf{k}}}. \tag{4.3}$$

Thus, since $k^\dagger = k^{\psi^\dagger}$, we have $Y_t^* = k_t^{\psi^\dagger}(1)$ for all $t \in \mathbb{R}_+$; in particular, the operator process Y^* is continuous and such that

$$Y_t^* = I + \int_0^t \widetilde{k}_s^\dagger\big(\psi^\dagger(1)\big)\, d\Lambda_s = I + \int_0^t \widetilde{Y}_s^* \widetilde{\jmath}_s(F^*)\, d\Lambda_s \quad (t \in \mathbb{R}_+).$$

It follows that Z is continuous and, by the quantum Itô product formula,

$$Y_tY_t^* = I + \int_0^t \Big\{\widetilde{\jmath}_s(F)\widetilde{Y}_s\widetilde{Y}_s^* + \widetilde{Y}_s\widetilde{Y}_s^*\widetilde{\jmath}_s(F^*) + \widetilde{\jmath}_s(F\Delta)\widetilde{Y}_s\widetilde{Y}_s^*\widetilde{\jmath}_s(\Delta F^*)\Big\}\, d\Lambda_s$$

for all $t \in \mathbb{R}_+$. The assumption $q(F^*) = 0$ now implies that

$$Z_t = \int_0^t \Big\{\widetilde{\jmath}_s(F)\widetilde{Z}_s + \widetilde{Z}_s\widetilde{\jmath}_s(F^*) + \widetilde{\jmath}_s(F\Delta)\widetilde{Z}_s\widetilde{\jmath}_s(\Delta F^*)\Big\}\, d\Lambda_s \quad (t \in \mathbb{R}_+).$$

A further application of the product formula yields the identity

$$l_t(a) = \int_0^t \Big\{\widetilde{\jmath}_s(F)\widetilde{l}_s\big(\mu_1(a)\big) + \widetilde{l}_s\big(\mu_2(a)\big) + \widetilde{\jmath}_s(F\Delta)\widetilde{l}_s\big(\mu_3(a)\big)\Big\}\, d\Lambda_s$$

for all $t \in \mathbb{R}_+$ and $a \in \mathsf{A}$, where $\mu_1 = \mu$, $\mu_2 = F^*\mu(\cdot) + \theta$ and $\mu_3 = \Delta F^*\mu(\cdot)$ for the map

$$\mu : a \mapsto I_{\widehat{\mathsf{k}}} \otimes a + \Delta\theta(a).$$

Hence, by the complete boundedness and normality of all the maps involved, for any Hilbert space h the process $l^{\mathsf{h}} := \big(\mathrm{id}_{B(\mathsf{h})} \overline{\otimes} l_t\big)_{t\geqslant 0}$ satisfies

$$l^{\mathsf{h}}_t(A) = \int_0^t \Big\{ \widetilde{j^{\mathsf{h}}_s}(F^{\mathsf{h}})\widetilde{l^{\mathsf{h}}_s}\big(\mu^{\mathsf{h}}_1(A)\big) + \widetilde{l^{\mathsf{h}}_s}\big(\mu^{\mathsf{h}}_2(A)\big) + \widetilde{j^{\mathsf{h}}_s}\big((F\Delta)^{\mathsf{h}}\big)\widetilde{l^{\mathsf{h}}_s}\big(\mu^{\mathsf{h}}_3(A)\big)\Big\}\, d\Lambda_s$$

for all $t \in \mathbb{R}_+$ and $A \in B(\mathsf{h}) \overline{\otimes} \mathsf{A}$, where $j^{\mathsf{h}} := \big(\mathrm{id}_{B(\mathsf{h})} \overline{\otimes} j_t\big)_{t\geqslant 0}$ and, in terms of the flip $\Sigma : B(\mathsf{h} \otimes \widehat{\mathsf{k}}) \overline{\otimes} \mathsf{A} \to B(\widehat{\mathsf{k}} \otimes \mathsf{h}) \overline{\otimes} \mathsf{A}$, the operator $G^{\mathsf{h}} := \Sigma(I_{\mathsf{h}} \otimes G)$ for $G \in B(\widehat{\mathsf{k}}) \overline{\otimes} \mathsf{A}$ and the map $\mu^{\mathsf{h}}_i := \Sigma \circ (\mathrm{id}_{B(\mathsf{h})} \overline{\otimes} \mu_i)$ for $i = 1, 2, 3$. Therefore, by the Fundamental Estimate after (1.10),

$$\|l^{\mathsf{h}}_t(A)\eta\varpi(f)\|^2 \leqslant 3C\big([0, t[, f\big)^2 \int_0^t \max_i \big\|\widetilde{l^{\mathsf{h}}_s}\big(\nu^{\mathsf{h}}_i(A)\big)\widehat{f}(s)\eta\varpi(f)\big\|^2 ds$$

for all $t \in \mathbb{R}_+$, $A \in B(\mathsf{h}) \overline{\otimes} \mathsf{A}$, $\eta \in \mathsf{h} \otimes \mathfrak{h}$ and $f \in \mathbb{S}$, where $\nu_1 = \|F\|\mu^{\mathsf{h}}_1$, $\nu_2 := \mu^{\mathsf{h}}_2$ and $\nu_3 := \|F\Delta\|\mu^{\mathsf{h}}_3$. Since $\widetilde{l^{\mathsf{h}}} = l^{\widehat{\mathsf{k}} \otimes \mathsf{h}}$, this estimate may be iterated; using the complete contractivity of l, n iterations yield the inequality

$$\|l_t(\cdot)E_{\varpi(f)}\| \leqslant \big\{\sqrt{3}\, C\big([0, t[, f\big) \max_i \|\nu_i\|_{\mathrm{cb}} \|\widehat{f}_{[0,t[}\|\big\}^n / \sqrt{n!}$$

for all $t \in \mathbb{R}_+$ and $f \in \mathbb{S}$; letting $n \to \infty$ shows that the left-hand side is identically zero. Thus $l = 0$, as required. □

Remark. This extends Proposition 3.6 of [GLW] and rectifies the proof given there, which neglected a term in its equation (3.10).

The next few results are relevant to the role of multiplier cocycles in providing multipliers for Feynman-Kac perturbations of the Markov semigroup of the free flow j. They also provide a link to the vacuum-adapted approach to such perturbations ([BLS]).

Proposition 4.7. *Let $F \in B(\widehat{\mathsf{k}}) \overline{\otimes} \mathsf{A}$ with block-matrix form $\begin{bmatrix} k & m \\ l & w-1 \end{bmatrix}$ and let Z be the vacuum projection cocycle* (2.2). *Then $Y^{j,F}Z = Y^{j,F'}$, where $F' = F\Delta^\perp - \Delta = \begin{bmatrix} k & 0 \\ l & -1 \end{bmatrix}$.*

Proof. Set $X = Y^{j,F}Z$. Then X inherits continuity from $Y^{j,F}$, thus $\widetilde{X}$ is QS integrable. Recall that Z strongly satisfies the quantum stochastic differential equation (2.3). For all $t \in \mathbb{R}$, in view of the unitality of j, we have $\Delta \otimes I_{\mathcal{F}} = \widetilde{\jmath}_t(\Delta)$. Therefore, by the quantum Itô formula, the process X satisfies the QS differential equation

$$dX_t = \big(\widetilde{\jmath}_t(F)\widetilde{X}_t - (\Delta \otimes I_{\mathcal{F}})\widetilde{X}_t - \widetilde{\jmath}_t(F)(\Delta \otimes I_{\mathcal{F}})\widetilde{X}_t\big)\, d\Lambda_t = \widetilde{\jmath}_t\big(F\Delta^\perp - \Delta\big)\widetilde{X}_t\, d\Lambda_t$$

with $X_0 = I$. It follows from uniqueness (in Theorem 4.1) that $X = Y^{j,F'}$. □

As an immediate consequence we have the following corollary.

Corollary 4.8. *If $F \in B(\widehat{\mathsf{k}}) \overline{\otimes} \mathsf{A}$ then*

$$Y^{j,F}_t E_{\varpi(0)} = Y^{j,F\Delta^\perp}_t E_{\varpi(0)} \quad (t \in \mathbb{R}_+).$$

In particular, if F has the block-matrix form $\left[\begin{smallmatrix} k & m \\ l & n \end{smallmatrix}\right]$ *then, for any contraction* $w \in B(\mathsf{k}) \overline{\otimes} \mathsf{A}$,

$$Y_t^{j,F} E_{\varpi(0)} = Y_t^{j,G} E_{\varpi(0)}, \qquad \textit{where} \qquad G = \begin{bmatrix} k & -l^* w \\ l & w - 1 \end{bmatrix}.$$

Remarks. Note that boundedness of the process $Y^{j,F}$ is not assumed here. However $G \in QC$, by part (b) of Corollary 4.4, so $Y^{j,G}$ is quasicontractive, by Theorem 4.5.

The following two instances of Corollary 4.8 are of particular relevance: for $F \in B(\widehat{\mathsf{k}}) \overline{\otimes} \mathsf{A}$, we have $Y^{j,F} E_{\varpi(0)} = Y^{j,F'} E_{\varpi(0)} = Y^{j,F''} E_{\varpi(0)}$, where

$$F' = \begin{bmatrix} k & 0 \\ l & -1 \end{bmatrix} \quad \text{and} \quad F'' = \begin{bmatrix} k & -l^* \\ l & 0 \end{bmatrix}. \tag{4.4}$$

5. Bounded perturbation processes are multiplier cocycles

In this section we show that, when bounded with locally uniform bounds, the perturbation processes constructed in Section 4 are multiplier cocycles for the free flow j.

The following lemma is an immediate consequence of adaptedness when j is implemented by a unitary QS operator cocycle, as it is in [LiS] and [BaP]. Recall the associated E_0-semigroup J of the free flow, defined at the start of Section 2.

Lemma 5.1. *Let X be a bounded operator process in $B(\widehat{\mathsf{k}}) \overline{\otimes} \mathsf{A}$ which is strongly continuous on $\widehat{\mathsf{k}} \otimes \mathfrak{h} \otimes \mathcal{F}$ and such that the process $\left(\int_0^t X_s \, d\Lambda_s\right)_{t \geqslant 0}$ is also bounded. For all $t \geqslant r \geqslant 0$ and $R \in B(\mathfrak{h}) \overline{\otimes} \mathsf{N}_{[0,r[} \otimes I_{[r,\infty[}$,*

$$J_r\Big(\int_0^{t-r} X_u \, d\Lambda_u\Big) R = \int_r^t \widetilde{J}_r(X_{s-r}) \widetilde{R} \, d\Lambda_s,$$

*where $\widetilde{R} := I_{\widehat{\mathsf{k}}} \otimes R$ and $\widetilde{J}_r$ denotes the unital *-homomorphism* $\mathrm{id}_{B(\widehat{\mathsf{k}})} \overline{\otimes} J_r$.

Proof. The right-hand side is well defined since $\big(\widetilde{J}_r(X_{s-r})\big)_{s \geqslant r}$ is strongly continuous and adapted. If $R \in B(\mathfrak{h}) \otimes I_{\mathcal{F}}$ the result follows from the extended First Fundamental Formula, Lemma 1.3, together with the identity

$$\Omega[f,g] \circ J_r = \Omega[f_{r)}, g_{r)}] \circ j_{r)} \circ \Omega[S_r^* f, S_r^* g],$$

which holds for all f, $g \in \mathbb{S}$ and $r \in \mathbb{R}_+$; here $f_{r)}$ and $g_{r)}$ denote the restrictions of the functions f and g to the interval $[0,r[$, $(S_r)_{r \geqslant 0}$ is the right-shift semigroup on $L^2(\mathbb{R}_+; \mathsf{k})$ and $\Omega[f,g] := \mathrm{id}_{\mathsf{A}} \overline{\otimes} \omega_{\varpi(f),\varpi(g)}$. The case of general R now follows from the First Fundamental Formula, taking R to be an ampliated Weyl operator at first. □

Remark. If $\mathfrak{h}$ and k are separable then the continuity assumption on the process X may be replaced by locally uniform boundedness.

Theorem 5.2. *Let $F \in B(\widehat{\mathsf{k}}) \overline{\otimes} \mathsf{A}$ and suppose that the process $Y^{j,F}$ is bounded with locally uniform bounds. Then $Y^{j,F}$ is a multiplier cocycle for j.*

Proof. Fix $r > 0$. We must show that

$$Y_t = J_r(Y_{t-r})Y_r \tag{5.1}$$

for all $t > r$, where $Y = Y^{j,F}$. To this end, define a process Z in A by letting

$$Z_s := \begin{cases} Y_s & \text{if } s \leqslant r, \\ J_r(Y_{s-r})Y_r & \text{if } s \geqslant r. \end{cases}$$

It follows from Proposition 3.2 that Y is strongly continuous on $\mathfrak{h} \otimes \mathcal{F}$ so, by Lemma 1.11, the bounded process Z is strongly continuous on $\mathfrak{h} \otimes \mathcal{F}$ too. Hence $\big(\widetilde{\jmath}_s(F)\widetilde{Z}_s\big)_{s \geqslant 0}$ is strongly continuous and thus QS integrable. Therefore, by Lemma 5.1 and the cocycle property of j,

$$\begin{aligned} Z_t &= J_r(Y_{t-r})Y_r \\ &= Y_r + J_r\Big(\int_0^{t-r} \widetilde{\jmath}_u(F)\widetilde{Y}_u \, d\Lambda_u\Big)Y_r \\ &= I + \int_0^r \widetilde{\jmath}_s(F)\widetilde{Y}_s \, d\Lambda_s + \int_r^t \widetilde{\jmath}_s(F)\widetilde{J}_r(\widetilde{Y}_{s-r})\widetilde{Y}_r \, d\Lambda_s \\ &= I + \int_0^t \widetilde{\jmath}_s(F)\widetilde{Z}_s \, d\Lambda_s, \end{aligned}$$

for all $t \geqslant r$. As this also holds for $t < r$ it follows, by the uniqueness part of Theorem 4.1, that $Z = Y$. In particular (5.1) holds, as required. □

By Proposition 2.3 and Theorem 4.5, the theorem above has the following immediate consequence.

Corollary 5.3. *Let F_1, $F_2 \in B(\widehat{\mathsf{k}})\overline{\otimes}\mathsf{A}$ and suppose that Y^{j,F_1} and Y^{j,F_2} are bounded with locally uniform bounds. Then the normal completely bounded process*

$$j^{F_1,F_2} := \big((Y_t^{j,F_1})^* j_t(\cdot) Y_t^{j,F_2}\big)_{t \geqslant 0} \tag{5.2}$$

is an ultraweakly continuous QS cocycle which is exponentially completely bounded. Moreover, if F_1, $F_2 \in QC$ then the QS cocycle j^{F_1,F_2} is completely quasicontractive.

We abbreviate $j^{F,F}$ to j^F.

Remark. The completely positive QS cocycle j^F is contractive if and only if $F \in QC_0$; it is unital if and only if $Y^{j,F}$ is isometric or, equivalently, if F has the block-matrix form

$$\begin{bmatrix} ih - \frac{1}{2}l^*l & -l^*w \\ l & w - 1 \end{bmatrix}, \quad \text{with } h \text{ selfadjoint and } w \text{ isometric;}$$

the QS cocycle j^F is *-homomorphic if the multiplier $Y^{j,F}$ is coisometric, for which a *necessary* condition is that F have the block-matrix form

$$\begin{bmatrix} ih - \frac{1}{2}mm^* & m \\ -wm^* & w - 1 \end{bmatrix}, \quad \text{with } h \text{ selfadjoint and } w \text{ coisometric.}$$

The next result provides considerable freedom in the choice of multiplier cocycle for obtaining a given perturbation of the Markov semigroup of the free flow j.

Theorem 5.4. *For $i = 1, 2$, let $F_i \in B(\widehat{\mathsf{k}}) \overline{\otimes} \mathsf{A}$ and suppose that Y^{j,F_i} is bounded; let F_i' and F_i'' be defined as in (4.4). Then the expectation semigroups of the ultraweakly continuous QS cocycles $j^{F_1',F_2'}$ and $j^{F_1'',F_2''}$ are both equal to that of j^{F_1,F_2}, namely $\big(\mathbb{E} \circ j_t^{F_1,F_2}\big)_{t \geqslant 0}$.*

Proof. By part (b) of Corollary 4.4, the operators F_1' and $F_2' \in QC$; thus Theorems 4.5 and 5.2 imply that $Y^{F_1'}$ and $Y^{F_2'}$ are quasicontractive multiplier cocycles for j. Therefore $j^{F_1',F_2'}$ is ultraweakly continuous, by Corollary 5.3, and so its expectation semigroup is too; by Corollary 4.8, the expectation semigroups of j^{F_1,F_2} and $j^{F_1',F_2'}$ are equal. Exactly the same argument applies to $j^{F_1'',F_2''}$ and the result follows. □

Remark. In Section 8 we shall see how the weak* generator of the expectation semigroup of j^{F_1,F_2} appears as a perturbation of that of the free flow j on the domain of its QS derivative.

6. Characterisation of multiplier cocycles

Under a Hölder-regularity assumption, every multiplier cocycle of the free flow j satisfies a quantum stochastic differential equation of the form (4.1), giving a partial converse to Theorem 5.2. This leads to a characterisation theorem for multiplier cocycles when the free flow is Markov regular.

Theorem 6.1. *Let Y be a locally uniformly bounded multiplier cocycle for j and suppose that the associated QS mapping cocycle $k := (j_t(\cdot)Y_t)_{t \geqslant 0}$ satisfies the condition*

$$\limsup_{t \to 0+} t^{-1/2} \big\{ \| k_{t,\varepsilon} - \iota^{\mathsf{A}}_{\mathcal{F},\varepsilon} \|_{\mathrm{cb}} + \| k^{\dagger}_{t,\varepsilon} - \iota^{\mathsf{A}}_{\mathcal{F},\varepsilon} \|_{\mathrm{cb}} \big\} < \infty \quad (\varepsilon \in \mathcal{E}). \tag{6.1}$$

Then $Y = Y^{j,F}$ for a unique operator $F \in B(\widehat{\mathsf{k}}) \overline{\otimes} \mathsf{A}$.

Proof. As k is completely bounded with locally bounded cb norm, $k = k^{\phi}$ for a completely bounded map $\phi : \mathsf{A} \to \mathsf{A} \overline{\otimes} B(\widehat{\mathsf{k}})$, by Theorem 1.8. Since $Y_t = k_t(1_{\mathsf{A}})$, it follows that Y strongly satisfies the quantum stochastic differential equation (4.1), with $F = \phi(1_{\mathsf{A}})$, and so $Y = Y^{j,F}$ by uniqueness. Uniqueness of F follows from Proposition 4.2. □

Remarks. When the free flow j is Markov regular, (6.1) is equivalent to

$$\limsup_{t \to 0+} t^{-1/2} \| (Y_t - I) E_{\varepsilon} \| < \infty \text{ and } \limsup_{t \to 0+} t^{-1/2} \| (Y_t^* - I) E_{\varepsilon} \| < \infty \ \ (\varepsilon \in \mathcal{E}). \tag{6.2}$$

This is because, in this case, j itself satisfies the inequality (6.1).

The first of the conditions in (6.2) is necessary for Y to be of the form $Y^{j,F}$; the second is too if Y is coisometric or if both Y is locally uniformly bounded and the spaces $\mathfrak{h}$ and k are separable, by Theorem 4.1.

Theorem 6.2. *Let Y be a quasicontractive multiplier cocycle for the free flow j and suppose that j is Markov regular. The following are equivalent:*

(i) *$Y = Y^{j,F}$ for an operator $F \in B(\widehat{\mathsf{k}}) \overline{\otimes} \mathsf{A}$;*
(ii) *the associated QS cocycle $k := (j_t(\cdot)Y_t)_{t \geqslant 0}$ is Markov regular.*

In this case $F = \psi(1)$, where ψ is the stochastic generator of k.

Proof. If (i) holds then, as shown in the proof of Proposition 4.6, $k = k^\psi$ for the completely bounded map (4.3) and so (ii) holds. Conversely, suppose that (ii) holds. Recall that $e^{\beta t} Y_t^F = Y_t^G$, where $G = F + \beta\Delta^\perp$. Therefore, replacing Y by $(e^{-\beta t} Y_t)_{t \geqslant 0}$ for a suitable $\beta \in \mathbb{R}$, we may assume without loss of generality that Y is contractive. Fixing $u \in \mathfrak{h}$ and $f \in \mathbb{S}$, let $\mathcal{Q}$ be the (c, c)-associated semigroup of k, as defined in (1.17), where $c = f(0)$. Then, by the contractivity of Y, when t is less than any point of discontinuity of the step function f,

$$\begin{aligned} \|(Y_t - I)u\varpi(f)\|^2 + \|(Y_t^* - I)u\varpi(f)\|^2 \\ &\leqslant 4\,\mathrm{Re}\langle u\varpi(f), (I - Y_t)u\varpi(f)\rangle \\ &= 4\,\mathrm{Re}\langle u, \big(1 - \mathcal{Q}_t(1)\big)u\rangle \leqslant 4\,\|\mathrm{id}_{\mathsf{A}} - \mathcal{Q}_t\|\,\|u\|^2. \end{aligned}$$

As k is Markov regular, $\mathcal{Q}$ is norm continuous and thus norm differentiable, so (6.2) holds. The result now follows from Theorem 6.1 and the first remark following it. □

Remark. This result simultaneously generalises Theorem 6.7 of [LW_1] (the case where j is trivial, Y is contractive, $\mathsf{A} = B(\mathfrak{h})$ and k is separable) and the main result of [Bra] (the case where Y is unitary, $\mathfrak{h}$ is separable and k has one dimension). Bradshaw uses the quantum martingale representation theorem ([PS_2]); his proof does not extend beyond the case of unitary Y and separable $\mathfrak{h}$.

7. Perturbations of flows

In [LiS] and [BaP] the free flow j is given by

$$j_t = \alpha_{B_t} \quad (t \in \mathbb{R}_+),$$

where $(\alpha_s)_{s\in\mathbb{R}}$ is a normal *-automorphism group on A and $(B_t)_{t\geqslant 0}$ is a standard one-dimensional Brownian motion. In this case j is governed by the classical stochastic differential equation

$$dj_t = j_t \circ \delta\, dB_t + \tfrac{1}{2} j_t \circ \delta^2 dt,$$

where δ is the derivation generating α. In these papers it is further assumed that the automorphism group, and so also the free flow, is unitarily implemented. Here we generalise to consider a genuinely quantum stochastic flow j, which is neither assumed to be unitarily implemented nor assumed to be driven by a classical Brownian motion.

Let θ be the ultraweak stochastic derivative of the free flow j given by Theorem 1.10, and let τ_0 be the ultraweak generator of the Markov semigroup of j. Then $\mathrm{Dom}\,\theta$ is a subspace of A contained in $\mathrm{Dom}\,\tau_0$, the algebra identity $1 \in \mathrm{Dom}\,\theta$ with $\theta(1) = 0$ and the map θ is real (that is, $\mathrm{Dom}\,\theta$ is *-invariant and $\theta(x)^* = \theta(x^*)$ for all $x \in \mathrm{Dom}\,\theta$). By the quantum Itô product formula,

$$\theta(x^* y) = \theta(x)^* \iota(y) + \iota(x)^* \theta(y) + \theta(x)^* \Delta \theta(y),$$

where $\iota = \iota_{\widehat{\mathsf{k}}}^{\mathsf{A}}$, for all x, $y \in \operatorname{Dom}\theta$ such that $x^*y \in \operatorname{Dom}\theta$. In terms of the block-matrix form $\begin{bmatrix} \tau_0 & \delta^\dagger \\ \delta & \pi-\iota \end{bmatrix}$ of θ, in which $\iota = \iota_{\mathsf{k}}^{\mathsf{A}}$,

$$\begin{aligned} \pi(x^*y) &= \pi(x)^*\pi(y), \\ \delta(x^*y) &= \delta(x)^*y + \pi(x)^*\delta(y) \\ \text{and} \quad \tau_0(x^*y) &= \tau_0(x)^*y + x^*\tau_0(y) + \delta(x)^*\delta(y) \end{aligned}$$

for all x, $y \in \operatorname{Dom}\theta$ such that $x^*y \in \operatorname{Dom}\theta$.

Theorem 7.1. *Let (j, θ) be as above, let F_1, $F_2 \in B(\widehat{\mathsf{k}})\overline{\otimes}\mathsf{A}$ and suppose that the perturbation process Y^{j,F_1} is bounded. Then the mapping process j^{F_1,F_2}, defined by* (5.2)*, weakly satisfies the quantum stochastic differential equation*

$$k_0 = \iota, \quad dk_t = \widetilde{k}_t \circ \phi \, d\Lambda_t, \tag{7.1}$$

where $\operatorname{Dom}\phi = \operatorname{Dom}\theta$, $\iota = \iota_{\mathcal{F}}^{\mathsf{A}}$ *and, for all* $x \in \operatorname{Dom}\phi$,

$$\phi(x) = \theta(x) + F_1^*\iota(x) + F_1^*\Delta\big(\theta(x) + \iota(x)\big)\Delta F_2 + \iota(x)F_2 + F_1^*\Delta\theta(x) + \theta(x)\Delta F_2.$$

If the adjoint process $\big((Y_t^{j,F_1})^*\big)_{t\geqslant 0}$ is strongly continuous on $\mathfrak{h}\otimes\mathcal{F}$ then j^{F_1,F_2} satisfies (7.1) *strongly.*

Proof. The first part is a straightforward consequence of the quantum Itô product formula applied to $(Y_t^{j,F_1})^* j_t(\cdot)$ and then the Second Fundamental Formula (1.9), in polarised form. If $\big((Y_t^{j,F_1})^*\big)_{t\geqslant 0}$ is strongly continuous on $\mathfrak{h} \otimes \mathcal{F}$ then, setting $k = j^{F_1,F_2}$, the process $\widetilde{k}\big(\phi(x)\big)$ is QS integrable for each $x \in \operatorname{Dom}\phi$ and so the second part follows too. □

Remarks. With regard to the hypotheses, recall that if $F \in QC$ then $Y^{j,F}$ is quasicontractive, by Theorem 4.5; if, in addition, $\mathfrak{h}$ and k are separable then $(Y^{j,F})^*$ is necessarily strongly continuous on $\mathfrak{h} \otimes \mathcal{F}$, by Theorem 4.1.

If $F_2 = F_1 = F$, $q(F) = 0$ and $Y^{j,F}$ is coisometric (for which the necessary condition $q(F^*) = 0$ is also sufficient when j is Markov regular) then j^F is a QS flow on A.

If F_1 and F_2 have block-matrix forms

$$\begin{bmatrix} k_1 & m_1 \\ l_1 & w_1 - 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} k_2 & m_2 \\ l_2 & w_2 - 1 \end{bmatrix},$$

respectively, then $\phi(x)$ equals

$$\begin{bmatrix} \tau_0(x) + l_1^*\delta(x) + l_1^*\pi(x)l_2 + \delta^\dagger(x)l_2 + k_1^*x + xk_2 & \big(\delta^\dagger(x) + l_1^*\pi(x)\big)w_2 + xm_2 \\ w_1^*\big(\delta(x) + \pi(x)l_2\big) + m_1^*x & w_1^*\pi(x)w_2 - x\otimes 1 \end{bmatrix}.$$

8. Feynman–Kac formulae

The cocycle-perturbation theorem of the previous section yields a general form of quantum Feynman–Kac formula. Let $\mathcal{P}^0$ be a Markov semigroup on A which is realised as the Markov semigroup of a QS flow j, in the sense that $\mathcal{P}_t^0 = \mathbb{E} \circ j_t$ for all $t \in \mathbb{R}_+$, with $\mathbb{E}$ the vacuum expectation defined in (1.3). Let $\theta = \begin{bmatrix} \tau_0 & \delta^\dagger \\ \delta & \pi-\iota \end{bmatrix}$ be the quantum stochastic derivative of j, as in Theorem 1.10, where τ_0 is the weak* generator of $\mathcal{P}^0$ and $\iota = \iota_{\mathsf{k}}^{\mathsf{A}}$.

Theorem 8.1. *Let (j, θ) be as in Section 7, and let l_1, $l_2 \in |\mathsf{k}\rangle \overline{\otimes} \mathsf{A}$ and k_1, $k_2 \in \mathsf{A}$. Then there is an ultraweakly continuous normal completely quasi-contractive semigroup $\mathcal{P}$ on A whose ultraweak generator is an extension of the operator with domain $\operatorname{Dom}\theta$ given by the prescription*

$$x \mapsto \tau_0(x) + l_1^*\delta(x) + l_1^*\pi(x)l_2 + \delta^\dagger(x)l_2 + k_1^*x + xk_2,$$

which is

(a) *contractive if $k_1^* + k_1 + l_1^*l_1 \leqslant 0$ and $k_2^* + k_2 + l_2^*l_2 \leqslant 0$;*
(b) *unital if $k_1^* + l_1^*l_2 + k_2 = 0$;*
(c) *completely positive if $l_1 = l_2$ and $k_1 = k_2$.*

Proof. Let

$$F_1 = \begin{bmatrix} k_1 & -l_1^* \\ l_1 & 0 \end{bmatrix} \quad \text{and} \quad F_2 = \begin{bmatrix} k_2 & -l_2^* \\ l_2 & 0 \end{bmatrix}.$$

By Corollary 4.4, F_1, $F_2 \in QC$ and so, by Corollary 5.3, j^{F_1,F_2} is an ultraweakly continuous normal completely quasicontractive QS cocycle; let $\mathcal{P}$ be its expectation semigroup and let ϕ be the map defined in Theorem 7.1. Then j^{F_1,F_2} is unital if $\phi(1_{\mathsf{A}}) = 0$, completely positive if $F_1 = F_2$ and contractive if $q(F_1) \leqslant 0$ and $q(F_2) \leqslant 0$, by Theorem 4.5. Therefore, since

$$\phi(1_{\mathsf{A}}) = \Delta^\perp \otimes (k_1^* + l_1^*l_2 + k_2) \quad \text{and} \quad q(F_i) = \Delta^\perp \otimes (k_i^* + k_i + l_i^*l_i)$$

for $i = 1$, 2, the result follows from Theorem 7.1. □

Remarks. The term $k_1^*x + xk_2$ contributes a bounded perturbation which can alternatively be realised via the Trotter product formula; see, for example, [Dav].

The *domain algebra* of $\mathcal{P}^0$ is the largest unital *-subalgebra of A contained in $\operatorname{Dom}\tau_0$, whose existence follows from an argument using the Kuratowski–Zorn Lemma. Note that the domain algebra is contained in the selfadjoint unital subspace $\mathcal{D}_2^{0,0}$ of A defined in (1.20). There are interesting examples in which the domain algebra fails to be ultraweakly dense in A, both commutative ([Fa$_2$]) and noncommutative ([Ar$_2$]), and interesting examples where it is ultraweakly dense ([Ar$_2$]). In the latter case $\operatorname{Dom}\theta$ is ultraweakly dense, by Theorem 1.10. When $\mathsf{A} = B(\mathfrak{h})$ and $\mathfrak{h}$ is separable, $\mathcal{D}_2^{0,0}$ is an algebra ([Ar$_2$]) and therefore equals the domain algebra of $\mathcal{P}^0$.

Thus any QS flow j dilating $\mathcal{P}^0$, in the sense that $\mathcal{P}^0 = \big(\mathbb{E} \circ j_t\big)_{t \geqslant 0}$, and any F_1, $F_2 \in B(\widehat{\mathsf{k}})\overline{\otimes}\mathsf{A}$ such that Y^{j,F_1} is bounded, give rise to a semigroup $\mathcal{P}$ whose generator is a noncommutative vector field-type perturbation of the generator τ_0 of $\mathcal{P}^0$, through the quantum Feynman–Kac formula

$$\mathcal{P}_t = \mathbb{E}\Big[(Y_t^{j,F_1})^* j_t(\cdot) Y_t^{j,F_2}\Big] \quad (t \in \mathbb{R}_+).$$

In terms of the perturbation generators F_1 and F_2, the semigroup $\mathcal{P}$ depends only on $F_1\Delta^\perp$ and $F_2\Delta^\perp$.

Specialising to Markov semigroups we have the following result, which considerably extends the class of Feynman–Kac formulae obtained in [BaP], where the free flow is obtained from an automorphism group of A randomised by a classical Brownian motion.

Corollary 8.2. *Let (j, θ) be as above, and let $l \in |\mathsf{k}\rangle \overline{\otimes} \mathsf{A}$ and $h = h^* \in \mathsf{A}$. There is a completely positive Markov semigroup $\mathcal{P}$ on A whose ultraweak generator is extension of the operator with domain $\operatorname{Dom}\theta$ given by the prescription*

$$x \mapsto \tau_0(x) + l^*\delta(x) + l^*\pi(x)l + \delta^\dagger(x)l + k^*x + xk,$$

where $k := ih - \frac{1}{2}l^*l$.

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Department of Mathematics and Statistics, Fylde College, Lancaster University, Lancaster LA1 4YF, United Kingdom

E-mail address: `a.belton@lancaster.ac.uk`

E-mail address: `j.m.lindsay@lancaster.ac.uk`

Institute of Mathematics, Polish Academy of Sciences, ul. Śniadeckich 8, 00-956 Warszawa, Poland

E-mail address: `a.skalski@impan.pl`