THE MODULI SPACE OF CURVES AND ITS TAUTOLOGICAL RING

RAVI VAKIL

The moduli space of curves has proven itself a central object in geometry. The past decade has seen substantial progress in understanding the moduli space of curves, involving ideas, for example, from geometry (algebraic, symplectic, and differential), physics, topology, and combinatorics. Many of the new ideas are related to the tautological ring of the moduli space, a subring of the cohomology (or Chow) ring that (i) seems highly structured, and (ii) seems to include all the geometrically natural classes.

The goal of this article is to give the reader an introduction to the moduli space of curves and some intuition into it and its structure. We do this by focusing on its tautological ring, and in particular on its beautiful combinatorial structure. As motivation, we consider the prototypical example of a moduli space, that of the Grassmannian (parameterizing k-dimensional subspaces of an n-dimensional space). Its cohomology ring has a elegant structure, and Mumford suggested studying the moduli space of curves in the same way. We introduce the moduli space of (genus g, n-pointed) curves, with enough information to give a feel for its basic geography. We next introduce the tautological cohomology (or Chow) classes. Finally, we describe some of what is known about the tautological ring, emphasizing combinatorial aspects. We will assume as little as possible, although some notions from geometry and topology will be used.

The selection of material is, of course, a personal one, and there are necessarily omissions of important work. However, we hope that enough has been given for the reader new to the field to appreciate the deep structure of this important geometrical object. For the expert, we hope that the view will be a refreshing one with perhaps some combinatorial surprises. Many important recent developments in this highly interdisciplinary field are beyond the scope of this article. For example, the most exciting news in recent times is Madsen and Weiss' proof of Madsen's generalization of Mumford's conjecture, based on earlier work of Madsen and Tillmann; the methods are topological and singularity-theoretic. Other relevant fields include Teichmüller theory, geometric topology, conformal field theory, and arithmetic geometry (Grothendieck's *dessins d'enfants*). Even within algebraic geometry, this discussion leaves out important recent work of Farkas, Gibney-Keel-Morrison, and many others.

We will use the algebro-geometric language of schemes, but readers should feel free to interpret these loosely as a certain type of (ringed, topological) space, and to think instead in their category of choice (such as algebraic varieties, analytic spaces, manifolds,

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Ravi Vakil is assistant professor in the Department of Mathematics at Stanford University. His work was supported in part by NSF grant DMS-0098769, an AMS Centennial Fellowship, and an Alfred P. Sloan Research Fellowship. 2000MSC: 14H10, 14H81, 30Fxx.

topological spaces, etc.). All dimensions will be complex (= algebraic), not real, unless otherwise specified.

The perspective here on moduli spaces is that of Grothendieck, Mumford, and Deligne. The highlights discussed are informed most heavily by the books of Arbarello, Cornalba, Griffiths, Harris, and Morrison, and by work of Faber, Getzler, Looijenga, Pandharipande, Pikaart, and others. In particular, the author is deeply grateful to Joe Harris for teaching him how to think about this beautiful subject. Suggestions from D. M. Jackson and many others have greatly improved the exposition of this article.

Prototypical example of a moduli space: The Grassmannian.

A moduli space is, informally, a parameter space for a certain type of object. This is best understood in terms of a prototypical example, that of the Grassmannian.

Fix a dimension n vector space $\mathbb{C}^n$. (We work over the complex numbers. However, the construction works over an arbitrary base field, and in much greater generality.) The Grassmannian $G(k,n)$ is the set of dimension k subspaces of $\mathbb{C}^n$. If V is a dimension k subspace, denote the corresponding point of $G(k,n)$ by $[V]$. The Grassmannian clearly has more structure than just that of a set; for example, there is an intuitive notion of "nearby k-spaces", so $G(k,n)$ should be a topological space. There is a natural way to give $G(k,n)$ the structure of a manifold (and indeed the structure of a variety), as follows.

Fix k general subspaces of dimension $n-k+1$, $W_1, \dots, W_k$. "Most" dimension k subspaces V intersect each W_i in a one-dimensional subspace (i.e. a point of $\mathbb{P}W_i$). Conversely, for "most" choices of points of $[\ell_1] \in \mathbb{P}W_1, \dots, [\ell_k] \in \mathbb{P}W_k$, the ℓ_i are linearly independent and hence span a k-dimensional subspace V, and $V \cap W_i = \ell_i$ for all i. Thus we have given a "large subset" of $G(k,n)$ the structure of a complex manifold (or a smooth variety). The choice of a different k-tuple $(W_i)_{1 \leq i \leq k}$ will give a different large subset. One may quickly check that these subsets cover $G(k,n)$, and that the manifold (or algebraic) structures are compatible. Hence $G(k,n)$ is a complex manifold, and indeed a smooth complex algebraic variety. (Exercise: if $k=1$, $G(k,n) \cong \mathbb{P}^{n-1}$ as manifolds, and the "usual" cover by n affine spaces $\mathbb{C}^{n-1}$ is of the form described above.)

There is a "universal family" over $G(k,n)$:

$$\begin{array}{ccc} U & \hookrightarrow & G(k,n) \times \mathbb{C}^n \\ \pi \downarrow & \swarrow & \\ G(k,n). & & \end{array} \tag{1}$$

Here U is a complex algebraic variety, and π is a morphism (of varieties). Furthermore, $\pi^{-1}[V] = V \subset \mathbb{C}^n$. In fact, U is a vector bundle over $G(k,n)$, sometimes called the *tautological bundle*.

There is an explicit sense in which this algebraic structure on the set $G(k,n)$ is the "right" one. (This is the rigorous definition of a moduli space.) A family of k-spaces $\rho : V \to B$ parameterized by a variety B induces a map of sets $B \to G(k,n)$ (taking a point $b \in B$ to the class of the fiber $\rho^{-1}(b)$). This map of sets is in fact an algebraic morphism. Moreover, the pullback of the universal family $U \to G(k,n)$ to B by this morphism is our original family $V \to B$. Hence we have a bijection between the set of families of k-planes parameterized by a variety B and morphisms $B \to G(k,n)$. This uniquely specifies the algebraic structure on $G(k,n)$. (Caveat: The notion of a "family of k-spaces parameterized by a base" needs to be appropriately defined. The appropriate algebraic version of continuity of the family is "flatness".)

In short, we have defined $G(k,n)$ in a categorical manner as follows. We have a "moduli functor" that is a contravariant functor from the category of varieties to the category of sets, that takes a variety B to the set of families $\mathrm{Fam}(B)$ of k-planes parameterized by B. A morphism of varieties $B' \to B$ induces a map of sets $\mathrm{Fam}(B) \to \mathrm{Fam}(B')$: a family over B pulls back to a family over B'. This functor is isomorphic to another functor, sending B to $\mathrm{Hom}(B, G(k,n))$. However, the Grassmannian can (and should) be thought of rather concretely. A point of the Grassmannian should be thought of as a k-plane; "nearby" points of the Grassmannian should be thought of as "nearby" k-planes in $\mathbb{C}^n$.

Cohomology classes of moduli spaces.

If $\mathcal{M}$ is a moduli space of a certain kind of object (or structure), then facts about $\mathcal{M}$ can be interpreted as universal statements about all families of such objects. In particular, cohomology classes of the moduli space are universal cohomology classes. For example, the classifying space of rank k complex vector bundles can be interpreted as a moduli space $\mathcal{M}_{\mathrm{VB}}$, and

(2) $$H^*(\mathcal{M}_{\mathrm{VB}}) = \mathbb{Z}[x_1, \dots, x_k],$$

where x_i has degree (i.e. real codimension) $2i$. Any vector bundle V over a base B induces a map $B \to \mathcal{M}_{\mathrm{VB}}$, and the pullback of x_i is the ith Chern class $c_i(V)$ of the bundle. Hence (2) can be interpreted as showing that the only universal cohomology classes (in the appropriate sense) coming from vector bundles are Chern classes, and there are no relations among the classes. (To make this discussion precise in the algebraic category, one needs the notion of Artin stacks.)

In analogy with Chern classes, the cohomology ring of a moduli space parameterizing a certain kind of object is of interest for many reasons, including the following.

(i) Specific questions about the objects in question can be reduced to calculations on the moduli space.
(ii) To prove general facts about about all families of objects, it suffices to prove a specific fact about a single (hopefully finite-dimensional) space.

The cohomology ring of the Grassmannian has a rich structure, connected to combinatorics, representation theory, and the theory of symmetric functions. (See [Fu] for a detailed description.) We observe first that $G(k,n)$ is smooth and compact. Hence we can associate cohomology classes with their Poincaré duals, and will do so.

It is a fundamental fact that the cohomology ring is generated (as a ring) by the Chern classes of a naturally defined vector bundle on $G(k,n)$: the universal family $U \to G(k,n)$ of (1).

A specific question: How many lines meet four given lines in three-space?.

An example of a specific question about k-planes that can be reduced to calculations in $H^*(G(k,n))$ is the following. Fix four general lines L_1, L_2, L_3, L_4 in $\mathbb{C}^3$. How many lines meet all four? Because this question is in some sense linear algebra, one might suspect its answer to be 0, 1, or ∞. In fact it is 2. (The reader may enjoy showing this.)

This is a favorite question of many mathematicians, because they will all agree that it is the baby case of an important idea. However, they won't agree on what the important idea is! (Their answer will depend on whether their primary interest is linear algebra, geometry, topology, representation theory of S_n and $GL(n)$, Gromov-Witten theory and quantum cohomology, combinatorics, symmetric functions, or something else.)

This question can be reduced to intersection theory on the Grassmannian as follows. Interpret $G(2,4)$ projectively, as the space of projective lines in projective three-space. It has (complex) dimension four. Given a fixed line L_i in $\mathbb{P}^3$, the locus of lines meeting L_i is (complex) codimension 1. Let $D \in H^2(G(2,4))$ be the cohomology class corresponding to this locus. (In fact $D = c_1(U)$.) Then this question translates to: compute D^4.

The moduli space of n-pointed genus g curves.

We next apply the machinery of moduli spaces to algebraic curves. (For a more detailed but still informal introduction, see [MirSym, Part 4]. For a more complete introduction, see [HM]. For excellent summaries of recent research, see [HL, FL].)

We take smooth curves to be Riemann surfaces (connected compact complex manifolds of dimension 1), although the following discussion applies over an arbitrary field (and indeed over an arbitrary base scheme). An n-pointed smooth curve is a smooth curve with n *distinct* points labeled p_1 through p_n. (In real dimension 2, there is a natural identification among Riemannian structures up to scaling, conformal structures, almost complex structures, complex structures, and complex algebraic structures. These equivalences do not hold for higher dimensional spaces.)

The genus of a smooth curve is its genus as a surface. For example, Figure 1 is a 3-pointed genus 6 curve. Perhaps the best way to get a handle on a general genus g curve is as a high-degree branched cover of the projective line $\mathbb{P}^1$. This was the perspective of Riemann, and perhaps surprisingly, many of the results proved in the last decade involve understanding the moduli of curves in terms of branched covers.

Fix a genus g, and a non-negative integer n. Exclude the cases $(g,n) = (0,0)$, $(0,1)$, $(0,2)$, and $(1,0)$ for technical reasons that will be described later; the cases $(1,1)$ and $(2,0)$ are also somewhat special. Let $\mathcal{M}_{g,n}$ be the set of smooth genus g n-pointed curves. (When $n = 0$, the notation $\mathcal{M}_g$ is often used.) As in the case of the Grassmannian, $\mathcal{M}_{g,n}$ has additional structure. Unfortunately, it cannot be given the structure of a complex manifold, or a variety. (More precisely: the moduli functor is not representable in these categories.) However, it can be given the structure of an orbifold, or more precisely (to avoid

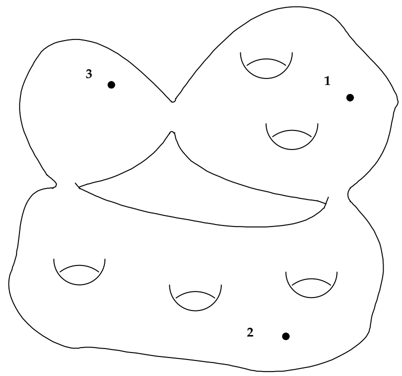

FIGURE 1. A 3-pointed genus 6 curve

vagaries of definition of orbifold that cause problems with $(1,1)$ and $(2,0)$), a smooth Deligne-Mumford stack. The general reader may think of $\mathcal{M}_{g,n}$ as a complex manifold without going too far wrong; the main caveat is that cohomology should be taken with $\mathbb{Q}$-coefficients, not $\mathbb{Z}$-coefficients. In order to camouflage this issue, we will call $\mathcal{M}_{g,n}$ a "space". Note for experts: each point of an orbifold has an associated finite group. For $\mathcal{M}_{g,n}$, the group corresponding to the pointed curve $(C, p_1, \dots, p_n)$ is the automorphism group of the pointed curve. If the group is $\{e\}$, $\mathcal{M}_{g,n}$ is a manifold near that point. For example, most genus 3 curves have only the trivial automorphism, but the general *hyperelliptic curve* given by the compactification of the locus in $\mathbb{C}^2$

$$y^2 = \prod_{i=1}^{8}(x - a_i) \qquad (a_i \neq a_j \text{ for } i \neq j)$$

has a nontrivial automorphism (given by $(x, y) \mapsto (x, -y)$), and the curve

$$x^3y + y^3z + z^3x = 0$$

in the projective plane has the most automorphisms of any genus 3 curve, 168 (exercise: find them all!).

We will need one fact about $\mathcal{M}_{g,n}$. Remarkably, in some sense this was known to Riemann.

Fact 1. The space $\mathcal{M}_{g,n}$ is smooth and irreducible (i.e. connected), of dimension $3g-3+n$.

The geometry of this space is rather subtle, and very few general statements are known. One example which is elementary to state is the following. Any complex curve of genus at most 10 can be arbitrarily well approximated by a rationally definable curve (i.e. one defined over $\mathbb{Q}$, described by equations with rational coefficients). However, an appropriate generalization of a conjecture of Lang would imply that in genus at least 24, the set of such curves has measure 0. (This follows from three facts: (i) rationally definable curves correspond to $\mathbb{Q}$-points of $\mathcal{M}_g$; (ii) $\mathcal{M}_{g,0}$ is "unirational" for $g \leq 10$; and (iii) $\mathcal{M}_{g,0}$ is of "general type" for $g \geq 24$, a celebrated result of Harris, Mumford, and Eisenbud.)

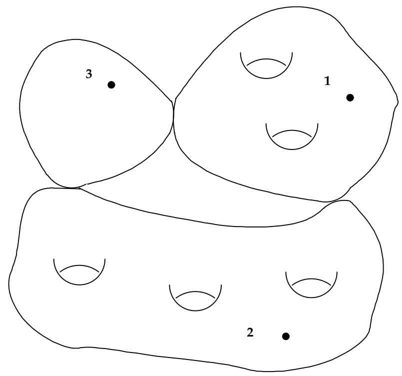

FIGURE 2. A 3-pointed genus 6 curve with three nodes

Another fact suggesting that the topology of this space has some elegant structure is Harer and Zagier's result that the (orbifold) Euler characteristic of $\mathcal{M}_{g,n}$ is given by:

$$\chi(\mathcal{M}_{g,n}) = (-1)^n \frac{(2g+n-3)!B_{2g}}{2g(2g-2)!} \tag{3}$$

if $g > 0$, where B_{2g} denotes the Bernoulli number. (If $g = 0$, $\chi(\mathcal{M}_{g,n}) = (-1)^{n+1}(2g+n-3)!$.)

Deligne and Mumford's compactification $\overline{\mathcal{M}}_{g,n}$ of $\mathcal{M}_{g,n}$.

Deligne and Mumford [DM] introduced a compactification $\overline{\mathcal{M}}_{g,n}$ of the moduli space $\mathcal{M}_{g,n}$. The additional points of the moduli space correspond to *nodal curves*. The points of a nodal curve are either smooth, or analytically isomorphic to a neighborhood of the origin of $xy = 0$ in $\mathbb{C}^2$. The curve is still required to be connected and compact. An n-pointed nodal curve is a nodal curve with n distinct smooth points labeled p_1 through p_n.

Informally, the *arithmetic genus* of a curve with nodes is the genus of its smoothing. For example, a smoothing of Figure 2 is Figure 1; hence Figure 2 has arithmetic genus 6. (Warning: the picture of the node is misleading, due to the difficulties of depicting a node in real two- or three-dimensional space. In fact, at each node the two branches are glued together transversely, and are not tangent.)

The *geometric genus* of a *component* of a nodal curve is found by "ungluing" the nodes, and finding the genus of the corresponding component. For example, the components of the curve of Figure 2 have genus 0, 2, and 3.

To each nodal curve with labeled points, we associate a *dual graph*. Vertices correspond to components; they are labeled with their geometric genus. For each node, we draw an edge joining the appropriate vertices. For each marked point, we draw a "half-edge" incident to the appropriate vertex, with the same label as the point. For example, the

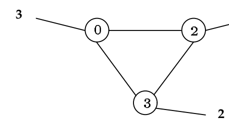

FIGURE 3. The dual graph corresponding to Figure 2

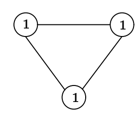

FIGURE 4. The stratum corresponding to this graph is isomorphic to $(\mathcal{M}_{1,2} \times \mathcal{M}_{1,2} \times \mathcal{M}_{1,2})/S_3$

dual graph corresponding to Figure 2 is shown in Figure 3. (Combinatorial exercise: if a nodal curve C has k components of geometric genera $g_1, \dots, g_k$, and δ nodes, then the arithmetic genus of C is $\sum_{i=1}^{k}(g_i - 1) + 1 + \delta$.)

A nodal curve is said to be *stable* if it has finite automorphism group. This is equivalent to a combinatorial condition: (i) each genus 0 vertex of the dual graph has valence at least three, and (ii) each genus 1 vertex has valence at least one. For example, the curve of Figure 2 is stable.

Now let $\overline{\mathcal{M}}_{g,n}$ be the moduli space of n-pointed genus g stable curves. (The reader may verify that, as we have assumed $(g,n) \neq (1,0)$, one need only check condition (i).)

Fact 2. The moduli space of stable curves $\overline{\mathcal{M}}_{g,n}$ is a smooth, compact, irreducible space (Deligne-Mumford stack), of dimension $3g - 3 + n$.

Moreover, the moduli space has a natural stratification.

Fact 3. $\overline{\mathcal{M}}_{g,n}$ is stratified by topological type, or equivalently, by dual graph.

In other words, to each stable dual graph there corresponds a stratum. For example, the stratum given by Figure 3 consists of (points corresponding to) all curves looking like Figure 2. To give such a curve is the same as giving a 3-pointed genus 0 curve, a 3-pointed genus 2 curve, and a 3-pointed genus 3 curve, and gluing them together as shown in Figure 2. Thus this stratum is isomorphic to $\mathcal{M}_{0,3} \times \mathcal{M}_{2,3} \times \mathcal{M}_{3,3}$. In the same way, one may show that each stratum is isomorphic to a product of moduli spaces of smooth pointed curves, modulo a finite group. (For an example with a nontrivial finite group: the stratum corresponding to the dual graph of Figure 4 is isomorphic to the quotient of $\mathcal{M}_{1,2} \times \mathcal{M}_{1,2} \times \mathcal{M}_{1,2}$ by the symmetric group S_3.) In particular, the next fact follows (after mild combinatorics) from the first three.

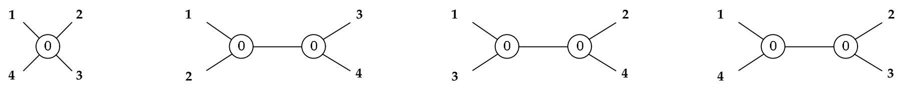

FIGURE 5. The strata of $\overline{\mathcal{M}}_{0,4}$

Fact 4. The codimension of a stratum is the number of edges of the corresponding dual graph. $\overline{\mathcal{M}}_{g,n}$ contains $\mathcal{M}_{g,n}$ as a dense open stratum. (The other strata are called *boundary strata*.) The 0-dimensional strata, or 0-strata, correspond to dual graphs where all vertices are trivalent of genus 0.

Examples: $\overline{\mathcal{M}}_{0,3}$, $\overline{\mathcal{M}}_{0,4}$, and $\mathcal{M}_{1,1}$.

It is a straightforward fact that any three points on $\mathbb{P}^1$ may be sent to $0, 1, \infty$ by a unique automorphism of $\mathbb{P}^1$. Hence $\mathcal{M}_{0,3}$ consists of a single point. One may quickly check that the only stable 3-pointed genus 0 graph is the obvious one (with one genus 0 vertex), so $\overline{\mathcal{M}}_{0,3}$ is a point as well.

If p_1, p_2, p_3, p_4 are distinct points on $\mathbb{P}^1$, then the automorphism of $\mathbb{P}^1$ of the previous paragraph (sending the first three points to 0, 1, and ∞ respectively) will send p_4 to some number $\lambda \neq 0, 1, \infty$. This map is the cross-ratio map:

$$\lambda = \frac{(p_4 - p_1)(p_2 - p_3)}{(p_4 - p_3)(p_2 - p_1)}.$$

Hence $\mathcal{M}_{0,4} \cong \mathbb{P}^1 - \{0, 1, \infty\}$. (In its modern guise of the map to $\mathcal{M}_{0,4}$, the lowly cross-ratio remains an important tool in geometry.) The compactified space $\overline{\mathcal{M}}_{0,4}$ has three boundary strata, the last three in Figure 5. In fact $\overline{\mathcal{M}}_{0,4} \cong \mathbb{P}^1$; the reader should try to associate the three boundary strata with the three degenerate values (0, 1, ∞) of λ . The three boundary strata are trivially rationally equivalent, as they are points on $\mathbb{P}^1$. This relation goes by many names; we will refer to it as the cross-ratio relation.

The reader may check that no stable n-pointed genus 0 curve has a non-trivial automorphism. Thus $\overline{\mathcal{M}}_{0,n}$ is a manifold (and indeed an algebraic variety).

The space $\mathcal{M}_{1,1}$ parameterizes genus 1 curves with a marked point, i.e. elliptic curves. To each elliptic curve is associated a j-invariant (a complex number), and to each complex number there is an associated elliptic curve with that j-invariant, so $\mathcal{M}_{1,1}$ looks a lot like $\mathbb{C}$. However, $\mathcal{M}_{1,1}$ is not a manifold (as every elliptic curve has an involution, so $\mathcal{M}_{1,1}$ has non-trivial stack structure). In a concrete sense, $\mathcal{M}_{1,1}$, not $\mathbb{C}$, is the right moduli space for elliptic curves. For example, modular forms are sections of a line bundle on the space $\overline{\mathcal{M}}_{1,1}$.

Natural morphisms between moduli spaces of curves.

There are natural morphisms between moduli spaces of curves. For example, given an n-pointed genus g curve ($n > 0$, $(g,n) \neq (0,3),(1,1)$), one may forget point p_n to obtain an $(n-1)$-pointed genus g curve. Hence there is a "forgetful morphism":

$$\text{(4)} \qquad \textit{Forgetful morphism:} \qquad \overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_{g,n-1}.$$

(Caution: when a point is forgotten, the resulting curve may not be stable. However, there is a natural way to "stabilize" an unstable curve. Exercise: if point p_3 is forgotten in Figure 2, what should be the resulting stable 2-pointed genus 6 curve?)

Also, given an (n_1+1)-pointed genus g_1 curve C_1 and an (n_2+1)-pointed genus g_2 curve C_2, one may glue one of the marked points of C_1 to one of the marked points of C_2 to obtain a new curve C of genus g_1+g_2 with n_1+n_2 marked points (two of the previously labeled points have disappeared). Similarly, given an $n+2$-pointed genus g curve C, one may glue two of the marked points of C together, obtaining a new curve of genus $g+1$ with n marked points. Hence we have:

$$\text{(5)} \quad \textit{Gluing morphisms:} \qquad \overline{\mathcal{M}}_{g_1,n_1+1} \times \overline{\mathcal{M}}_{g_2,n_2+1} \to \overline{\mathcal{M}}_{g_1+g_2,n_1+n_2}, \quad \overline{\mathcal{M}}_{g,n+2} \to \overline{\mathcal{M}}_{g+1,n}.$$

The gluing morphisms came up implicitly in our earlier discussion of boundary strata as products of smaller moduli spaces of curves.

Call the forgetful and gluing morphisms (4), (5) the *natural morphisms* between moduli spaces of curves.

The tautological ring.

In algebraic geometry, rather than considering the cohomology ring, we consider the algebraic version of it, the Chow ring $A^*(\overline{\mathcal{M}}_{g,n})$. The Chow groups are generated by algebraic cycles. Relations in Chow groups are generated by the atomic relation that two points in $\mathbb{P}^1$ are "homologous" (the analogous terminology in Chow is *rationally equivalent*; we reserve the word "homologous" for cohomology), and the condition that relations pull back and push forward under appropriate morphisms. As both the generators and relations of the Chow ring are a subset of those generating cohomology, Chow is neither stronger nor weaker than simplicial cohomology. For example, as the generators of Chow groups are algebraic and hence of even real dimension, Chow groups do not see odd-graded cohomology. On the other hand, two distinct points on a smooth curve of positive genus are not equivalent in Chow, while they are in homology. The reader may think instead about cohomology, although some information will be lost, as will be explained soon. There is a natural map $A^k \to H^{2k}$. (The index for Chow corresponds to algebraic (= complex) codimension, hence its doubling in the map to cohomology.) Because we are considering Deligne-Mumford stacks or orbifolds, we take rational rather than integral coefficients.

In some sense, it appears that the Chow ring $A^*(\overline{\mathcal{M}}_{g,n})$ is not the right object to study. Instead, attention has focused on a certain "tautological" subring, denoted by $R^*(\overline{\mathcal{M}}_{g,n}) \subset A^*(\overline{\mathcal{M}}_{g,n})$, for three admittedly vague reasons: (i) many interesting questions in geometry boil down to the tautological subring; (ii) no natural questions seem to reduce to questions in the non-tautological part of the Chow ring; and (iii) the tautological subring has, at least conjecturally, a great deal of structure. For example, $A^*(\overline{\mathcal{M}}_{g,n})$ is probably usually huge except in special cases (see the discussion of $\overline{\mathcal{M}}_{1,11}$ below). $R^*(\overline{\mathcal{M}}_{g,n})$ is much smaller.

However, in complex codimension 1, Arbarello and Cornalba showed using results of Harer that the tautological ring is equal to the cohomology ring.

As a first informal definition, $R^*(\overline{\mathcal{M}}_{g,n})$ consists of *all the classes naturally coming from geometry*. By this "definition", any geometric calculation that can be translated to the Chow ring will require only knowledge of the tautological ring. (For an example of how straightforward facts in the tautological ring can lead to solutions of longstanding classical problems, see [V].) An easy example of a class coming from geometry is the class of a boundary stratum.

A second example of classes are Chern classes of certain natural vector bundles on the moduli space. For example, there are n natural line bundles on $\overline{\mathcal{M}}_{g,n}$, denoted by $\mathbb{L}_1, \ldots, \mathbb{L}_n$. The fiber of $\mathbb{L}_i$ at the point $[C, p_1, \ldots, p_n] \in \overline{\mathcal{M}}_{g,n}$ is the (one-complex-dimensional) cotangent space to C at p_i. Define $\psi_i = c_1(\mathbb{L}_i)$; these are called the *$\psi$-classes*, and will be central players in the rest of this exposition.

Another natural vector bundle on $\mathcal{M}_{g,n}$ is the *Hodge bundle*: to a smooth curve C one associates the g-dimensional vector space of differentials. This definition can be extended over the boundary, giving a rank g vector bundle $\mathbb{E}$ over $\overline{\mathcal{M}}_{g,n}$. Define $\lambda_k = c_k(\mathbb{E})$ $(1 \leq k \leq g)$. These *λ-classes* will come up again later.

There are many other natural classes, including the Miller-Morita-Mumford κ-classes, $\kappa_1, \kappa_2, \ldots$ (where κ_i is codimension i). For brevity, we omit their definition, although they will arise again in Faber's conjecture. Mumford's original definition of the tautological ring of $\mathcal{M}_g$ was in terms of κ-classes. The κ-classes were also independently defined by Morita and Miller.

The generalization of the tautological ring to $\overline{\mathcal{M}}_{g,n}$ is due to Hain and Looijenga. The following description is due to Faber and Pandharipande.

Definition. The system of tautological rings $\left(R^*(\overline{\mathcal{M}}_{g,n}) \subset A^*(\overline{\mathcal{M}}_{g,n})\right)_{g,n}$ is the smallest system of $\mathbb{Q}$-algebras satisfying:

(i) $\psi_1, \ldots, \psi_n \in R^*(\overline{\mathcal{M}}_{g,n})$, and
(ii) the system is closed under pushforwards by natural morphisms ((4) and (5)).

For example, the boundary strata are obtained by pushing forward the fundamental class by gluing morphisms (5). All other "natural" geometric classes are in this ring, including λ-classes, κ-classes, and pullbacks of tautological classes under natural morphisms.

Definition. Define the tautological ring on any open subset of $\overline{\mathcal{M}}_{g,n}$ by its restriction from $\overline{\mathcal{M}}_{g,n}$.

The striking (often conjectural) combinatorial structure of the tautological ring comes in two related flavors:

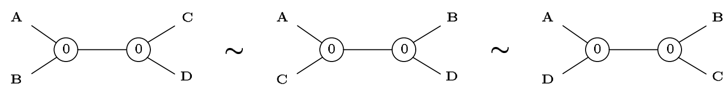

FIGURE 6. The cross-ratio equivalences of boundary strata

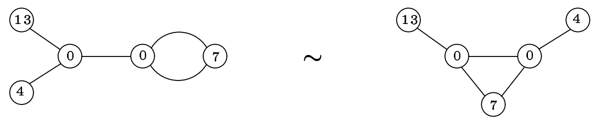

FIGURE 7. An example of a cross-ratio equivalence (in $\overline{\mathcal{M}}_{25}$)

- structure related to boundary strata (and hence graphs); and
- structure of top intersections in the tautological ring, often called *Hodge integrals*; especially top intersections of ψ-classes.

Before describing what is known about these two structures in arbitrary genus, we will first recapitulate what is known in low genus. In genus 0, the structure is exceptionally elegant and combinatorial. In higher genus, odder behavior starts to happen; examples often involve $\overline{\mathcal{M}}_{1,11}$.

Genus zero.

The gluing morphisms (5) combined with the cross-ratio relation give equivalences among boundary strata (in cohomology), depicted in Figure 6. In the Figure, A, B, C, and D are taken to be graphs with one "root". We will also refer to these as *cross-ratio relations*. The equivalence shown in Figure 7 is an example of a cross-ratio relation.

Keel has proved that in genus 0 (for any number of points), the cohomology and Chow groups are generated by boundary strata, and that the relations are generated by the cross-ratio relations. In particular, $H^* = A^{*/2} = R^{*/2}$. (In fact, the cohomology ring is generated as an algebra by complex codimension 1 strata.) Multiplication of boundary strata in cohomology can be described combinatorially as well, so the cohomology ring of $\overline{\mathcal{M}}_{0,n}$ has a beautiful combinatorial structure.

A different aspect of the combinatorial structure of the cohomology/Chow/tautological ring of $\overline{\mathcal{M}}_{0,n}$ is visible in terms of ψ-classes:

$$\int_{\overline{\mathcal{M}}_{0,n}} \psi_1^{a_1} \cdots \psi_n^{a_n} = \binom{n-3}{a_1, \ldots, a_n}.$$

We are using ordinary multiplication to represent cup product. Define $\int_X y$ (where $y \in H^{2i}(X)$ or $A^i(X)$) to be the degree of y if $i = \dim X$, and to be 0 otherwise. As $\dim \overline{\mathcal{M}}_{0,n} = n-3$, $\sum_i a_i$ must be $n-3$ in order to produce a 0-cycle.

This result is even more beautiful in the language of generating functions. Let

$$E_{g,n} = \sum_{a_1,\dots,a_n} \left(\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{a_1} \cdots \psi_n^{a_n} \right) t_1^{a_1} \cdots t_n^{a_n} \tag{6}$$

where $t_1, \dots, t_n$ are formal variables. This is a homogeneous polynomial of degree $\dim \overline{\mathcal{M}}_{g,n} = 3g-3+n$. Then

$$E_{0,n} = (t_1 + \cdots + t_n)^{n-3}. \tag{7}$$

Genus one: The strange case of $\overline{\mathcal{M}}_{1,11}$.

Before we discuss how well genus 1 is understood, we begin with a pathology: $\overline{\mathcal{M}}_{1,11}$. In genus 0, the map $A^{*/2} \to H^*$ is an isomorphism. One blunt reason for $A^{*/2}(X) \to H^*(X)$ *not* to be an isomorphism for a complex projective manifold X is if X has odd cohomology: the algebraic classes are generated by algebraic (hence complex analytic) cycles, and hence are even-dimensional. (But the Euler characteristic of the moduli space of curves may be negative by (3), odd cohomology definitely arises.) There is a more refined obstruction as well: Hodge theory dictates that there is a sub-vector space of the even-dimensional cohomology where the algebraic cohomology must lie; we will call this the cohomology of *Tate type*.

The 22-real-dimensional space $\overline{\mathcal{M}}_{1,11}$ has odd cohomology, in its middle dimension. The existence of this cohomology is a consequence of the existence of a non-zero 11-form on this 11-complex-dimensional orbifold. (This uses the existence of the famous weight 12 modular form Δ.) Hence the natural map from the Chow ring of $\overline{\mathcal{M}}_{1,n}$ to its cohomology ring $H^*(\overline{\mathcal{M}}_{1,n})$ cannot be a surjection for $n = 11$, and indeed for $n \geq 11$.

Also, the existence of this non-zero 11-form implies that $A_0(\overline{\mathcal{M}}_{1,n})$ is huge (for $n \geq 11$) — for example, it is uncountably generated. (A more precise statement and proof of this folklore fact appear in [GrV1].) As $R_0(\overline{\mathcal{M}}_{1,n})$ (i.e. $R^{\dim \overline{\mathcal{M}}_{1,n}}(\overline{\mathcal{M}}_{1,n})$) is countably generated (indeed one-dimensional, see below), we see that $R^0(\overline{\mathcal{M}}_{1,n}) \to A^0(\overline{\mathcal{M}}_{1,n})$ cannot be an isomorphism. On the other hand, for $n \leq 10$, the maps $R^{*/2}(\overline{\mathcal{M}}_{1,n}) \to A^{*/2}(\overline{\mathcal{M}}_{1,n}) \to H^*(\overline{\mathcal{M}}_{1,n})$ are isomorphisms, by work of Belorousski.

Hence the cohomology ring has more information than the Chow ring (for example, it has odd cohomology), and the Chow ring has more information than the cohomology ring (for example, it can better distinguish 0-cycles). In genus 1, the tautological ring is well-understood, and it is "better-behaved" in that it sees neither of these two extra types of information, as we shall soon see.

As in genus 0, the tautological groups are generated by boundary strata. (This follows from the definition of the tautological ring using ψ-classes, and the fact that on $\overline{\mathcal{M}}_{1,n}$, ψ_i

is rationally equivalent to a sum of boundary strata.) Intersections between boundary strata are quite straightforward. There is a new relation in codimension 2 on $\overline{\mathcal{M}}_{1,4}$ due to Getzler; the relation is complicated, and omitted here. It is perhaps surprising that this relation was not discovered earlier, given the fact that $\overline{\mathcal{M}}_{1,4}$ is a compact complex fourfold, parameterizing simple objects of long-standing interest (essentially, four points on an elliptic curve). Furthermore, Getzler has announced that his relation and the cross-ratio relation generate *all* relations on $\overline{\mathcal{M}}_{1,n}$, and that the map from the tautological ring to the cohomology ring is an injection. (More precisely, the map $R^{*/2} \to H^*$ gives an isomorphism of the tautological ring with the even cohomology. As a consequence, for example, any even-dimensional class is homologous to a tautological class.)

There is an analog to (7), involving the logarithm of a certain generating function, divided by 24.

Genus greater than one.

Starting in genus 2, the boundary strata do not generate the tautological ring: ψ_1 on the threefold $\overline{\mathcal{M}}_{2,1}$ is not homologous to boundary strata. (This is in some sense the only example in genus 2; all others are derived from this via the gluing morphism.) The Chow ring of $\overline{\mathcal{M}}_{2,n}$ is known for small n, but many open questions remain. Belorousski and Pandharipande have proved a new relation in codimension 2 on $\overline{\mathcal{M}}_{2,3}$. Polito showed in her thesis that there are no additional codimension 2 relations for $\overline{\mathcal{M}}_{g,n}$, but surely there are more to be found in higher codimension. An explicit algebraic cycle, not homologous to a tautological class, defined over $\mathbb{Q}$, was found on $\overline{\mathcal{M}}_{2,22}$ by Graber and Pandharipande; it is the first in a family of examples, and its construction uses $\overline{\mathcal{M}}_{1,11}$.

In genus greater than 2, the full Chow ring is well-understood in a few cases, including: $A^*(\overline{\mathcal{M}}_3)$ (work of Faber; it contains only tautological elements), $A^*(\mathcal{M}_4) \cong \mathbb{Q}[\lambda_1]/(\lambda_1^3)$ (Faber again), and $A^*(\mathcal{M}_5) \cong \mathbb{Q}[\lambda_1]/(\lambda_1^4)$ (work of Izadi combined with work of Faber). Pikaart has shown that for large g (for given n), the cohomology of $\overline{\mathcal{M}}_{g,n}$ is not all of Tate type and hence not all algebraic (let alone tautological); his argument involves $\overline{\mathcal{M}}_{1,11}$. There is also a higher-genus version of (7), given by a "genus-expansion" ansatz of the physicists Itzykson and Zuber.

The tautological ring in arbitrary genus: top intersections of ψ-classes.

The reason for interest in top intersections of ψ-classes is that knowing these values allows us to find the top intersections of any classes in the tautological ring, by appropriately unwinding the definition of the tautological ring. (Based on his earlier work, Faber has written a remarkable MAPLE program that will explicitly perform intersections of many different types of classes in the tautological ring [F2].) Furthermore, these top intersections of ψ-classes have a striking structure which allows them to be computed, using Witten's conjecture or Hurwitz numbers.

Witten's conjecture (proved by Kontsevich, and later by Okounkov and Pandharipande) was the catalyst for much of the interest in the tautological ring in recent years. (See [L] for a detailed exposition.) Define F_g by

$$F_g := \sum_{n \geq 0} \frac{1}{n!} \sum_{k_1, \dots, k_n} \left(\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{k_1} \cdots \psi_n^{k_n} \right) t_{k_1} \cdots t_{k_n},$$

the generating function for all top intersections of ψ-classes in genus g. (Compare this to (6).) Define a generating function for all such intersections in *all* genus by

$$F := \sum_g F_g \lambda^{2g-2}.$$

This is *Witten's free energy* or the *Gromov-Witten potential of a point*. For convenience, we will set $\lambda = 1$.

Witten's conjecture (Kontsevich's Theorem) gives a recursion for top intersections of ψ-classes in the form of a partial differential equation satisfied by F. Using this differential equation along with a straightforward geometric fact known as the *string equation* and the trivial fact (the "initial condition") $\int_{\overline{\mathcal{M}}_{0,3}} 1 = 1$ (as $\overline{\mathcal{M}}_{0,3}$ is a point), all top intersections are quickly recursively determined.

Witten's Conjecture (Kontsevich's Theorem).

$$(2n+1)\frac{\partial^3}{\partial t_n \partial t_0^2} F = \left(\frac{\partial^2}{\partial t_{n-1} \partial t_0} F \right) \left(\frac{\partial^3}{\partial t_0^3} F \right) + 2 \left(\frac{\partial^3}{\partial t_{n-1} \partial t_0^2} F \right) \left(\frac{\partial^2}{\partial t_0^2} F \right) + \frac{1}{4} \frac{\partial^5}{\partial t_{n-1} \partial t_0^4} F.$$

Kontsevich's proof of Witten's conjecture is by combinatorializing these top intersections, expressing them in terms of sums over "ribbon-graphs". Okounkov and Pandharipande's more recent proof is by way of Hurwitz numbers, which is a second way of combinatorializing top intersections of ψ-classes, and is described below.

Witten's conjecture indicates a deep connection between moduli spaces such as these, and integrable systems. This link is still mysterious. Another link to integrable systems is via the Virasoro Lie algebra. Dijkgraaf, Verlinde, and Verlinde showed that Witten's conjecture (along with the string equation) implies that e^F is annihilated by a sequence of differential operators L_{-1}, L_0, L_1, ... , corresponding to part of the Virasoro algebra. The first differential equation $L_{-1}e^F = 0$ is the string equation, and the second is geometrically obvious, but the geometric meaning of the rest is obscure. One of the fundamental open questions in Gromov-Witten theory is a generalization of this formulation of Witten's conjecture, and deals with maps of curves to an arbitrary (smooth projective) target manifold: the *Virasoro conjecture* (due to the physicists Eguchi, Hori, and Xiong, as well as S. Katz). This question has recently been settled in two important open cases: that of projective spaces (by Givental) and that of smooth curves (by Okounkov and Pandharipande).

Hurwitz numbers.

Another combinatorial description of top intersections of ψ-classes, which has been alluded to above, is via *Hurwitz numbers*. Hurwitz numbers count, essentially, ordered factorizations of a permutation into transpositions, and these factorizations may be regarded as combinatorial objects. More precisely, fix a positive integer d, a non-negative integer r, and a conjugacy class of S_d given by a partition α (i.e. $\alpha_1 + \cdots + \alpha_n = d$). Define g by $2 - 2g = d + n - r$. Then define

$$\tilde{H}^g_\alpha = \# \{\sigma_1, \ldots, \sigma_r \in \operatorname{tran}(S_d) : \sigma_1 \cdots \sigma_r \in C(\alpha)\} /d! \tag{8}$$

where $\operatorname{tran}(S_d)$ is the set of transpositions in S_d (elements of the form (ij)), and $C(\alpha) \subset S_d$ is the conjugacy class defined by α.

This number can be interpreted as the answer to a problem in enumerative geometry, as follows. Fix $r + 1$ distinct points $\{p_1, \ldots, p_r, \infty\}$ on the sphere, interpreted as the complex projective line $\mathbb{CP}^1$, and count the number of degree d covers of the sphere, unbranched away from the $r + 1$ points, with the simplest possible branching above each of the first r points (i.e. two of the sheets come together, analytically the projection of the parabola $y^2 = x$ to the x-axis in $\mathbb{C}^2$; the remaining $d-2$ sheets are unbranched), and with branching above ∞ given by α. Then the cover also has a complex structure by the Riemann existence theorem, and hence is a Riemann surface, although possibly with many components.

Define the Hurwitz number H^g_α as the number of such covers where the cover has *one* component. In this case, the covering curve has genus g. The analog in (8) is to require that the σ_i generate S_d. Equivalently, if one constructs a graph with vertices labeled 1 through d, and draws an edge between (jk) if $\sigma_i = (jk)$, then this "monodromy graph" must be connected. The simplest way to describe the relationship between the numbers H^g_α and $\tilde{H}^g_\alpha$ is to construct generating functions H and $\tilde{H}$ for these numbers in a straightforward way; then $\tilde{H} = e^H$.

Hurwitz numbers were first studied by Hurwitz. In recent times, they have been studied by combinatorialists, representation theorists, physicists, the Arnol'd school of geometers, symplectic and algebraic geometers interested in Gromov-Witten theory, and more. For a summary of this rich history, see [ELSV, GJV] .

Hurwitz numbers are obviously easy to compute. If you play with them for a while, you may soon come to the following conjecture:

$$H^g_\alpha = \frac{r!}{\# \operatorname{Aut}(\alpha)} \prod_{i=1}^n \frac{\alpha_i^{\alpha_i}}{\alpha_i!} P(\alpha_1, \ldots, \alpha_n) \tag{9}$$

where P is some mysterious symmetric polynomial of total degree between $2g-3+n$ and $3g - 3 + n$. This conjecture is a baby form of a much more sophisticated polynomiality conjecture of the combinatorialists Goulden and Jackson, which can be interpreted as a "structure theorem" for Hurwitz numbers. The number $3g-3+n$ is extremely suggestive: it is the dimension of $\overline{\mathcal{M}}_{g,n}$, and there is indeed a genus g curve with n marked points in the picture: the cover has genus g, and has n pre-images over ∞.

Equation (9) is explained by the following truly remarkable formula:

Theorem (The "ELSV formula", Ekedahl, Lando, Shapiro, and Vainshtein).

$$P(\alpha_1, \ldots, \alpha_n) = \int_{\overline{\mathcal{M}}_{g,n}} \frac{1 - \lambda_1 + \cdots + (-1)^g \lambda_g}{(1 - \alpha_1 \psi_1) \cdots (1 - \alpha_n \psi_n)}. \tag{10}$$

This equation should be interpreted as follows: formally invert the denominator, expand everything out, and keep only the terms of codimension $\dim \overline{\mathcal{M}}_{g,n} = 3g - 3 + n$. Hence for fixed g and n,

$$P(\alpha_1, \ldots, \alpha_n) = \sum_{\substack{k_1 + \cdots + k_n + j = 3g-3+n \\ 0 \leq j \leq g}} \left((-1)^g \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{k_1} \psi_2^{k_2} \cdots \psi_n^{k_n} \lambda_j \right) \alpha_1^{k_1} \alpha_2^{k_2} \cdots \alpha_n^{k_n},$$

immediately identifying the mysterious polynomial P. Note in particular that the highest-degree portion of P is a generating function for top intersections of ψ-classes (and is, upon replacement of α_i by t_i, equ. (6)). This gives a straightforward way of computing these abstractly-defined numbers: it is not hard to recover the coefficient of a polynomial P of known degrees by computing enough values, and Hurwitz numbers are easy to compute. Hence, remarkably, the combinatorics of the symmetric group lead inexorably to the tautological ring of the moduli space of curves!

There are many other consequences; see [ELSV, GJV] for surveys. For example, (i) Goulden and Jackson's full polynomiality conjecture can be proved, (ii) conjectured recursions for Hurwitz numbers can be proved, and new ones found, (iii) after a non-trivial change of variables, the generating function H becomes (a mild generalization of) Witten's free energy of a point, and (iv) this combinatorial description of intersections of ψ-classes leads to Okounkov and Pandharipande's proof of Witten's conjecture via asymptotic combinatorics.

We remark in passing that the ELSV formula is an equality not just of numbers, but in Chow (in $R_0(\overline{\mathcal{M}}_{g,n}) := R^{3g-3+n}(\overline{\mathcal{M}}_{g,n})$). Interpreted properly, this yields:

Theorem [GrV1]. $R_0(\overline{\mathcal{M}}_{g,n}) \cong \mathbb{Q}$.

In other words, any two tautological 0-cycles on $\overline{\mathcal{M}}_{g,n}$ are commensurate. This is trivial in cohomology, but far from obvious in the Chow ring (where A_0, as described earlier, can be huge). Also, Witten's conjecture can in some sense be interpreted as a recursion not just of numbers, but in Chow.

The ELSV formula can be proved in Gromov-Witten theory; this interpretation leads to connections with the second, related, source of combinatorial structure in the tautological ring, the boundary strata.

The tautological ring in arbitrary genus: cycles of arbitrary dimension.

In some sense we have a handle on 0-cycles. By Faber's algorithms, we also know how to intersect any two tautological cycles. In cohomology, this would be enough for us to determine the full structure of the ring, thanks to Poincaré duality. It has been conjectured that "Poincaré duality" also holds for $\overline{\mathcal{M}}_{g,n}$:

Poincaré duality conjecture (or speculation) for stable curves (Hain-Looijenga). $R^*(\overline{\mathcal{M}}_{g,n})$ satisfies Poincaré duality.

In other words, if $d = \dim \overline{\mathcal{M}}_{g,n} = 3g - 3 + n$:

(I) $R^i(\overline{\mathcal{M}}_{g,n}) = 0$ for $i > d$, which is obvious for dimensional reasons;
(II) $R^d(\overline{\mathcal{M}}_{g,n}) \cong \mathbb{Q}$;
(III) the natural pairing $R^i(\overline{\mathcal{M}}_{g,n}) \times R^{d-i}(\overline{\mathcal{M}}_{g,n}) \to R^d(\overline{\mathcal{M}}_{g,n}) \cong \mathbb{Q}$ is perfect.

For convenience, we will call such a ring a *dimension d Poincaré duality ring*, and (I) and (II) the *socle* conditions. (This terminology is from the theory of Gorenstein rings.)

There is no *a priori* reason to expect the tautological ring to satisfy Poincaré duality. (For example, the Chow ring certainly doesn't, as the hugeness of $A_0(\overline{\mathcal{M}}_{1,11})$ shows.) At this point, there is a great deal of low-genus evidence for this conjecture; however, the only genus-free evidence to date is (II), described earlier.

This conjecture was motivated by an earlier conjecture by Faber.

Faber's conjecture (for smooth curves) [F1, Conj. 1]**.**

(a) $R^*(\mathcal{M}_g)$ is a dimension $g-2$ Poincaré duality ring.
(b) The $[g/3]$ classes $\kappa_1, \dots, \kappa_{[g/3]}$ generate the ring, with no relations in degrees $[g/3]$.
(c) There exist explicit formulas for the proportionalities in degree $g-2$, which may be given as follows.

$$\psi_1^{d_1+1}\psi_2^{d_2+1}\cdots\psi_k^{d_k+1} = \frac{(2g-3+k)!(2g-1)!!}{(2g-1)!\prod_{j=1}^k(2d_j+1)!!}\kappa_{g-2} = \sum_{\sigma\in S_k}\kappa_\sigma.$$

Here $(2a-1)!!$ is a shorthand for $1\cdot 3\cdot\dots\cdot(2a-1)$; $\kappa_\sigma = \kappa_{|\alpha_1|}\kappa_{|\alpha_2|}\cdots\kappa_{|\alpha_{\nu(\sigma)}|}$ for a decomposition $\sigma = \alpha_1\alpha_2\cdots\alpha_{\nu(\sigma)}$ of the permutation σ into disjoint cycles; and $|\alpha|$ is the sum of the elements in the cycle α, where S_k acts on the k-tuples with entries $d_1, \dots, d_k$. (Faber conjectured furthermore that $R^*(\mathcal{M}_g)$ "behaves like" the algebraic cohomology ring of a smooth projective manifold of dimension $g-2$; the additional properties required will not be discussed here.) Faber verified the conjecture for genus up to 15 — a highly nontrivial calculation on difficult-to-handle spaces. Part (c) in particular highlights the combinatorial structure of the tautological ring. Significant progress has been made on this conjecture, in the form of three theorems.

Theorem.

(a) (Looijenga) $R^i(\mathcal{M}_g) = 0$ for $i > g-2$, $R^{g-2}(\mathcal{M}_g) = \mathbb{Q}$. (The latter statement uses a calculation of Faber [F1, Thm. 2].)
(b) (Morita, Ionel) Part (b) of Faber's conjecture is true in cohomology. Ionel's proof should extend to the Chow ring without difficulty.
(c) (Getzler, Pandharipande, Givental) Part (c) of Faber's conjecture is true.

(A word of explanation of (c): Getzler and Pandharipande showed that the Virasoro conjecture for $\mathbb{P}^2$ implies this part of Faber's conjecture. As described earlier, Givental has since proved the Virasoro conjecture for all projective spaces. This proof for (c) is quite round-about and complicated; it would be good to have a better proof, and this seems within reach.)

Hence all that remains is the "perfect pairing" part of the conjecture; while there is some prospect for proving cases of this open problem (for example, the case $i = 1$ or possibly 2), there are no current hopes for approaching the general problem.

Faber and Pandharipande have proposed two other generalizations of Faber's conjecture, to curves with "rational tails" (curves whose dual graph contains a vertex of genus g), and curves of "compact type" (curves whose dual graph has no loops). For example, the curve of Figure 1 is of both types, but the curve of Figure 2 is neither.

Boundary strata and vanishing of tautological classes.

These conjectures are closely related to the combinatorial structure of the tautological ring involving boundary strata, by way of the following result.

Tautological vanishing theorem [GrV2]. Any tautological class of codimension i is trivial away from strata satisfying

$$\text{\# genus 0 vertices } \geq i - g + 1.$$

For example, the "vanishing" part of Looijenga's theorem (a) above (that $R^i(\mathcal{M}_g) = 0$ for $i > g - 2$) is a consequence as follows. If $i \geq g$, then any codimension i class vanishes on the open set corresponding to curves with no genus 0 vertices, and in paricular on $\mathcal{M}_g$. If $i = g - 1$, then the theorem does not directly apply. However, from the definition of the tautological ring, we see that any tautological class on $\mathcal{M}_g$ is pushed forward from $\mathcal{M}_{g,1}$; hence a codimension $g - 1$ class on $\mathcal{M}_g$ is pushed forward from a codimension g class on $\mathcal{M}_{g,1}$, which vanishes by the same argument. The reader may like to extend this argument to address the case of "curves with rational tails" described above (part of the "rational tail" conjecture of Faber and Pandharipande).

As another example, the reader may enjoy proving the analog of part (a) of Faber's conjecture for the moduli space of "compact type" curves $\mathcal{M}^c_{g,n} \subset \overline{\mathcal{M}}_{g,n}$ by showing the following sequence of facts. If $i > 2g - 3 + n$, there is no compact type (i.e. no-loop) stable dual graph of genus g with n points. If $i = 2g - 3 + n$, then there are: they are strata containing only trivalent genus 0 vertices, g genus 1 "leaves", and n "half-edges". These strata are of codimension precisely $2g - 3 + n$, so $R^{2g-3+n}(\mathcal{M}^c_{g,n})$ is generated by these strata. Any two of these strata are related by cross-ratio relations. It is a straightforward geometric fact that any one of these strata is non-trivial in cohomology (and hence the Chow ring), so we have shown that $R^i(\overline{\mathcal{M}}^c_{g,n})$ vanishes for $i > 2g - 3 + n$, and is one-dimensional for $i = 2g - 3 + n$.

The analogous theorem for stable curves, the fact $R_0(\overline{\mathcal{M}}_{g,n}) \cong \mathbb{Q}$ described earlier, follows similarly; there are many extensions along these lines.

We conclude with another remarkable consequence, which indeed partially motivated the vanishing theorem. As described earlier, in genus 1, ψ_i is rationally equivalent to a sum of boundary strata. In genus 2, Mumford and Getzler proved that on $\overline{\mathcal{M}}_{2,n}$, $\psi_i \psi_j$ is also supported on the boundary (i.e. is trivial on $\mathcal{M}_{2,n}$). Based on this, Getzler conjectured the following statement, which was recently proved by Ionel. (Ionel states the result in cohomology, but her proof should carry through in the Chow ring.)

Ionel's theorem (formerly Getzler's conjecture). If $g > 0$, all degree g monomials in ψ_1, ..., ψ_n vanish on $\mathcal{M}_{g,n}$.

Conclusion.

Since Kontsevich's proof of Witten's conjecture, there has been a great deal of progress in understanding the geometry and topology of the moduli space of curves, and in particular its tautological ring. The objects in question are quite classical, and the fundamental perspective behind the proofs of the results described above, that of understanding curves as covers of the projective line, is very much so. However, the recent progress is highly motivated by work in other fields (such as string theory, combinatorics, representation theory, topology, symplectic geometry, integrable systems and more), and some of the techniques are quite modern (such as those of Gromov-Witten theory). As new ideas continue to flow into algebraic geometry from these neighboring fields, there should be even more exciting developments in the near future.

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