PRODUCTS OF UNBOUNDED NORMAL OPERATORS

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ABSTRACT. The present paper partly constitutes an "unbounded" follow-up of a paper by I. Kaplansky dealing with bounded products of normal operators. Results on the normality of unbounded products are also included.

1. Introduction

In this paper we are concerned with the normality of the products AB and BA for two operators, where B is unbounded. The problems of this sort lie among the most fundamental questions in the Hilbert space theory and were explored by many authors, especially in the bounded case (see [7, 8, 19, 20]). See also the paper [3] and the references therein. Nonetheless, only a few shy attempts where made in the unbounded case. See the very recent papers [6] and [**?**].

We also cite the reference [11] where the following result (among others) has been obtained

Theorem 1.

(1) *Assume that B is a unitary operator. Let A be an unbounded normal operator. If B and A commute (i.e. $BA \subset AB$), then BA is normal.*
(2) *Assume that A is a unitary operator. Let B be an unbounded normal operator. If A and B commute (i.e. $AB \subset BA$), then BA is normal.*

For results involving normality and self-adjointness of unbounded operator products, see [9, 10, 11, 6]. For similar papers on the sum of two normal operators, see [12] and [14].

One purpose of this paper is to try to get an analog for unbounded operators of the following result

Theorem 2. *[Kaplansky, [7]] Let A and B be two bounded operators on a Hilbert space such that AB and A are normal. Then B commutes with AA^* iff BA is normal.*

In order to do that and to allow a broader audience to read the present paper, we recall basic definitions and results on unbounded operators. Some important references are [1, 4, 5, 17].

All operators are assumed to be densely defined (i.e. having a dense domain) together with any operation involving them or their adjoints. Bounded operators are assumed to be defined on the whole Hilbert space. If A and B are two unbounded operators with domains $D(A)$ and $D(B)$ respectively, then B is called an extension of A, and we write $A \subset B$, if $D(A) \subset D(B)$ and if A and B coincide on $D(A)$. We

2000 *Mathematics Subject Classification.* Primary 47A05. Secondary 47B15, 47B20.
Key words and phrases. Product of operators. Normal, hyponormal, subnormal operators. Fuglede-Putnam theorem.

write $A \subseteq B$ if $A \subset B$ or $A = B$ (meaning that $D(A) = D(B)$ and $Ax = Bx$ for all $x \in D(A)$).

If $A \subset B$, then $B^* \subset A^*$. An unbounded operator A is said to be closed if its graph is closed; self-adjoint if $A = A^*$ (hence from known facts self-adjoint operators are automatically closed); normal if it is closed and $AA^* = A^*A$ (this implies that $D(AA^*) = D(A^*A)$).

A densely defined operator A is said to be hyponormal if $D(A) \subset D(A^*)$ and $\|A^*x\| \leq \|Ax\|$ for $x \in D(A)$. We say that a densely defined operator S in a Hilbert space H is subnormal if there is another Hilbert space $L \supset H$ and a normal operator N in L such that $S \subset N$. It is wellknown that each subnormal operator is hyponormal and that each hyponormal operator is closable

The Fuglede-Putnam theorem (see [2] and [16]) is important to prove our results so we recall it here

Theorem 3. *Let A be a bounded operator. Let N and M be two unbounded normal operators. If $AN \subseteq MA$, then $AN^* \subseteq M^*A$.*

We digress a little bit to say that a new version of this famous theorem has been obtained by the author where all the operators involved are unbounded. See [13].

It is also convenient to recall the following theorem which appeared in [18], but we state it in the form we need.

Theorem 4. *If T is a closed subnormal (resp. closed hyponormal) operator and S is a closed hyponormal (resp. closed subnormal) operator verifying $XT^* \subset SX$ where X is a bounded operator, then both S and T^* are normal once $\ker X = \ker X^* = \{0\}$.*

2. Main Results

We start by giving a counterexample that shows that the same assumptions, as in Theorem 2, would not yield the same results if B is an unbounded operator, let alone the case where both operators are unbounded.

What we want is a normal bounded operator A and an unbounded (and closed) operator B such that BA is normal, $A^*AB \subset BA^*A$ but AB is not normal.

Example 1. *Let*

$$Bf(x) = e^{x^2} f(x) \text{ and } Af(x) = e^{-x^2} f(x)$$

on their respective domains

$$D(B) = \{f \in L^2(\mathbb{R}) : \ e^{x^2} f \in L^2(\mathbb{R})\} \text{ and } D(A) = L^2(\mathbb{R}).$$

Then A is bounded and self-adjoint (hence normal). B is self-adjoint (hence closed).

Now AB is not normal for it is not closed as $AB \subset I$. BA is normal as $BA = I$ (on $L^2(\mathbb{R})$). Hence $AB \subset BA$ which implies that

$$AAB \subset ABA \Longrightarrow AAB \subset ABA \subset BAA.$$

Now, we state and prove the generalization of Theorem 2. We have

Theorem 5. *Let B be an unbounded closed operator and A a bounded one such that AB (resp. BA) and A are normal. Then*

$$BA \text{ normal (resp. } AB) \Longrightarrow A^*AB \subset BA^*A.$$

Proof. Since AB and BA are normal, the equation

$$A(BA) = (AB)A$$

implies that

$$A(BA)^* = (AB)^*A$$

by the Fuglede-Putnam theorem. Hence

$$AA^*B^* \subset B^*A^*A \text{ or } A^*AB \subset BA^*A.$$

□

We already observed in Example 1 that the converse in the previous theorem does not hold. An extra hypothesis combined with a result by Stochel [18] yield the following

Theorem 6. *If B is an unbounded closed operator and and if A is a bounded one such that AB and A are normal, and if further BA is hyponormal (resp. subnormal), then*

$$BA \text{ normal} \Longleftarrow A^*AB \subset BA^*A.$$

Proof. The idea of proof is similar in core to Kaplansky's ([7]). Let $A = UR$ be the polar decomposition of A, where U is unitary and R is positive (remember that they also commute and that $R = \sqrt{A^*A}$), then one may write

$$U^*ABU = U^*URBU = RBU \subset BRU = BA$$

or

$$U^*AB = U^*\overline{AB} = U^*((AB)^*)^* \subset BAU^*$$

(by the closedness of AB). Since $(AB)^*$ is normal, it is closed and subnormal. Since B is closed and A is bounded, BA is closed. Since it is hyponormal, Theorem 4 applies and yields the normality of BA as U is invertible.

The proof is very much alike in the case of subnormality. □

Now, if we assume that A is unitary, then we have the following interesting result (cf Theorem 1) that bypasses the commutativity of operators. Besides, this constitutes generalization of Theorem 2 with the assumption A unitary.

Theorem 7. *If A is unitary and B is an unbounded normal operator, then*

$$BA \text{ is normal} \Longleftrightarrow AB \text{ is normal.}$$

Proof. First, recall that self-adjoint operators are maximally symmetric, that is, if T is self-adjoint and S is symmetric, then $T \subset S \Rightarrow T = S$ (we shall call this the MS-property). Second, if T is closed, then T^*T and TT^* are both self-adjoint (see [17] for both results).

Now, assume that BA is normal. To show that AB is normal observe that it is first closed (which is essential) thanks to the invertibility of A and the closedness of B. We then have

$$(AB)^*AB = B^*A^*AB = B^*B$$

and

$$AB(AB)^* = ABB^*A^*$$

and we must show the equality of the two quantities. We have

$$A^*B^*BA \subset (BA)^*BA \text{ and } BB^* \subset BA(BA)^*.$$

By the closedness of B and that of BA, and the MS-property

$$BB^* = BA(BA)^*.$$

By the normality of BA, we obtain

$$A^*B^*BA \subset BB^* \text{ or } A^*B^*B \subset BB^*A^*.$$

Hence

$$AB(AB)^* = ABB^*A^* \supset AA^*B^*B = B^*B = (AB)^*AB.$$

Therefore, and by the MS-property again, we must have

$$AB(AB)^* = (AB)^*AB.$$

To prove the converse, we may argue similarly with some minor changes. Suppose that AB is normal and let us show that BA is normal too.

Firstly, note that BA is closed, but this time, since B is closed and A is bounded.

Secondly, and by the normality of AB, we have

$$(AB)^*AB = B^*B = AB(AB)^* = ABB^*A^*$$

and hence

$$ABB^* = B^*BA.$$

Accordingly, we have

$$(BA)^*BA \supset A^*B^*BA = A^*ABB^* = BB^*$$

and

$$BA(BA)^* \supset BB^*.$$

By the MS-property, we must have

$$(BA)^*BA = BB^* \text{ and } BA(BA)^* = BB^*,$$

proving the normality of BA. The proof is complete.

□

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