Fractional diffusion limit for collisional kinetic equations: a Hilbert expansion approach

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Abstract

We develop a Hilbert expansion approach for the derivation of fractional diffusion equations from the linear Boltzmann equation with heavy tail equilibria.

1 Setting of the result

1.1 Introduction

The linear Boltzmann equation is a simple kinetic equation describing the evolution of a particle distribution function. It couples free transport and scattering phenomena (due to collisions with the background). The properties of the scattering process determines the long time behavior of the particle distribution function. It fixes in particular the profile of the equilibrium velocity distribution.

Asymptotic analysis of such an equation is a very classical problem. Typically, assuming that the mean free path (i.e. distance between two collisions) is very small and the time scale is very large, it is possible to derive a hydrodynamic type equation describing the evolution of the density of particles. Often, the thermodynamical equilibrium is given by a Maxwellian distribution function, and the mean square displacement of the particle is a linear function of time. The correct time/space scaling ($x = \varepsilon x'$ and $t = \varepsilon^2 t'$) then leads to a diffusion equation for the density of particles. This type of diffusion approximation for kinetic equations has been widely studied in several papers, see for instance [1, 4, 10, 7] (and references therein).

In recent works (see [14], [13], [3]) similar asymptotic analysis are performed when the equilibrium function is not a Maxwellian distribution, but rather a heavy tail function. In this case, the diffusion coefficient appearing in the classical diffusion limit is no longer well defined and one has to modify the time scale to recover an equation for the particle density. This is known in the literature as anomalous diffusion phenomena (the mean square displacement of the particle is not a linear function of time), and it leads to fractional diffusion equations.

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Denoting by ε the ratio of the microscopic length scale (mean free path) and macroscopic length scale, we scale the space variable as $x = \varepsilon x'$ and the time variable as $t = \varepsilon^\alpha t'$. The starting point of this paper is then the following rescaled Boltzmann equation:

$$\varepsilon^\alpha \partial_t f^\varepsilon + \varepsilon v \cdot \nabla_x f^\varepsilon = Q(f^\varepsilon) \tag{1}$$

where the particle distribution function $f^\varepsilon(t,x,v)$ depends on the time $t > 0$, the space variable $x \in \mathbb{R}^n$ and the velocity variable $v \in \mathbb{R}^n$. The collision operator appearing in the right hand side is a linear integral operator of the form:

$$Q(f) = \int_{\mathbb{R}^n} \sigma(x,v,v') \big[f(v')F(v) - f(v)F(v') \big] dv' \tag{2}$$

with

$$\sigma(x,v,v') = \sigma(x,v',v).$$

Note that the operator Q is the sum of a gain term

$$Q^+(f) = \int_{\mathbb{R}^n} \sigma(x,v,v') f(v')\, dv'\, F(v)$$

and a loss term

$$Q^-(f) = -\nu(v) f(v) \quad \text{with } \nu(v) = \int_{\mathbb{R}^n} \sigma(x,v,v') F(v') dv'.$$

This operator has a one dimensional kernel spanned by the function $F(v)$ and it preserves the total mass:

$$\int_{\mathbb{R}^n} Q(f)\, dv = 0 \qquad \text{for all } f. \tag{3}$$

The usual diffusion limit corresponds to $\alpha = 2$. It can be studied using the so-called *Hilbert expansion method*, which is based on a formal expansion of the solution in the form

$$f^\varepsilon = f^0 + \varepsilon f^1 + \varepsilon^2 f^2 + r^\varepsilon.$$

Inserting this expansion in (1) and identifying the terms of same order in ε yields:

$$\begin{aligned} Q(f^0) &= 0 && (4)\\ Q(f^1) &= v \cdot \nabla_x f^0 && (5)\\ Q(f^2) &= \partial_t f^0 + v \cdot \nabla_x f^1 && (6) \end{aligned}$$

and the remainder r^ε solves

$$\partial_t r^\varepsilon + \varepsilon^{-1} v \cdot \nabla_x r^\varepsilon = \varepsilon^{-2} Q(r^\varepsilon) + \varepsilon \partial_t f^1 + \varepsilon^2 \partial_t f^2 + \varepsilon v \cdot \nabla_x f^2.$$

Equation (4) implies that $f^0(t,x,v) = \rho(t,x)F(v)$ and (5) then yields $f^1 = Q^{-1}(v \cdot \nabla_x f^0) = Q^{-1}(vF(v)) \cdot \nabla_x \rho$ (assuming that such an inverse exists, see Proposition 2.2). Finally, in view of (3), the equation (6) for f^2 gives (integrating with respect to v):

$$\partial_t \rho + \nabla_x \cdot j = 0$$

where

$$j = \int_{\mathbb{R}^n} v f^1\, dv = \int_{\mathbb{R}^n} v \otimes Q^{-1}(vF(v))\, dv \nabla_x \rho.$$

We deduce that for f^2 to exist, ρ must solve

$$\partial_t \rho - \nabla_x \cdot D \nabla_x \rho = 0$$

with diffusion matrix

$$D = -\int_{\mathbb{R}^n} v \otimes Q^{-1}(vF(v))\, dv.$$

Now, assuming that ρ is a (smooth solution) of this diffusion equation, we can define f^0, f^1 and f^2 solutions of (4), (5) and (6), and show that the remainder r^ε converges strongly to 0 in some L^2 space. The main drawback of this method, compared with the moment method, is that it requires stronger regularity assumptions on the initial data in order to get the appropriate bounds on f^0, f^1 and f^2. On the other hand the method yields strong convergence of the solution (rather than weak convergence for the moment method). Hilbert expansion type methods have also proved very useful to study diffusion limits for some non linear collision operators, see [12, 9, 11].

In this paper we are considering a situation in which this limit fails because the diffusion matrix D above is infinite. This is clearly the case when the equilibrium function F, rather than a Maxwellian distribution function is a heavy tail distribution function, satisfying

$$F(v) \sim \frac{\kappa_0}{|v|^{n+\alpha}} \qquad \text{as } |v| \to \infty$$

for some $\alpha \in (0, 2)$ (note that this α will be the same as the time scale in (1)). Such a profile has been mentioned in several context such as astrophysical plasmas ([17, 15] or in the study of granular plasmas ([5, 8] for inelastic Maxwell models, [16] for inelastic Kac model and [6] for mixture of gases with Maxwellian collision kernel). A similar study is undertaken in [14] using a relatively simple method based on the Fourier transform. This method does not generalize easily to more complicated situation involving space dependent collision kernel σ or non linear operator. In [13], an alternative method is developed which is based on the weak formulation of the Boltzmann equation (1) and the choice of particular test functions defined via an auxiliary equation. This method is somewhat analogous to the well known *moment method* for the usual diffusion limit. Both approaches provide the weak convergence of f^ε to ρF. The aim of the present paper is to develop a Hilbert expansion approach for this anomalous diffusion regime. One of the advantages of this new method (compared with [14, 13]) will be to give strong convergence results. We also believe that this method can be used to address some nonlinear problems, such as models involving the Boltzmann-Pauli collision operator.

The first difficulty in developing such an expansion method is that since $\alpha \in (0, 2)$, it is not obvious how one should identify the terms with same order (some terms have a fractional order). The originality of this expansion is thus the splitting of the different terms of equation (1) and their distribution in the different levels of the expansion. This splitting follows a heuristic based on the choice of the auxiliary function in [13]. Indeed, we notice that the advection term and the gain loss of the collision operator are of the same order whereas the gain term is of higher order (in term of ε). This leads to a rearrangement in the Hilbert expansion as explained in section 3.

We are now going to list our assumptions on σ and F and give the main result of this paper (Theorem 1.1 below). In Section 2, we will present a proof of this result in the simpler case $\sigma = 1$. In this case, the expansion is relatively simple, and we will explain why in the more general case, a more complicated expansion needs to be introduced. The main theorem is then proved in Section 3.

1.2 Main result

In order to simplify the computation, we will assume that there exist some constants σ_0 and σ_1 such that

$$0 < \sigma_0 \leq \sigma(x, v, v') = \sigma(x, v', v) \leq \sigma_1 < \infty \quad \text{for all } (x,\, v,\, v') \in \mathbb{R}^{3n}. \tag{7}$$

In particular, this implies that the collision frequency

$$\nu(x, v) = \int_{\mathbb{R}^n} \sigma(x, v, v') F(v') dv'$$

satisfies

$$\sigma_0 \leq \nu(x, v) \leq \sigma_1 \quad \text{for all } (x, v) \in \mathbb{R}^{2n}.$$

We also need to assume that the collision frequency is regular enough and has a nice asymptotic behavior for large v. For simplicity, we will assume:

$$\begin{aligned} &\partial_x^k \nu \in L^\infty(\mathbb{R}^n \times \mathbb{R}^n) \text{ for all multi-indice } k \text{ such that } |k| \leq 4 \\ &\nu(x, v) = \nu_0(x) \quad \forall |v| > 1 \end{aligned} \tag{8}$$

(this last assumption can of course be replaced with $\lim_{|v|\to\infty} \nu(x, v) = \nu_0(x)$).

The most important assumption concerns the behavior of the equilibrium $F(v)$ for large v. We assume:

$$F(v) = \frac{\kappa}{|v|^{n+\alpha}} \qquad \text{for } |v| \geq 1 \tag{9}$$

for some $\alpha \in (0, 2)$ and

$$F(v) \leq \kappa \qquad \text{for all } v \in \mathbb{R}^n. \tag{10}$$

Finally, we assume that the following symmetry conditions are satisfied:

$$F(-v) = F(v)\,, \qquad \nu(x, -v) = \nu(x, v) \quad \text{for all } (x\, v) \in \mathbb{R}^{2n}. \tag{11}$$

This implies in particular that the flux $\int_{\mathbb{R}^n} vF(v)\, dv = 0$ (when it is defined, that is when $\alpha > 1$), which we know is a necessary condition for the derivation of diffusion type equation.

We can now state our main result:

Theorem 1.1. *Assume that (7), (8), (9), (10), (11) hold and let f^ε be the solution of (1) with initial condition $f_{in}(x, v) = \rho_{in}(x)F(v)$ such that $\rho_{in} \in H^4(\mathbb{R}^n)$, then*

$$||f^\varepsilon - \rho_0 F||_{L^\infty(0,\infty;L^2_{F^{-1}}(\mathbb{R}^n\times\mathbb{R}^n))} \longrightarrow 0, \qquad \text{as } \varepsilon \to 0, \tag{12}$$

where ρ_0 satisfies

$$\begin{cases} \partial_t \rho_0 + L(\rho_0) = 0 & \text{in } [0, \infty) \times \mathbb{R}^n, \\ \rho_0(0, x) = \rho_{in}(x) & \text{in } \mathbb{R}^n \end{cases} \tag{13}$$

and L is an elliptic operator defined by the singular integral:

$$L(\rho) = \text{P.V.} \int_{\mathbb{R}^n} \gamma(x, y) \frac{\rho(x) - \rho(y)}{|y - x|^{n+\alpha}}\, dy. \tag{14}$$

with

$$\gamma(x, y) = \nu_0(x)\, \nu_0(y) \int_0^\infty z^\alpha e^{-z \int_0^1 \nu_0((1-s)x+sy)\, ds}\, dz$$

Note that $\gamma(x,y) = \gamma(y,x)$ and that condition (7) yields

$$0 < \gamma_1 < \gamma(x,y) \le \gamma_2.$$

In particular the operator L is a self-adjoint elliptic operator of order α (comparable to $(-\Delta)^{\alpha/2}$).

Before giving the proof of this theorem in the general case (Section 3), we will present two simpler cases that can be handled with a more direct Hilbert method based on the Fourier Transform and for which the result is obtained in a L^1 framework as well as in a L^2 framework with less regularity assumptions on the initial data.

2 Space homogeneous cross sections

2.1 Constant cross section

In this section, we prove Theorem 1.1 in the simpler framework where the cross section σ is constant (for the sake of simplicity we take $\sigma = 1$). We then have

$$Q(f) = \int_{\mathbb{R}^n} f(v')\, dv'\, F(v) - f(v). \tag{15}$$

In that case, the use of the Fourier transform (as in [14]) allows for explicit computations and a much simpler proof.

We have the following proposition:

Proposition 2.1.

Let f^ε be the solution of (1) with Q given by (15) where F satisfies (9-11) and with initial condition $f_{in}(x,v) = \rho_{in}(x)F(v)$ such that $\rho_{in} \in W^{3,1}(\mathbb{R}^n)$. Then

$$||f^\varepsilon - \rho_0 F||_{L^\infty(0,\infty;L^1(\mathbb{R}^n\times\mathbb{R}^n))} \longrightarrow 0, \qquad \text{as } \varepsilon \to 0 \tag{16}$$

where ρ_0 satisfies

$$\begin{cases} \partial_t \rho_0 + \kappa(-\Delta)^{\frac{\alpha}{2}}\rho_0 = 0 & \text{in } [0,\infty)\times\mathbb{R}^n \\ \rho_0(0,x) = \rho_{in}(x) & \text{in } \mathbb{R}^n \end{cases} \tag{17}$$

with

$$\kappa = \int_{\mathbb{R}^n} \frac{(w\cdot e)^2}{1+(w\cdot e)^2}\frac{1}{|w|^{n+\alpha}}\, dw. \tag{18}$$

If instead we have $\rho_{in} \in H^3(\mathbb{R}^n)$, then

$$||f^\varepsilon - \rho_0 F||_{L^\infty(0,\infty;L^2_{F^{-1}}(\mathbb{R}^n\times\mathbb{R}^n))} \longrightarrow 0, \qquad \text{as } \varepsilon \to 0. \tag{19}$$

The proof relies on the following Hilbert expansion for f^ε:

$$f^\varepsilon = f_0^\varepsilon + g_1^\varepsilon + g_2^\varepsilon + r^\varepsilon,$$

with

$$Q(f_0^\varepsilon) = 0 \tag{20}$$

$$(1+\varepsilon v\cdot\nabla_x) g_1^\varepsilon = -\varepsilon v\cdot\nabla_x f_0^\varepsilon \tag{21}$$

$$Q(g_2^\varepsilon) = \varepsilon^\alpha \partial_t f_0^\varepsilon - Q^+(g_1^\varepsilon) \tag{22}$$

and r^ε solution of

$$\partial_t r^\varepsilon + \varepsilon^{1-\alpha} v \cdot \nabla_x r^\varepsilon = \varepsilon^{-\alpha} Q(r^\varepsilon) - \partial_t g_1^\varepsilon - \partial_t g_2^\varepsilon - \varepsilon^{1-\alpha} v \cdot \nabla g_2^\varepsilon. \tag{23}$$

In order to investigate these equations, we recall the following result, which holds for general operators of the form (2), whenever (7) holds (see [7]):

Proposition 2.2.

i) The operators Q^+ and Q are bounded operator on $L^2_{F^{-1}}(\mathbb{R}^n)$.

ii) The kernel of Q has dimension 1 and is spanned by F.

iii) There exists a constant $c > 0$ such that

$$-\int_{\mathbb{R}^n} Q(f) f \frac{1}{F}\, dv \geq c \int_{\mathbb{R}^n} |f - \rho F|^2 \frac{1}{F}\, dv.$$

iv) For all $h \in L^2_{F^{-1}}(\mathbb{R}^n)$, the equation $Q(g) = h$ has a solution if and only if $\int h(v)\, dv = 0$, and there is a unique such solution (denoted $Q^{-1}(h)$) satisfying $\int_{\mathbb{R}^n} g(v)\, dv = 0$.

v) The operator Q^{-1} is bounded in $L^2_{F^{-1}}(\mathbb{R}^n)$:

$$\|Q^{-1}(h)\|_{L^2_{F^{-1}}} \leq C \|h\|_{L^2_{F^{-1}}} \qquad \text{for all } h \text{ such that } \int_{\mathbb{R}^n} h(v)\, dv = 0. \tag{24}$$

vi) $\displaystyle\int Q(f) sgn(f)\, dv \leq 0$ for all f.

Equation (20) then yields (using (ii)):

$$f_0^\varepsilon(x, v, t) = \rho^\varepsilon(x, t) F(v).$$

and so equation (21) becomes

$$\mathcal{T}_\varepsilon g_1^\varepsilon = -\varepsilon v \cdot \nabla_x \rho^\varepsilon F$$

where we introduced the operator

$$\mathcal{T}_\varepsilon = 1 + \varepsilon v \cdot \nabla_x.$$

Assuming (for the time being) that this operator can be inverted, we can then write

$$g_1^\varepsilon = -\mathcal{T}_\varepsilon^{-1}(\varepsilon v \cdot \nabla_x \rho^\varepsilon) F. \tag{25}$$

Finally, the solvability condition for (22) (using (iv)) is

$$\int \left[\varepsilon^\alpha \partial_t \rho^\varepsilon F - Q^+(g_1^\varepsilon)\right]\, dv = 0$$

which reads

$$\varepsilon^\alpha \partial_t \rho^\varepsilon - \int g_1^\varepsilon\, dv = 0.$$

Using (25), we see that this is a condition on ρ^ε, which we can write as

$$\varepsilon^\alpha \partial_t \rho^\varepsilon + \int \mathcal{T}_\varepsilon^{-1}(\varepsilon v \cdot \nabla_x \rho^\varepsilon) F\, dv = 0. \tag{26}$$

Because of the simple form of the operator Q, this condition also implies that the right hand side of (22) is actually 0 (rather than only having zero integral), and thus

$$g_2^\varepsilon = 0.$$

Remarks 2.3. *The usual Hilbert expansion consists in writing* $f^\varepsilon = \rho F + f_1^\varepsilon + f_2^\varepsilon + r^\varepsilon$ *where*

$$\nu(x,v) f_1^\varepsilon = Q^+(f_1^\varepsilon) - \varepsilon v\cdot\nabla_x \rho F,$$

$$Q(f_2^\varepsilon) + \varepsilon v\cdot\nabla_x f_1^\varepsilon = \varepsilon^\alpha \partial_t \rho F.$$

We can see that the scheme that we use here consists in reshuffling the terms $Q^+(f_1^\varepsilon)$ *and* $\varepsilon v\cdot\nabla_x f_1^\varepsilon$ *in the first order and second order equation.*

Note also that while in the usual Hilbert expansion, the first term is solution of the asymptotic equation, here we have ρ^ε *which is solution of an approximation of the limiting model (26).*

In order to prove Proposition 2.1, we thus have to show that ρ^ε, solution of (26), exists and that g_1^ε and r^ε go to zero in some appropriate norm.

This is best done using Fourier transform (with respect to x): We denote by $\widehat{h}(\xi)$ the Fourier transform, with respect to x, of a function h. Taking the Fourier transform in (21), we get:

$$(1 + i\varepsilon v\cdot\xi)\widehat{g_1^\varepsilon} = -i\varepsilon v\cdot\xi \widehat{f_0}$$

and so

$$\widehat{g_1^\varepsilon} = -\frac{i\varepsilon v\cdot\xi}{1+i\varepsilon v\cdot\xi} F(v)\,\widehat{\rho^\varepsilon}. \tag{27}$$

We can thus rewrite (26) as

$$\partial_t \widehat{\rho^\varepsilon} + a^\varepsilon(\xi)\,\widehat{\rho^\varepsilon} = 0. \tag{28}$$

with

$$a^\varepsilon(\xi) = \varepsilon^{-\alpha}\int_{\mathbb{R}^n} \frac{i\varepsilon v\cdot\xi}{1+i\varepsilon v\cdot\xi} F(v)\,dv.$$

We now need the following lemma (see also [14]):

Lemma 2.4. *For all* ξ, $a^\varepsilon(\xi)\geq 0$ *and*

$$|a^\varepsilon(\xi) - \kappa|\xi|^\alpha| \leq C\varepsilon^{2-\alpha}|\xi|^2 \longrightarrow 0, \tag{29}$$

with κ *given by (18). Furthermore, there exist positive constants* c_1 *and* c_2 *such that,*

$$c_1 \min(\varepsilon^{-\alpha}, |\xi|^\alpha) \leq a^\varepsilon(\xi) \leq c_2(|\xi|^\alpha + \varepsilon^{(2-\alpha)}|\xi|^2) \tag{30}$$

Proof of Lemma 2.4. A simple computation (using the properties of F) yields

$$\begin{aligned}\int_{\mathbb{R}^n} \frac{i\varepsilon v\cdot\xi}{1+i\varepsilon v\cdot\xi} F(v)\,dv &= \int_{\mathbb{R}^n} \frac{(\varepsilon v\cdot\xi)^2}{1+(\varepsilon v\cdot\xi)^2} F(v)\,dv \\ &= \int_{|v|\leq 1} \frac{(\varepsilon v\cdot\xi)^2}{1+(\varepsilon v\cdot\xi)^2} F(v)\,dv \\ &\quad + \int_{|v|\geq 1} \frac{(\varepsilon v\cdot\xi)^2}{1+(\varepsilon v\cdot\xi)^2} \frac{1}{|v|^{n+\alpha}}\,dv.\end{aligned}$$

The first integral is clearly bounded by $C\varepsilon^2|\xi|^2$, and using the change of variable $w = \varepsilon v|\xi|$ we can write the second integral as

$$\int_{|v|\geq 1} \frac{(\varepsilon v\cdot\xi)^2}{1+(\varepsilon v\cdot\xi)^2} \frac{1}{|v|^{n+\alpha}}\,dv = \varepsilon^\alpha|\xi|^\alpha \int_{|w|\geq\varepsilon|\xi|} \frac{(w\cdot e)^2}{1+(w\cdot e)^2} \frac{1}{|w|^{n+\alpha}}\,dw$$

We deduce

$$\begin{aligned}|\varepsilon^\alpha a^\varepsilon(\xi) - \varepsilon^\alpha \kappa|\xi|^\alpha| \quad &\leq \quad C\varepsilon^2|\xi|^2 + \varepsilon^\alpha|\xi|^\alpha \int_{|w|\leq\varepsilon|\xi|} \frac{(w\cdot e)^2}{1+(w\cdot e)^2}\frac{1}{|w|^{n+\alpha}}\,dw \\ &\leq C\varepsilon^2|\xi|^2\end{aligned}$$

This implies (29) and the second inequality in (30). The first inequality in (30) is obtained by distinguishing two cases:

If $\varepsilon|\xi| \leq 1$, then the computation above yields $\int_{|v|\geq 1} \frac{(\varepsilon v\cdot\xi)^2}{1+(\varepsilon v\cdot\xi)^2}\frac{1}{|v|^{n+\alpha}}\,dv \geq c\varepsilon^\alpha|\xi|^\alpha$ and the result follows.

If $\varepsilon|\xi| \geq 1$, then we have

$$\begin{aligned}\int_{|v|\geq 1} \frac{(\varepsilon v\cdot\xi)^2}{1+(\varepsilon v\cdot\xi)^2}\frac{1}{|v|^{n+\alpha}}\,dv \quad &\geq \quad \varepsilon^\alpha|\xi|^\alpha \int_{|w|\geq\varepsilon|\xi|} \frac{1}{2}\frac{1}{|w|^{n+\alpha}}\,dw \\ &\geq \quad 1\end{aligned}$$

and so

$$a^\varepsilon \geq \varepsilon^{-\alpha}.$$

□

We deduce

Corollary 2.5. *For any ρ_{in} and $\varepsilon > 0$, there exists a unique ρ^ε solution of (28), given by*

$$\widehat{\rho^\varepsilon}(t,\xi) = e^{-ta^\varepsilon(\xi)}\widehat{\rho_{in}}(\xi)$$

Furthermore, ρ^ε satisfies

$$||\,|\xi|^k\widehat{\rho^\varepsilon}(t)||_{L^\infty(\mathbb{R}^n)} \leq ||\,|\xi|^k\widehat{\rho_{in}}||_{L^\infty(\mathbb{R}^n)}$$

and

$$||\,|\xi|^k\partial_t\widehat{\rho^\varepsilon}(t)||_{L^\infty(\mathbb{R}^n)} \leq ||\,|\xi|^{k+2}\widehat{\rho_{in}}||_{L^\infty(\mathbb{R}^n)}$$

for all $k \geq 0$. Finally we have

$$||\rho^\varepsilon - \rho_0||_{L^\infty(0,T;L^1(\mathbb{R}^n))} \leq CT||\rho_{in}||_{W^{2,1}(\mathbb{R}^n)}$$

and

$$||\rho^\varepsilon - \rho_0||_{L^\infty(0,T;L^1(\mathbb{R}^n))} \leq CT||\rho_{in}||_{H^2(\mathbb{R}^n)}$$

where ρ_0 is the unique solution of (17).

Proof. We only prove the last assertion: We note that

$$\partial_t(\widehat{\rho^\varepsilon} - \widehat{\rho_0}) + a^\varepsilon(\xi)(\widehat{\rho^\varepsilon} - \widehat{\rho_0}) = (\kappa|\xi|^\alpha - a^\varepsilon(\xi))\widehat{\rho_0}$$

and so Lemma 2.4 yields

$$\frac{d}{dt}|\widehat{\rho^\varepsilon} - \widehat{\rho_0}| \leq C\varepsilon^{2-\alpha}|\xi|^2\widehat{\rho_0}.$$

We deduce

$$\frac{d}{dt}||\rho^\varepsilon - \rho_0||_{L^1(\mathbb{R}^n)} \leq C\varepsilon^{2-\alpha}||\rho_0||_{W^{2,1}(\mathbb{R}^n)}$$

which gives the $L^1(\mathbb{R}^n)$ convergence of ρ^ε. A similar computation (multiplying by $\widehat{\rho^\varepsilon} - \widehat{\rho_0}$) gives the $L^2(\mathbb{R}^n)$ convergence.

□

Proof of Proposition 2.1. We now notice that (27) yields

$$|\widehat{g_1^\varepsilon}| \leq \frac{|\varepsilon v\cdot\xi|}{\sqrt{1+(\varepsilon v\cdot\xi)^2}}F(v)\widehat{\rho^\varepsilon}$$

and thus

$$\begin{aligned}||\widehat{g_1^\varepsilon}||_{L^\infty_\xi(\mathbb{R}^n;L^1_v(\mathbb{R}^N)))} &\leq \sup_\xi\left\{\int \frac{|\varepsilon v\cdot\xi|}{\sqrt{1+(\varepsilon v\cdot\xi)^2}}F(v)\,dv\widehat{\rho^\varepsilon}(\xi)\right\}\\ &\leq \sup_\xi\left\{C(\varepsilon^\alpha|\xi|^\alpha+\varepsilon|\xi|)\widehat{\rho^\varepsilon}\right\}\\ &\leq C\,(\varepsilon^\alpha+\varepsilon)\,||(1+|\xi|^2)\widehat{\rho^\varepsilon}||_{L^\infty(\mathbb{R}^n)},\end{aligned}$$

and similarly

$$\begin{aligned}||\partial_t\widehat{g_1^\varepsilon}||_{L^\infty_\xi(\mathbb{R}^n;L^1_v(\mathbb{R}^N)))} &\leq \sup_\xi\left\{C(\varepsilon^\alpha|\xi|^\alpha+\varepsilon|\xi|)\partial_t\widehat{\rho^\varepsilon}\right\}\\ &\leq \sup_\xi\left\{C(\varepsilon^\alpha|\xi|^\alpha+\varepsilon|\xi|)a^\varepsilon(\xi)\widehat{\rho^\varepsilon}\right\}\end{aligned}$$

so, using Lemma 2.4, we deduce

$$||\partial_t\widehat{g_1^\varepsilon}||_{L^\infty_\xi(\mathbb{R}^n;L^1_v(\mathbb{R}^N)))} \leq (\varepsilon^\alpha+\varepsilon)||(1+|\xi|^3)\widehat{\rho^\varepsilon}||_{L^\infty(\mathbb{R}^n)}.$$

It follows that g_1^ε and $\partial_t g_1^\varepsilon$ converge to 0 strongly in $L^\infty(0,\infty;L^1(\mathbb{R}^n\times\mathbb{R}^n))$. Finally, multiplying the equation for the remainder r^ε, (23), by $\mathrm{sign}(r^\varepsilon)$ (and using Proposition 2.2 (vi)), we get

$$\partial_t||r^\varepsilon||_{L^1(\mathbb{R}^n)} \leq ||\partial_t g_1^\varepsilon||_{L^1(\mathbb{R}^n)}.$$

Using the fact that $r^\varepsilon(t=0)=-g_1^\varepsilon(t=0)$, we deduce

$$||r^\varepsilon||_{L^\infty(0,T;L^1(\mathbb{R}^{2n}))} \longrightarrow 0$$

and so

$$||f^\varepsilon-\rho^\varepsilon F||_{L^\infty(0,T;L^1(\mathbb{R}^n\times\mathbb{R}^n))} \longrightarrow 0.$$

Similarly, we have (using the computations of Lemma 2.4):

$$\begin{aligned}\int_{\mathbb{R}^n}|\widehat{g_1^\varepsilon}|^2\frac{1}{F(v)}\,dv &\leq \int_{\mathbb{R}^n}\frac{|\varepsilon v\cdot\xi|^2}{1+(\varepsilon v\cdot\xi)^2}F(v)\,dv\widehat{\rho^\varepsilon}^2\\ &\leq C(\varepsilon^\alpha|\xi|^\alpha+\varepsilon^2|\xi|^2)\widehat{\rho^\varepsilon}^2\end{aligned}$$

and so

$$||\widehat{g_1^\varepsilon}||_{L^2_{F^{-1}}(\mathbb{R}^n\times\mathbb{R}^n)} \leq C\varepsilon^\alpha||\rho^\varepsilon||_{H^1(\mathbb{R}^n)}$$

and in a same way

$$||\partial_t\widehat{g_1^\varepsilon}||_{L^2_{F^{-1}}(\mathbb{R}^n\times\mathbb{R}^n)} \leq C\varepsilon^\alpha||\rho^\varepsilon||_{H^3(\mathbb{R}^n)}.$$

A similar argument (multiplying (23) by r^ε/F) gives the strong convergence in $L^\infty(0,\infty;L^2_{F^{-1}}(\mathbb{R}^n\times\mathbb{R}^n))$. □

2.2 Case of space homogeneous cross section

We now show, in a still relatively simple case, why the general case is more complicated. In this section, we assume that Q is given by:

$$Q(f) = \int_{\mathbb{R}^n} \sigma(v,v')[f(v')F(v) - f(v)F(v')]\,dv' \tag{31}$$

with σ depending only on v and v' (and not on x), and satisfying

$$0 < \sigma_0 \leq \sigma(v,v') = \sigma(v',v) \leq \sigma_1.$$

We might try to proceed as in the previous section, writing

$$f^\varepsilon = f_0^\varepsilon + g_1^\varepsilon + g_2^\varepsilon + r^\varepsilon,$$

with

$$\begin{aligned} f_0^\varepsilon(x,v,t) &= \rho^\varepsilon(x,t)F(v), \\ g_1^\varepsilon &= -\mathcal{T}_\varepsilon^{-1}(\varepsilon v\cdot\nabla_x\rho^\varepsilon), F \\ Q(g_2^\varepsilon) &= \varepsilon^\alpha\partial_t\rho^\varepsilon F - Q^+(g_1^\varepsilon). \end{aligned}$$

where the operator $\mathcal{T}_\varepsilon$ is now defined by $\mathcal{T}_\varepsilon(f) = \nu f + \varepsilon v\cdot\nabla_x f$.

The solvability condition for g_2^ε (obtained by integrating this last equation with respect to v) reads

$$\begin{aligned} \partial_t\rho^\varepsilon &= \varepsilon^{-\alpha}\int_{\mathbb{R}^n} Q^+(g_1^\varepsilon)\,dv \\ &= \varepsilon^{-\alpha}\int_{\mathbb{R}^n} \nu g_1^\varepsilon\,dv \\ &= -\varepsilon^{-\alpha}\int_{\mathbb{R}^n} \nu(v)\mathcal{T}_\varepsilon^{-1}(\varepsilon v\cdot\nabla_x\rho^\varepsilon)F\,dv. \end{aligned}$$

We thus obtain that ρ^ε must solve

$$\partial_t\rho^\varepsilon + L^\varepsilon(\rho^\varepsilon) = 0$$

with

$$L^\varepsilon(\rho) = \varepsilon^{-\alpha}\int_{\mathbb{R}^n} \nu(v)\mathcal{T}_\varepsilon^{-1}(\varepsilon v\cdot\nabla_x\rho^\varepsilon)F\,dv.$$

Once again, we can identify the nature of this operator using Fourier transform: We get

$$\begin{aligned} \widehat{L^\varepsilon(\rho)} &= \varepsilon^{-\alpha}\int_{\mathbb{R}^n} \frac{i\varepsilon v\cdot\xi}{\nu(v)+i\varepsilon v\cdot\xi}\nu(v)F(v)\,dv\widehat{\rho^\varepsilon} \\ &= a^\varepsilon(\xi)\widehat{\rho^\varepsilon}. \end{aligned}$$

However, unlike the previous section, we do not have $Q(g_2^\varepsilon) = 0$. Instead, we have

$$Q(g_2^\varepsilon) = -\varepsilon^\alpha L^\varepsilon(\rho^\varepsilon)F - Q^+(g_1^\varepsilon)$$

which, in Fourier, reads:

$$Q(\widehat{g_2^\varepsilon}) = -\varepsilon^\alpha\int_{\mathbb{R}^n} \frac{i\varepsilon v'\cdot\xi}{\nu(v')+i\varepsilon v'\cdot\xi}[\nu(v') - \sigma(v,v')]F(v')\,dv'\widehat{\rho^\varepsilon}F(v)$$

and so

$$\begin{aligned}|Q(\widehat{g_2^\varepsilon})| &\leq 2\sigma_1\varepsilon^\alpha \int_{\mathbb{R}^n} \frac{|\varepsilon v'\cdot\xi|}{\sqrt{\sigma_0^2+(\varepsilon v'\cdot\xi)^2}} F(v')\,dv' |\widehat{\rho^\varepsilon}| F(v)\\ &\leq C\varepsilon^\alpha(\varepsilon^\alpha|\xi|^\alpha+\varepsilon|\xi|)|\widehat{\rho^\varepsilon}|F(v).\end{aligned}$$

This clearly implies that g_2^ε goes to zero as ε goes to 0, but it also means that it will be difficult to control the term $\varepsilon^{1-\alpha}v\cdot\nabla g_2^\varepsilon$ because of the lack of integrability of vF (at least when $\alpha<1$).

For this reason, we need to add a term in the expansion, as explained in the next section.

3 Proof of Theorem 1.1

3.1 A reshuffled Hilbert expansion

In order to prove Theorem 1.1, we use a Hilbert type expansion and some techniques similar to those first introduced in [13].

We recall that f^ε is a solution of

$$\varepsilon^\alpha\partial_t f^\varepsilon+\varepsilon v\cdot\nabla_x f^\varepsilon = Q(f^\varepsilon) \tag{32}$$

where

$$Q(f)=Q^+(f)-\nu f \quad\text{with}\quad Q^+(f)=\int_{\mathbb{R}^n}\sigma(x,v,v')F(v)f(v')dv',$$

and we write the following expansion

$$f^\varepsilon=\rho^\varepsilon(t,x)F(v)+g_1^\varepsilon(t,x,v)+g_2^\varepsilon(t,x,v)+g_3^\varepsilon(t,x,v)+r^\varepsilon(t,x,v) \tag{33}$$

where the terms g_1^ε, g_2^ε, g_3^ε satisfy

$$(\nu(x,v)+\varepsilon v\cdot\nabla_x)g_1^\varepsilon = -\varepsilon v\cdot\nabla_x(\rho^\varepsilon F) \tag{34}$$

$$Q(g_2^\varepsilon) = -Q^+(g_1^\varepsilon)-\varepsilon^\alpha\partial_t\rho^\varepsilon F \tag{35}$$

$$(\nu(x,v)+\varepsilon v\cdot\nabla_x)g_3^\varepsilon = -\varepsilon v\cdot\nabla_x g_2^\varepsilon. \tag{36}$$

The remainder solves

$$\partial_t r^\varepsilon+\varepsilon^{1-\alpha}v\cdot\nabla_x r^\varepsilon=\frac{Q(r^\varepsilon)}{\varepsilon^\alpha}-\partial_t g_1^\varepsilon-\partial_t g_2^\varepsilon-\partial_t g_3^\varepsilon+\frac{Q^+(g_3^\varepsilon)}{\varepsilon^\alpha}. \tag{37}$$

We recall that we defined the operator

$$\mathcal{T}_\varepsilon(g)=\nu g+\varepsilon v\cdot\nabla_x g. \tag{38}$$

A simple computation (see [13]) shows that this operator is invertible with

$$\mathcal{T}_\varepsilon^{-1}(f)=\int_0^{+\infty} e^{-\int_0^z \nu(x-\varepsilon vs,v)\,ds} f(x-\varepsilon vz,v)\,dz \tag{39}$$

for any function $f(x,v)$. In particular, equations (34) and (36) are solvable without additional conditions, but (35) requires the following solvability condition (obtained by integrating (35) with respect to v):

$$\partial_t\rho^\varepsilon+L^\varepsilon(\rho^\varepsilon)=0 \tag{40}$$

with

$$
\begin{aligned}
L^\varepsilon(\rho) &= -\varepsilon^{-\alpha}\int_{\mathbb{R}^n} Q^+(g_1^\varepsilon)dv \\
&= -\varepsilon^{-\alpha}\int_{\mathbb{R}^n} \nu(x,v)g_1^\varepsilon dv \\
&= \varepsilon^{-\alpha}\int_{\mathbb{R}^n} \nu(x,v)\,\mathcal{T}_\varepsilon^{-1}(\varepsilon v\cdot\nabla_x\rho^\varepsilon)\,F\,dv. \qquad (41)
\end{aligned}
$$

In view of (37), it is clear that in order to prove Theorem 1.1, we need to show that g_1^ε, g_2^ε and g_3^ε can be defined and that the terms $\partial_t g_1^\varepsilon$, $\partial_t g_2^\varepsilon$ $\partial_t g_3^\varepsilon$ and $\frac{Q^+(g_3^\varepsilon)}{\varepsilon^\alpha}$ in the right hand side of (37) go to 0 in $L^1(0,\infty;L^2_{F^{-1}}(\mathbb{R}^n\times\mathbb{R}^n))$.

The first step, detailed in the next section, will be to study the properties of this operator L^ε and show that the equation (40) has a smooth solution ρ^ε. In section 3.3, we will derive some important estimate on $Q^{-1}(h)$ that will be needed to get some estimates on g_2^ε. In Section 3.4, we derive estimates on g_1^ε, g_2^ε and g_3^ε and their derivatives and show that $f^\varepsilon-\rho^\varepsilon F$ converges to zero strongly. The final step (Section 3.5) is to show that ρ^ε, solution of (40), converges strongly to ρ_0 solution of (13).

3.2 Properties of the operator L^ε and the equation (40)

First, of all, we note that for any function $\rho(x)$, (39) and (41) imply

$$
\begin{aligned}
L^\varepsilon(\rho) &= \varepsilon^{-\alpha}\int_{\mathbb{R}^n} \nu(x,v)\,\mathcal{T}_\varepsilon^{-1}(\varepsilon v\cdot\nabla_x\rho^\varepsilon)\,F\,dv \\
&= \varepsilon^{-\alpha}\int_{\mathbb{R}^n}\int_0^{+\infty} e^{-\int_0^z \nu(x-\varepsilon vs,v)\,ds}\,\varepsilon v\cdot\nabla_x\rho(x-\varepsilon vz)\,F(v)\,dz\,dv.
\end{aligned}
$$

Alternatively, we can also write

$$
\begin{aligned}
&\mathcal{T}_\varepsilon^{-1}(\varepsilon v\cdot\nabla_x\rho) \\
&\quad=\int_0^{+\infty} e^{-\int_0^z \nu(x-\varepsilon vs,v)\,ds}\,\varepsilon v\cdot\nabla_x\rho(x-\varepsilon vz)\,dz \\
&\quad=-\int_0^{+\infty} e^{-\int_0^z \nu(x-\varepsilon vs,v)\,ds}\,\frac{d}{dz}\big[\rho(x-\varepsilon vz)\big]\,dz \\
&\quad=-\int_0^{+\infty} e^{-\int_0^z \nu(x-\varepsilon vs,v)\,ds}\,\frac{d}{dz}\big[\rho(x-\varepsilon vz)-\rho(x)\big]\,dz \\
&\quad=-\int_0^{+\infty} e^{-\int_0^z \nu(x-\varepsilon vs,v)\,ds}\nu(x-\varepsilon vz,v)\,\big[\rho(x-\varepsilon vz)-\rho(x)\big]\,dz
\end{aligned}
$$

which leads to

$$
L^\varepsilon(\rho)=-\varepsilon^{-\alpha}\int_{\mathbb{R}^n}\int_0^{+\infty} e^{-\int_0^z \nu(x-\varepsilon vs,v)\,ds}\nu(x-\varepsilon vz,v)\,\big[\rho(x-\varepsilon vz)-\rho(x)\big]\,F(v)\,dz\,dv \qquad (42)
$$

which is the formula found in [13].

We can now prove the following lemma:

Lemma 3.1. *For all k integer, $k\geq 2$, the operator L^ε is a bounded operator from $H^k(\mathbb{R}^n)$ to $H^{k-2}(\mathbb{R}^n)$:*

$$
||L^\varepsilon(\rho)||_{H^{k-2}(\mathbb{R}^n)}\leq C||\rho||_{H^k(\mathbb{R}^n)} \quad \textit{for all } \rho\in H^k(\mathbb{R}^n). \qquad (43)
$$

Proof. We first prove that

$$||L^\varepsilon(\rho)||_{L^2(\mathbb{R}^n)} \le C||\rho||_{H^2(\mathbb{R}^n)} \qquad \text{for all } \rho \in H^2(\mathbb{R}^n). \tag{44}$$

A similar computation is performed in [13], and recalled here for the reader's convenience.

We write

$$\begin{aligned} L^\varepsilon(\rho) &= \varepsilon^{-\alpha}\int_{\mathbb{R}^n} \nu(x,v)\,\mathcal{T}_\varepsilon^{-1}(\varepsilon v\cdot\nabla_x\rho)\,F\,dv \\ &= I_1^\varepsilon + I_2^\varepsilon + I_3^\varepsilon \end{aligned}$$

where

$$I_1^\varepsilon = \varepsilon^{-\alpha}\int_{|v|\le 1} \nu(x,v)\,\mathcal{T}_\varepsilon^{-1}(\varepsilon v\cdot\nabla_x\rho)\,F\,dv,$$

$$I_2^\varepsilon = \varepsilon^{-\alpha}\int_{1\le|v|\le 1/\varepsilon} \nu(x,v)\,\mathcal{T}_\varepsilon^{-1}(\varepsilon v\cdot\nabla_x\rho)\,F\,dv,$$

$$I_3^\varepsilon = \varepsilon^{-\alpha}\int_{|v|\ge 1/\varepsilon} \nu(x,v)\,\mathcal{T}_\varepsilon^{-1}(\varepsilon v\cdot\nabla_x\rho)\,F\,dv.$$

We have

$$\begin{aligned} \mathcal{T}_\varepsilon^{-1}(\varepsilon v\cdot\nabla_x\rho) &= \int_0^{+\infty} e^{-\int_0^z \nu(x-\varepsilon vs,v)\,ds}\,\varepsilon v\cdot\nabla_x\rho(x-\varepsilon vz)\,dz \\ &= \int_0^{+\infty}\left[e^{-\int_0^z \nu(x-\varepsilon vs,v)\,ds} - e^{-\nu(x,v)z}\right]\varepsilon v\cdot\nabla_x\rho(x-\varepsilon vz)\,dz \\ &\quad + \int_0^{+\infty} e^{-\nu(x,v)z}\,\left[\varepsilon v\cdot\nabla_x\rho(x-\varepsilon vz) - \varepsilon v\cdot\nabla_x\rho(x)\right]dz \\ &\quad + \int_0^{+\infty} e^{-\nu(x,v)z}\,\varepsilon v\cdot\nabla_x\rho(x)\,dz \end{aligned}$$

and so an integration by parts yields

$$\begin{aligned} \mathcal{T}_\varepsilon^{-1}(\varepsilon v\cdot\nabla_x\rho) &= \int_0^{+\infty}\left[e^{-\int_0^z \nu(x-\varepsilon vs,v)\,ds} - e^{-\nu(x,v)z}\right]\varepsilon v\cdot\nabla_x\rho(x-\varepsilon vz)\,dz \\ &\quad + \int_0^{+\infty} e^{-\nu(x,v)z}\,\frac{1}{\nu(x,v)}\varepsilon^2 v\cdot D_x^2\rho(x-\varepsilon vz)\cdot v\,dz \\ &\quad + \int_0^{+\infty} e^{-\nu(x,v)z}\,\varepsilon v\cdot\nabla_x\rho(x)\,dz. \end{aligned}$$

Using the symmetry assumption on F and ν, we deduce:

$$\begin{aligned} I_1^\varepsilon &= \varepsilon^{-\alpha}\int_{|v|\le 1}\nu(x,v)\int_0^{+\infty}\left[e^{-\int_0^z \nu(x-\varepsilon vs,v)\,ds} - e^{-\nu(x,v)z}\right]\varepsilon v\cdot\nabla_x\rho(x-\varepsilon vz)\,F(v)\,dz\,dv \\ &\quad +\varepsilon^{-\alpha}\int_{|v|\le 1}\int_0^{+\infty} e^{-\nu(x,v)z}\varepsilon^2 v\cdot D_x^2\rho(x-\varepsilon vz)\cdot v\,dz F(v)\,dv \end{aligned}$$

and since F is bounded, and

$$\left|e^{-\int_0^z \nu(x-s\varepsilon v,v)ds} - e^{-z\nu(x,v)}\right| \le Cz^2 e^{-\sigma_1 z}\varepsilon|v|$$

we deduce

$$||I_1^\varepsilon||_{L^2(\mathbb{R}^n)} \leq C\varepsilon^{2-\alpha}||\rho||_{H^2(\mathbb{R}^n)}$$

Next, Assumption (9) and the change of variable $w = \varepsilon v$ yields:

$$\begin{aligned} I_2^\varepsilon &= \varepsilon^{-\alpha} \int_{1\leq|v|\leq 1/\varepsilon} \int_0^{+\infty} e^{-\int_0^z \nu(x-\varepsilon vs,v)\,ds} \nu(x,v)\ \varepsilon \cdot \nabla_x \rho(x-\varepsilon vz) \frac{1}{|v|^{n+\alpha}}\,dz\,dv \\ &= \int_{\varepsilon\leq|w|\leq 1} \int_0^{+\infty} e^{-\int_0^z \nu(x-ws,w/\varepsilon)\,ds} \nu(x,w/\varepsilon)\ w\cdot \nabla_x \rho(x-wz) \frac{1}{|w|^{n+\alpha}}\,dz\,dw \end{aligned}$$

and proceeding as with I_1^ε, we deduce

$$\begin{aligned} I_2^\varepsilon &= \int_{\varepsilon\leq|w|\leq 1} \nu(x,w/\varepsilon) \int_0^{+\infty} \left[e^{-\int_0^z \nu(x-ws,w/\varepsilon)\,ds} - e^{-\nu(x,w/\varepsilon)z}\right]\ w\cdot\nabla_x\rho(x-wz)\ \frac{1}{|w|^{n+\alpha}}\,dz\,dw \\ &\quad + \int_{\varepsilon\leq|w|\leq 1} \int_0^{+\infty} e^{-\nu(x,w/\varepsilon)z} w\cdot D_x^2\rho(x-wz)\cdot w\,dz \frac{1}{|w|^{n+\alpha}}\,dw \end{aligned}$$

and so

$$||I_2^\varepsilon||_{L^2(\mathbb{R}^n)} \leq C||\rho||_{H^2(\mathbb{R}^n)}.$$

Finally, to estimate I_3^ε, we use the formula

$$\mathcal{T}_\varepsilon^{-1}(\varepsilon v\cdot\nabla_x\rho) = -\int_0^{+\infty} e^{-\int_0^z \nu(x-\varepsilon vs,v)\,ds} \nu(x-\varepsilon vz,v)\left[\rho(x-\varepsilon vz)-\rho(x)\right]dz.$$

Assumption (9) and the change of variable $w = \varepsilon v$ then yields:

$$I_3^\varepsilon = -\int_{|w|\geq 1} \int_0^{+\infty} e^{-\int_0^z \nu(x-ws,w/\varepsilon)\,ds} \nu(x,w/\varepsilon)\,\nu(x-wz,v)\left[\rho(x-wz)-\rho(x)\right]\frac{1}{|w|^{n+\alpha}}\,dz\,dw$$

and so

$$||I_3^\varepsilon||_{L^2(\mathbb{R}^n)} \leq C||\rho||_{L^2(\mathbb{R}^n)}.$$

We now show how to obtain higher order estimates: We have

$$\begin{aligned} \partial_{x_i} L^\varepsilon(\rho^\varepsilon) &= \varepsilon^{-\alpha}\int_{\mathbb{R}^n} \partial_{x_i}\nu(x,v)\mathcal{T}_\varepsilon^{-1}(\varepsilon v\cdot\nabla_x\rho)F(v)dv \\ &\quad -\varepsilon^{-\alpha}\int_{\mathbb{R}^n} \nu(x,v)\mathcal{T}_\varepsilon^{-1}(\partial_{x_i}\nu\mathcal{T}_\varepsilon^{-1}(\varepsilon v\cdot\nabla_x\rho))F(v)\,dv \\ &\quad +\varepsilon^{-\alpha}\int_{\mathbb{R}^n} \nu(x,v)\mathcal{T}_\varepsilon^{-1}(\varepsilon v\cdot\nabla_x\partial_{x_i}\rho)F(v)dv. \end{aligned}$$

The last term is just $L^\varepsilon(\partial_{x_i}\rho)$ which can thus be bounded by the previous step. The first term is treated in a same way as $L^\varepsilon(\rho)$ except that we need to use the fact that $\frac{\partial_{x_i}\nu}{\nu}$ is even and bounded.

Therefore, we will only detail the computations concerning the second term. We split it into three integrals corresponding respectively to $|v| < 1$, $1 < |v| < \frac{1}{\varepsilon}$ and $|v| > \frac{1}{\varepsilon}$.

First of all, we handle the $|v| > \frac{1}{\varepsilon}$ part. For that purpose, we use (9) and the usual change of variable. It leads to

$$
\begin{aligned}
I_1^\varepsilon &= -\varepsilon^{-\alpha} \int_{|v|>\frac{1}{\varepsilon}} \nu(x,v)\mathcal{T}_\varepsilon^{-1}(\partial_{x_i}\nu\mathcal{T}_\varepsilon^{-1}(\varepsilon v\cdot\nabla_x\rho))F(v)\,dv \\
&= -\int_{|w|>1} \nu(x,\frac{w}{\varepsilon}) \int_0^{+\infty} e^{-\int_0^z \nu(x-ws,\frac{w}{\varepsilon})\,ds} \partial_{x_i}\nu(x-wz,\frac{w}{\varepsilon}) \int_0^{+\infty} e^{-\int_0^{\overline{z}} \nu(x-w(s+z),\frac{w}{\varepsilon})\,ds} \\
&\qquad \nu(x-w(z+\overline{z}),\frac{w}{\varepsilon})[\rho(x-w(z+\overline{z}))-\rho(x-\varepsilon vz)]\frac{1}{|w|^{n+\alpha}}dvdzd\overline{z}.
\end{aligned}
$$

Thus,

$$||I_1^\varepsilon||_{L^2(\mathbb{R}^n)} \leq C||\rho||_{L^2(\mathbb{R}^n)}.$$

Next, for $|v|<1$, we write

$$
\begin{aligned}
I_2^\varepsilon &= -\varepsilon^{-\alpha} \int_{|v|<1} \nu(x,v)\mathcal{T}_\varepsilon^{-1}(\partial_{x_i}\nu\mathcal{T}_\varepsilon^{-1}(\varepsilon v\cdot\nabla_x\rho))F(v)\,dv \\
&= -\varepsilon^{-\alpha} \int_{|v|<1} \nu(x,v) \int_0^{+\infty} e^{-\int_0^z \nu(x-\varepsilon vs,v)\,ds} \partial_{x_i}\nu(x-\varepsilon vz,v) \int_0^{+\infty} e^{-\int_0^{\overline{z}} \nu(x-\varepsilon v(s+z),v)\,ds} \\
&\qquad \varepsilon v\cdot\nabla_x\rho(x-\varepsilon v(z+\overline{z}))F(v)dvdzd\overline{z}
\end{aligned}
$$

which can be written

$$
\begin{aligned}
I_2^\varepsilon &= -\varepsilon^{-\alpha} \int_{|v|<1} \nu(x,v) \int_0^{+\infty} e^{-\int_0^z \nu(x-\varepsilon vs,v)\,ds} \partial_{x_i}\nu(x-\varepsilon vz,v) \\
&\qquad \int_0^{+\infty} [e^{-\int_0^{\overline{z}} \nu(x-\varepsilon v(s+z),v)\,ds} - e^{-\overline{z}\nu(x,v)}]\varepsilon v\cdot\nabla_x\rho(x-\varepsilon v(z+\overline{z}))F(v)dvdzd\overline{z} \\
&\quad -\varepsilon^{-\alpha} \int_{|v|<1} \nu(x,v) \int_0^{+\infty} e^{-\int_0^z \nu(x-\varepsilon vs,v)\,ds} \partial_{x_i}\nu(x-\varepsilon vz,v) \int_0^{+\infty} e^{-\overline{z}\nu(x,v)} \\
&\qquad \varepsilon v\cdot\nabla_x\rho(x-\varepsilon v(z+\overline{z}))F(v)dvdzd\overline{z}.
\end{aligned}
$$

Since

$$[e^{-\int_0^{\overline{z}} \nu(x-\varepsilon v(s+z),v)\,ds} - e^{-\overline{z}\nu(x,v)}] \leq \varepsilon|v|(z+\overline{z})^2 e^{-\sigma_1\overline{z}},$$

we get

$$||I_2^\varepsilon||_{L^2(\mathbb{R}^n)} \;\leq\; ||I_{21}^\varepsilon||_{L^2(\mathbb{R}^n)} + \varepsilon^{2-\alpha}||\nabla\rho||_{L^2(\mathbb{R}^n)}$$

where

$$I_{21}^\varepsilon = -\varepsilon^{-\alpha} \int_{|v|<1} \nu(x,v) \int_0^{+\infty} e^{-\int_0^z \nu(x-\varepsilon vs,v)\,ds} \partial_{x_i}\nu(x-\varepsilon vz,v) \int_0^{+\infty} e^{-\overline{z}\nu(x,v)}\varepsilon v\cdot\nabla_x\rho(x-\varepsilon v(z+\overline{z}))F(v)dvdzd\overline{z}.$$

On the other hand

$$
\begin{aligned}
I_{21}^{\varepsilon} &= -\varepsilon^{-\alpha} \int_{|v|<1} \nu(x,v) \int_0^{+\infty} e^{-\int_0^z \nu(x-\varepsilon vs, v)\, ds} \partial_{x_i}\nu(x-\varepsilon vz, v) \int_0^{+\infty} e^{-\overline{z}\nu(x,v)}\, \varepsilon v \cdot \nabla_x \rho(x-\varepsilon v(z+\overline{z}))F(v) dv dz d\overline{z} \\
&= -\varepsilon^{-\alpha} \int_{|v|<1} \nu(x,v) \int_0^{+\infty} e^{-\int_0^z \nu(x-\varepsilon vs, v)\, ds} [\partial_{x_i}\nu(x-\varepsilon vz, v) - \partial_{x_i}\nu(x,v)] \int_0^{+\infty} e^{-\overline{z}\nu(x,v)}\, \varepsilon v \cdot \nabla_x \rho(x-\varepsilon v(z+\overline{z}))F(v) dv dz d\overline{z} \\
&\quad -\varepsilon^{-\alpha} \int_{|v|<1} \nu(x,v) \int_0^{+\infty} e^{-\int_0^z \nu(x-\varepsilon vs, v)\, ds} \partial_{x_i}\nu(x,v) \int_0^{+\infty} e^{-\overline{z}\nu(x,v)}\, \varepsilon v \cdot \nabla_x \rho(x-\varepsilon v(z+\overline{z}))F(v) dv dz d\overline{z}
\end{aligned}
$$

and then, since

$$
[\partial_{x_i}\nu(x-\varepsilon vz, v) - \partial_{x_i}\nu(x,v)] \leq \varepsilon |v| ||\nu||_{W^{2,\infty}},
$$

we have

$$
||I_{21}^{\varepsilon}||_{L^2(\mathbb{R}^n)} \leq ||I_{22}||_{L^2(\mathbb{R}^n)} + \varepsilon^{2-\alpha} ||\nabla \rho||_{L^2(\mathbb{R}^n)}
$$

where

$$
I_{22}^{\varepsilon} = \varepsilon^{-\alpha} \int_{|v|<1} \nu(x,v) \int_0^{+\infty} e^{-\int_0^z \nu(x-\varepsilon vs, v)\, ds} \partial_{x_i}\nu(x,v) \int_0^{+\infty} e^{-\overline{z}\nu(x,v)}\, \varepsilon v \cdot \nabla_x \rho(x-\varepsilon v(z+\overline{z}))F(v) dv dz d\overline{z}.
$$

In a same way

$$
||I_{22}^{\varepsilon}||_{L^2(\mathbb{R}^n)} \leq ||I_{23}^{\varepsilon}||_{L^2} + \varepsilon^{2-\alpha} ||\nabla \rho||_{L^2(\mathbb{R}^n)}
$$

with

$$
I_{23}^{\varepsilon} = -\varepsilon^{-\alpha} \int_{|v|<1} \nu(x,v) \int_0^{+\infty} e^{-z\nu(x,v)}\, \partial_{x_i}\nu(x,v) \int_0^{+\infty} e^{-\overline{z}\nu(x,v)}\, \varepsilon v \cdot \nabla_x \rho(x-\varepsilon v(z+\overline{z}))F(v) dv dz d\overline{z}
$$

An integration by part in z gives

$$
\begin{aligned}
I_{23}^{\varepsilon} \quad = \quad & -\varepsilon^{-\alpha} \int_{|v|<1} \int_0^{+\infty} e^{-z\nu(x,v)}\, \partial_{x_i}\nu(x,v) \int_0^{+\infty} e^{-\overline{z}\nu(x,v)}\, \varepsilon^2 v \cdot D^2 \rho(x-\varepsilon v(z+\overline{z})) v F(v) dv dz d\overline{z} \\
& -\varepsilon^{-\alpha} \int_{|v|<1} \partial_{x_i}\nu(x,v) \int_0^{+\infty} e^{-\overline{z}\nu(x,v)}\, \varepsilon v \cdot \nabla_x \rho(x-\varepsilon v(\overline{z}))F(v) dv dz d\overline{z}.
\end{aligned}
$$

Another integration by parts gives

$$
\begin{aligned}
& -\varepsilon^{-\alpha} \int_{|v|<1} \partial_{x_i}\nu(x,v) \int_0^{+\infty} e^{-\overline{z}\nu(x,v)}\, \varepsilon v \cdot \nabla_x \rho(x-\varepsilon v(\overline{z}))F(v) dv dz d\overline{z} \\
= \quad & -\varepsilon^{-\alpha} \int_{|v|<1} \frac{\partial_{x_i}\nu(x,v)}{\nu(x,v)} \int_0^{+\infty} e^{-\overline{z}\nu(x,v)}\, \varepsilon^2 v \cdot D^2 \rho(x-\varepsilon v(\overline{z})) v F(v) dv dz d\overline{z} \\
& -\varepsilon^{-\alpha} \int_{|v|<1} \frac{\partial_{x_i}\nu(x,v)}{\nu(x,v)} \varepsilon v \cdot \nabla_x \rho(x) F(v) dv
\end{aligned}
$$

in which the last term vanishes (by symmetries).

Finally, we get

$$
||I_2^{\varepsilon}||_{L^2(\mathbb{R}^n)} \leq C ||\rho||_{H^2(\mathbb{R}^n)}
$$

and the same computations for $\varepsilon < |w| < 1$ where w is the change of variable $w = \dfrac{v}{\varepsilon}$ reads

$$
||I_3^{\varepsilon}||_{L^2(\mathbb{R}^n)} \leq C ||\rho||_{H^2(\mathbb{R}^n)}
$$

where

$$I_3^\varepsilon = -\varepsilon^{-\alpha} \int_{1<|v|<\frac{1}{\varepsilon}} \nu(x,v) \mathcal{T}_\varepsilon^{-1}(\partial_{x_i} \nu \mathcal{T}_\varepsilon^{-1}(\varepsilon v \cdot \nabla_x \rho)) F(v)\, dv.$$

We obtain finally,

$$||\partial_{x_i} L^\varepsilon(\rho)||_{L^2(\mathbb{R}^n)} \leq C(||L^\varepsilon(\rho)||_{L^2(\mathbb{R}^n)} + C(\sum_{|k|=2} ||\partial_x^k \rho||_{L^2(\mathbb{R}^n)} + \sum_{|k|=3} ||\partial^k \rho||_{L^2(\mathbb{R}^n)}).$$

Proceeding similarly, we can show (43) for all $k \geq 2$. □

Proposition 3.2. *For all $\rho_{in} \in H^k(\mathbb{R}^n)$ (for some $k \geq 0$), the equation*

$$\begin{cases} \partial_t \rho + L^\varepsilon(\rho) = 0 & \text{in } \mathbb{R}_+ \times \mathbb{R}^n \\ \rho(0,x) = \rho_{in} & \text{in } \mathbb{R}^n \end{cases} \tag{45}$$

has a unique solution solution $\rho^\varepsilon \in L^\infty(0,\infty; H^k(\mathbb{R}^n))$ for all $\varepsilon > 0$. Furthermore, if $k \geq 2$, then

$$||\rho^\varepsilon||_{L^\infty([0,\infty),H^k(\mathbb{R}^n))} \leq C \quad \text{and}$$

$$||L^\varepsilon(\rho^\varepsilon)||_{L^\infty([0,\infty),H^{k-2}(\mathbb{R}^n))} = ||\partial_t \rho^\varepsilon||_{L^\infty([0,\infty),H^{k-2}(\mathbb{R}^n))} \leq C,$$

with C independent of ε.

Proof. First, we have to prove that ρ^ε exists for $\varepsilon > 0$. For a fixed $\varepsilon > 0$, the operator L^ε (defined by (42)) is clearly a bounded operator on $L^2(\mathbb{R}^n)$. Furthermore, we have:

$$\begin{aligned}
\int_{\mathbb{R}^n} L^\varepsilon(\rho)\rho dx &= \varepsilon^{-\alpha} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \rho\, \nu \mathcal{T}_\varepsilon^{-1}(\varepsilon v \cdot \nabla_x \rho) F\, dv\, dx \\
&= -\varepsilon^{-\alpha} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \rho\, \varepsilon v \cdot \nabla_x \big(\mathcal{T}_\varepsilon^{-1}(\varepsilon v \cdot \nabla_x \rho)\big) F\, dv\, dx \\
&\quad + \varepsilon^{-\alpha} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \rho\, \mathcal{T}_\varepsilon \mathcal{T}_\varepsilon^{-1}(\varepsilon v \cdot \nabla_x \rho) F\, dv\, dx \\
&= \varepsilon^{-\alpha} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \varepsilon v \cdot \nabla_x \rho\, \mathcal{T}_\varepsilon^{-1}(\varepsilon v \cdot \nabla_x \rho) F\, dv\, dx \\
&= \varepsilon^{-\alpha} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \mathcal{T}_\varepsilon(g_1)\, g_1 \frac{1}{F}\, dv\, dx
\end{aligned}$$

where $g_1 = -\mathcal{T}_\varepsilon^{-1}(\varepsilon v \cdot \nabla_x \rho) F$. Hence

$$\begin{aligned}
\int_{\mathbb{R}^n} L^\varepsilon(\rho)\, \rho\, dx &= \varepsilon^{-\alpha} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |g_1|^2 \frac{\nu}{F}\, dv\, dx + \varepsilon^{-\alpha} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} (\varepsilon v \cdot \nabla g_1)\, g_1 \frac{1}{F}\, dv\, dx \\
&= \varepsilon^{-\alpha} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |g_1|^2 \frac{\nu}{F}\, dv\, dx \\
&\geq 0.
\end{aligned} \tag{46}$$

In particular, the operator L^ε is monotone and Lax Milgram's theorem implies that for any $f \in L^2(\mathbb{R}^n)$, the equation

$$\rho + L^\varepsilon(\rho) = f \tag{47}$$

has a solution in $L^2(\mathbb{R}^n)$. We deduce that L^ε is maximal monotone, and thus Hille-Yosida's theorem implies that there exists a unique ρ^ε in $C^0(\mathbb{R}^+, L^2(\mathbb{R}^n)) \cap C^1(\mathbb{R}^{+*}, L^2(\mathbb{R}^n))$ solution of (45).

We now prove the uniform estimates on $||\rho^\varepsilon||_{L^\infty([0,\infty],H^k(\mathbb{R}^n))}$. Note that since $\partial_t\rho^\varepsilon = -L^\varepsilon(\rho^\varepsilon)$ satisfy the same equation as ρ^ε with initial condition $\partial_t\rho^\varepsilon(0,\cdot) = -L^\varepsilon(\rho_0) \in H^{k-2}(\mathbb{R}^n)$, the other estimates will then follow.

Multiplying equation Equation (41) by ρ^ε and using (46), we now get:

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}||\rho^\varepsilon||^2_{L^2(\mathbb{R}^n)} &= \int_{\mathbb{R}^n} \partial_t\rho^\varepsilon\rho^\varepsilon dx \\
&= -\int_{\mathbb{R}^n} L^\varepsilon(\rho^\varepsilon)\,\rho^\varepsilon\,dx \\
&= -\varepsilon^{-\alpha}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |g_1^\varepsilon|^2\frac{\nu}{F}\,dv\,dx.
\end{aligned}$$

Since ρ_0^ε is bounded in $L^2(\mathbb{R}^n)$, this implies in particular

$$||\rho^\varepsilon||_{L^\infty(0,\infty;L^2(\mathbb{R}^n))} \leq C$$

and

$$||g_1^\varepsilon||_{L^2(0,\infty;L^2_{F^{-1}}(\mathbb{R}^n\times\mathbb{R}^n))} \leq C\varepsilon^{\frac{\alpha}{2}}. \tag{48}$$

Next, we show how one gets a similar estimate on $\nabla_x\rho^\varepsilon$: We write

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}||\nabla_x\rho^\varepsilon||^2_{L^2(\mathbb{R}^n)} &= \int_{\mathbb{R}^n}\partial_t\nabla_x\rho^\varepsilon\cdot\nabla_x\rho^\varepsilon dx = -\int_{\mathbb{R}^n}\nabla_x\rho^\varepsilon\cdot\nabla_x L^\varepsilon(\rho^\varepsilon)dx \\
&= -\varepsilon^{-\alpha}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\nabla_x\rho^\varepsilon\cdot\nabla_x\left(\nu\mathcal{T}_\varepsilon^{-1}(\varepsilon v\cdot\nabla_x\rho^\varepsilon)\right)F\,dv\,dx \\
&= -\varepsilon^{-\alpha}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\nabla_x\rho^\varepsilon\cdot\nabla_x\left(\mathcal{T}_\varepsilon\mathcal{T}_\varepsilon^{-1}(\varepsilon v\cdot\nabla_x\rho^\varepsilon)\right)F\,dv\,dx \\
&\quad +\varepsilon^{-\alpha}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\nabla_x\rho^\varepsilon\cdot\nabla_x\left(\varepsilon v\cdot\nabla_x(\mathcal{T}_\varepsilon^{-1}(\varepsilon v\cdot\nabla_x\rho^\varepsilon))\right)F\,dv\,dx \\
&= -\varepsilon^{-\alpha}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\nabla_x(\varepsilon v\cdot\nabla_x\rho^\varepsilon)\cdot\nabla_x\left(\mathcal{T}_\varepsilon^{-1}(\varepsilon v\cdot\nabla_x\rho^\varepsilon)\right)F\,dv\,dx.
\end{aligned}$$

Which gives (since $g_1^\varepsilon = \mathcal{T}_\varepsilon^{-1}(\varepsilon v\cdot\nabla_x\rho^\varepsilon)F$):

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}||\nabla_x\rho^\varepsilon||^2_{L^2(\mathbb{R}^n)} &= -\varepsilon^{-\alpha}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\nabla_x((\nu+\varepsilon v\cdot\nabla_x)g_1^\varepsilon)\cdot\nabla_x g_1^\varepsilon\frac{1}{F}\,dv\,dx \\
&= -\varepsilon^{-\alpha}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\nabla_x(\nu g_1^\varepsilon)\cdot\nabla_x g_1^\varepsilon\frac{1}{F}\,dv\,dx \\
&= -\varepsilon^{-\alpha}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}(\nu|\nabla_x g_1^\varepsilon|^2+\nabla_x\nu\cdot\nabla_x g_1^\varepsilon g_1^\varepsilon)\frac{1}{F}\,dv\,dx \\
&\leq -\frac{\sigma_0}{2}\varepsilon^{-\alpha}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|\nabla_x g_1^\varepsilon|^2\frac{1}{F}\,dv\,dx + C\varepsilon^{-\alpha}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}g_1^{\varepsilon 2}\frac{1}{F}\,dv\,dx
\end{aligned}$$

where we used (7) and (8). Using (48), we deduce:

$$||\nabla_x\rho^\varepsilon||^2_{L^\infty(0,\infty;L^2(\mathbb{R}^n))} \leq C,$$

and

$$||\nabla_x g_1^\varepsilon||_{L^2(0,\infty;L^2_{F^{-1}}(\mathbb{R}^n\times\mathbb{R}^n))} \leq C\varepsilon^{\frac{\alpha}{2}}.$$

Proceeding similarly, we can show that for any multi-indice $k=(k_1,k_2,...,k_n)$, we have:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}||\partial_x^k\rho^\varepsilon||^2_{L^2(\mathbb{R}^n)} &\leq -\varepsilon^{-\alpha}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\nu|\partial_x^k g_1^\varepsilon|^2\,\frac{1}{F}\,dv\,dx + C\sum_{1\leq|i|\leq|k|}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|\partial_x^i\nu||\partial_x^{k-i}g_1^\varepsilon||\partial_x^k g_1^\varepsilon)|\,\frac{1}{F}\,dv\,dx\\ &\leq -\frac{\sigma_0}{2}\varepsilon^{-\alpha}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|\partial_x^k g_1^\varepsilon|^2\,\frac{1}{F}\,dv\,dx + C\sum_{1\leq|i|\leq|k|}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|\partial_x^{k-i}g_1^\varepsilon|^2\,\frac{1}{F}\,dv\,dx.\end{aligned}$$

By induction on $|k|$, we deduce that

$$||\partial_x^k\rho^\varepsilon||^2_{L^\infty(0,\infty;L^2(\mathbb{R}^n))} \leq C,$$

and

$$||\partial_x^k g_1^\varepsilon||_{L^2(0,\infty;L^2_{F^{-1}}(\mathbb{R}^n\times\mathbb{R}^n))} \leq C\varepsilon^{\frac{\alpha}{2}}.$$

□

Remarks 3.3. *Note that as usual with the heat equation, it does not seem possible to derive estimates in $L^1(\mathbb{R}^n)$ (and such estimate do not follow from L^2 estimates in unbounded domain). This is the reason why in our main result, Theorem 1.1, we obtain convergence in $L^2_{F^{-1}}$ rather than L^1.*

3.3 Some estimates for the collision operator

In this section, we establish a couple of auxiliary results concerning the operator Q that will be needed in the next section. First, we prove the following consequence of Proposition 2.2:

Corollary 3.4. *Let $h(v)$ be such that $\int_{\mathbb{R}^n} h(v)dv = 0$ and $h/F\in L^\infty(\mathbb{R}^n)$, and let $g=Q^{-1}(h)$. Then $g/F\in L^\infty(\mathbb{R}^n)$ and there exists a constant K such that*

$$\|g/F\|_{L^\infty(\mathbb{R}^n)} \leq K\|h/F\|_{L^\infty(\mathbb{R}^n)}.$$

Proof. We have

$$\|h\|_{L^2_{F^{-1}}(\mathbb{R}^n)} \leq \|h/F\|_{L^\infty(\mathbb{R}^n)},$$

and so Proposition 2.2 implies that $g=Q^{-1}(h)$ exists and satisfies

$$\|g\|_{L^2_{F^{-1}}(\mathbb{R}^n)} \leq C\|h\|_{L^2_{F^{-1}}(\mathbb{R}^n)} \leq C\|h/F\|_{L^\infty(\mathbb{R}^n)}.$$

Next, we can write

$$g=\frac{1}{\nu}\left[\int_{\mathbb{R}^n}\sigma(x,v,v')g(v')\,dv'\,F(v)+h(v)\right].$$

We then deduce that

$$\begin{aligned}|g(v)| &\leq C(\|g\|_{L^1(\mathbb{R}^n)}F(v)+|h(v)|)\\ &\leq C(\|g\|_{L^2_{F^{-1}}(\mathbb{R}^n)}F(v)+|h(v)|)\\ &\leq C\|h/F\|_{L^\infty(\mathbb{R}^n)}F(v),\end{aligned}$$

which completes the proof. □

Next, we prove:

Proposition 3.5. *There exists C such that for all function $h(x,v)$ we have*

$$\left\| \frac{Q^+(\mathcal{T}_\varepsilon^{-1}(\varepsilon v\cdot\nabla_x h)F)}{F}\right\|_{L^2_x(\mathbb{R}^n;L^\infty_v(\mathbb{R}^n))} \leq C\varepsilon^\alpha ||h||_{L^2_x(\mathbb{R}^n;L^\infty_v(\mathbb{R}^n))} + C(\varepsilon^\alpha+\varepsilon)||\nabla_x h||_{L^2_x(\mathbb{R}^n;L^\infty_v(\mathbb{R}^n))}$$

for all $\varepsilon > 0$.

Proof. We start by noticing that

$$\begin{aligned} &Q^+(\mathcal{T}_\varepsilon^{-1}(\varepsilon v\cdot\nabla_x h)F)(x,v)\\ &= \int_{\mathbb{R}^n}\sigma(x,v,v')\int_0^{+\infty} e^{-\int_0^z \nu(x-\varepsilon s v',v')\,ds}\varepsilon v'\cdot\nabla_x h(x-\varepsilon z v',v')F(v')dzdv'F(v)\end{aligned}$$

is very similar to $\varepsilon^\alpha L^\varepsilon(h)$ (which is of order ε^α), though the lack of symmetry with respect to v prevents us from using the result directly (and explain why this term is of smaller order).

In particular, it is still natural to split this integral into three parts:

$$Q^+(\mathcal{T}_\varepsilon^{-1}(\varepsilon v\cdot\nabla_x h)F)(x,v) = (I_1^\varepsilon + I_2^\varepsilon + I_3^\varepsilon)F(v)$$

where

$$\begin{aligned} I_1^\varepsilon &= \int_{|v'|\leq 1}\sigma(x,v,v')\,\mathcal{T}_\varepsilon^{-1}(\varepsilon v'\cdot\nabla_x h)\,F(v')\,dv',\\ I_2^\varepsilon &= \int_{1\leq|v'|\leq 1/\varepsilon}\sigma(x,v,v')\,\mathcal{T}_\varepsilon^{-1}(\varepsilon v'\cdot\nabla_x h)\,F(v')\,dv',\\ I_3^\varepsilon &= \int_{|v'|\geq 1/\varepsilon}\sigma(x,v,v')\,\mathcal{T}_\varepsilon^{-1}(\varepsilon v'\cdot\nabla_x h)\,F(v')\,dv'.\end{aligned}$$

We then note that

$$I_1^\varepsilon = \int_{|v'|\leq 1}\sigma(x,v,v')\int_0^{+\infty} e^{-\int_0^z \nu(x-\varepsilon v' s,v')\,ds}\,\varepsilon v'\cdot\nabla_x h(x-\varepsilon v' z,v)\,F(v')\,dz\,dv'$$

and so

$$|I_1^\varepsilon(v)| \leq C\varepsilon\int_{|v'|\leq 1}\int_0^{+\infty} e^{-\sigma_0 z}\,||\nabla_x h(x-\varepsilon v' z,\cdot)||_{L^\infty_v}\,dz\,dv'.$$

We deduce

$$||I_1^\varepsilon||_{L^2_x(\mathbb{R}^n;L^\infty_v(\mathbb{R}^n)))} \leq C\varepsilon||\nabla_x h||_{L^2_x(\mathbb{R}^n;L^\infty_v(\mathbb{R}^n))}.$$

Next, Assumption (9) and the change of variable $w'=\varepsilon v'$ yields:

$$\begin{aligned} I_2^\varepsilon &= \int_{1\leq|v'|\leq 1/\varepsilon}\int_0^{+\infty}\sigma(x,v,v')e^{-\int_0^z \nu(x-\varepsilon v' s,v')\,ds}\,\varepsilon v'\cdot\nabla_x h(x-\varepsilon v' z,v')\frac{1}{|v'|^{n+\alpha}}\,dz\,dv'\\ &= \varepsilon^\alpha\int_{\varepsilon\leq|w'|\leq 1}\sigma(x,v,\frac{w'}{\varepsilon})\int_0^{+\infty} e^{-\int_0^z \nu(x-w's,\frac{w'}{\varepsilon})\,ds}\,w'\cdot\nabla_x h(x-w'z,\frac{w'}{\varepsilon})\frac{1}{|w'|^{n+\alpha}}\,dz\,dw',\end{aligned}$$

and so

$$|I_2^\varepsilon(v)| \;=\; C\varepsilon^\alpha \int_{\varepsilon\leq|w'|\leq 1}\int_0^{+\infty} e^{-\sigma_0 z}||\nabla_x h(x-w'z,\cdot)||_{L^\infty_v} \frac{|w'|}{|w'|^{n+\alpha}}\,dz\,dw'$$

Using the fact that $\int_{\varepsilon\leq|w'|\leq 1}\frac{|w'|}{|w'|^{n+\alpha}}\,dw' \leq C(1+\varepsilon^{1-\alpha})$, we deduce

$$\begin{aligned}||I_2^\varepsilon||_{L^2_x(\mathbb{R}^n;L^\infty_v(\mathbb{R}^n))} &\leq C\varepsilon^\alpha\left(\int_{\varepsilon\leq|w'|\leq 1}\frac{|w'|}{|w'|^{n+\alpha}}\,||\nabla_x h||^2_{L^2_x(\mathbb{R}^n;L^\infty_v(\mathbb{R}^n))}\,dw'\right)^{1/2}\\ &\qquad\times\left(\int_{\varepsilon\leq|w'|\leq 1}\frac{|w'|}{|w'|^{n+\alpha}}\,dw'\right)^{1/2}\\ &\leq C(\varepsilon+\varepsilon^\alpha)||\nabla_x h||_{L^2_x(\mathbb{R}^n;L^\infty_v(\mathbb{R}^n))}.\end{aligned}$$

Finally, to estimate I_3^ε, we use the formula

$$\mathcal{T}_\varepsilon^{-1}(\varepsilon v\cdot\nabla_x h) = -\int_0^{+\infty} e^{-\int_0^z \nu(x-\varepsilon v s,v)\,ds}\nu(x-\varepsilon v z,v)\left[h(x-\varepsilon v z,v)-h(x)\right]dz.$$

Assumption (9) and the change of variable $w=\varepsilon v$ then yields:

$$I_3^\varepsilon = -\varepsilon^\alpha\int_{|w'|\geq 1}\int_0^{+\infty} e^{-\int_0^z \nu(x-w's,\frac{w'}{\varepsilon})\,ds}\nu(x,\frac{w'}{\varepsilon})\,\nu(x-w'z,v)\left[h(x-w'z,\frac{w'}{\varepsilon})-h(x,\frac{w'}{\varepsilon})\right]\frac{1}{|w'|^{n+\alpha}}\,dz\,dw'$$

and so

$$||I_3^\varepsilon||_{L^2_x(\mathbb{R}^n;L^\infty_v(\mathbb{R}^n))} \leq C\varepsilon^\alpha||h||_{L^2_x(\mathbb{R}^n;L^\infty_v(\mathbb{R}^n))}.$$

□

3.4 Convergence of $f^\varepsilon - \rho^\varepsilon F$

We are now ready to prove that $f^\varepsilon-\rho^\varepsilon F$ converges to zero. More precisely, we are going to show the following proposition:

Proposition 3.6. *Let f^ε be a solution of (32) with $f^\varepsilon(0,x,v)=\rho_{in}(x)F(v)$ and ρ^ε be the solution of (45). Then*

$$||f^\varepsilon-\rho^\varepsilon F(v)||_{L^\infty(0,\infty;L^2_{F^{-1}}(\mathbb{R}^n\times\mathbb{R}^n))} \leq C(\varepsilon^{\alpha/2}+\varepsilon^{1-\alpha/2})||\rho||_{L^\infty(0,\infty;H^4(\mathbb{R}^n))}.$$

Proof. The expansion (33) gives

$$||f^\varepsilon-\rho^\varepsilon F(v)||_{L^\infty(0,\infty;L^2_{F^{-1}}(\mathbb{R}^{2n}))} \;\leq\; ||g_1^\varepsilon||_{L^\infty L^2_{F^{-1}}}+||g_2^\varepsilon||_{L^\infty L^2_{F^{-1}}}+||g_3^\varepsilon||_{L^\infty L^2_{F^{-1}}}+||r^\varepsilon||_{L^\infty L^2_{F^{-1}}} \tag{49}$$

Furthermore, the remainder r^ε can be estimated by multiplying the equation (37) by r^ε/F and integrating with respect to x and v. Proposition 2.2 (iii) then yields

$$\begin{aligned}||r^\varepsilon||_{L^\infty L^2_{F^{-1}}} &\leq ||\partial_t g_1^\varepsilon||_{L^1(0,\infty;L^2_{F^{-1}}(\mathbb{R}^{2n}))}+||\partial_t g_2^\varepsilon||_{L^1(0,\infty;L^2_{F^{-1}}(\mathbb{R}^{2n}))}\\ &\quad+||\partial_t g_3^\varepsilon||_{L^1(0,\infty;L^2_{F^{-1}}(\mathbb{R}^{2n}))}+\varepsilon^{-\alpha}||Q^+(g_3^\varepsilon)||_{L^1(0,\infty;L^2_{F^{-1}}(\mathbb{R}^{2n}))} \qquad (50)\\ &\quad+||g_1^\varepsilon(0,\cdot)||_{L^2_{F^{-1}}(\mathbb{R}^{2n})}+||g_2^\varepsilon(0,\cdot)||_{L^2_{F^{-1}}(\mathbb{R}^{2n})}+||g_3^\varepsilon(0,\cdot)||_{L^2_{F^{-1}}(\mathbb{R}^{2n})}.\end{aligned}$$

We now see that Proposition 3.6 will be proved if we can show that all the norms arising in the right hand side of (49) and (50) go to zero.

All the necessary estimates are collected in a sequence of Lemmas below, which altogether complete the proof of Proposition 3.6. □

Lemma 3.7. *There exists a constant C such that*

$$||g_1^\varepsilon||_{L^\infty(0,\infty;L^2_{F^{-1}}(\mathbb{R}^{2n}))} \leq C\varepsilon^{\frac{\alpha}{2}}||\rho^\varepsilon||_{L^\infty(0,\infty;H^1(\mathbb{R}^n))}$$

and

$$||\partial_t g_1^\varepsilon||_{L^\infty(0,\infty;L^2_{F^{-1}}(\mathbb{R}^{2n}))} \leq C\varepsilon^{\frac{\alpha}{2}}||\rho^\varepsilon||_{L^\infty(0,\infty;H^3(\mathbb{R}^n))}.$$

Lemma 3.8. *There exists a constant C such that*

$$\begin{aligned} ||g_2^\varepsilon||_{L^\infty(0,\infty;L^2_{F^{-1}}(\mathbb{R}^{2n}))} &\leq ||g_2^\varepsilon/F||_{L^\infty(0,\infty;L^2_x(\mathbb{R}^n;L^\infty_v(\mathbb{R}^n)))} \\ &\leq C(\varepsilon^\alpha+\varepsilon)||\rho^\varepsilon||_{L^\infty(0,\infty;H^2(\mathbb{R}^n))} \end{aligned}$$

and

$$||\partial_t g_2^\varepsilon||_{L^\infty(0,\infty;L^2_{F^{-1}}(\mathbb{R}^{2n}))} \leq C(\varepsilon^\alpha+\varepsilon)||\rho^\varepsilon||_{L^\infty(0,\infty;H^4(\mathbb{R}^n))}.$$

Lemma 3.9. *There exists a constant C such that*

$$||g_3^\varepsilon||_{L^\infty(0,\infty;L^2_{F^{-1}}(\mathbb{R}^{2n}))} \leq C(\varepsilon^\alpha+\varepsilon)||\rho^\varepsilon||_{L^\infty(0,\infty;H^2(\mathbb{R}^n))}$$

$$||\partial_t g_3^\varepsilon||_{L^\infty(0,\infty;L^2_{F^{-1}}(\mathbb{R}^{2n}))} \leq C(\varepsilon^\alpha+\varepsilon)||\rho^\varepsilon||_{L^\infty(0,\infty;H^4(\mathbb{R}^n))}$$

and

$$||Q^+(g_3^\varepsilon)||_{L^\infty(0,\infty;L^2_{F^{-1}}(\mathbb{R}^{2n}))} \leq C\varepsilon^\alpha[\varepsilon^{\alpha/2}+\varepsilon^{1-\alpha/2}]||\rho^\varepsilon||_{L^\infty(0,\infty;H^2(\mathbb{R}^n))}.$$

The proof of this last lemma will require the following estimate:

Lemma 3.10. *There exists a constant C such that*

$$\left\|\frac{\nabla_x g_2^\varepsilon}{F}\right\|_{L^\infty(0,\infty;L^2_x(\mathbb{R}^n;L^\infty_v(\mathbb{R}^n))} \leq C(\varepsilon^\alpha+\varepsilon)||\rho^\varepsilon||_{L^\infty(0,\infty;H^2(\mathbb{R}^n))} + C\varepsilon^{\alpha/2}||\rho^\varepsilon||_{L^\infty(0,\infty;H^1(\mathbb{R}^n))}.$$

The rest of this section is devoted to the proof of these lemmas.

Proof of Lemma 3.7. We recall that (48) yields

$$||g_1^\varepsilon||_{L^2(0,\infty;L^2_{F^{-1}}(\mathbb{R}^{2n})))} \leq C\varepsilon^{\frac{\alpha}{2}}.$$

However, we have to work a bit more to get an estimate uniformly in time.

We recall that

$$\begin{aligned} g_1^\varepsilon(t,x,v) &= \mathcal{T}_\varepsilon^{-1}(\varepsilon v\cdot\nabla_x\rho^\varepsilon)F \\ &= -\varepsilon\int_0^{+\infty} e^{-\int_0^z(\nu(x-\varepsilon sv,v)\,ds}v\cdot\nabla_x\rho^\varepsilon(x-\varepsilon zv)F(v)dz. \end{aligned} \tag{51}$$

Therefore

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{g_1^\varepsilon(t,x,v)^2}{F} dvdx$$

$$\leq C\varepsilon^\alpha \int_{\mathbb{R}^n}\int_{|w|>\varepsilon} \left(\int_0^{+\infty} e^{-\int_0^z (\nu(x-sw,\frac{w}{\varepsilon})\,ds} w\cdot\nabla_x\rho^\varepsilon(x-zw)dz\right)^2 F\left(\frac{w}{\varepsilon}\right)\frac{dw\,dx}{\varepsilon^{n+\alpha}}$$

$$+\,\varepsilon^2 \int_{\mathbb{R}^n}\int_{|v|\leq 1} \left(\int_0^{+\infty} e^{-\int_0^z (\nu(x-\varepsilon sv,v)\,ds} v\cdot\nabla_x\rho^\varepsilon(x-\varepsilon zv)dz\right)^2 F(v)dv\,dx.$$

Splitting the first integral and integrating by parts, we get:

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{g_1^\varepsilon(t,x,v)^2}{F} dvdx$$

$$\leq \varepsilon^\alpha \int_{\mathbb{R}^n}\int_{|w|>1} \left(\int_0^{+\infty} \nu(x-zw,\frac{w}{\varepsilon}) e^{-\int_0^z (\nu(x-sw,\frac{w}{\varepsilon})\,ds}[\rho^\varepsilon(x-zw)-\rho^\varepsilon(x)]dz\right)^2 \frac{1}{|w|^{n+\alpha}}dw\,dx$$

$$+\,\varepsilon^\alpha \int_{\mathbb{R}^n}\int_{1>|w|>\varepsilon} \left(\int_0^{+\infty} e^{-\int_0^z (\nu(x-sw,\frac{w}{\varepsilon})\,ds} w\cdot\nabla_x\rho^\varepsilon(x-zw)dz\right)^2 \frac{1}{|w|^{n+\alpha}}dw\,dx$$

$$+\,\varepsilon^2 C \int_{\mathbb{R}^n}\int_{|v|\leq 1} \left(\int_0^{+\infty} e^{-\sigma_0 z}|\nabla_x\rho^\varepsilon(x-\varepsilon zv)|dz\right)^2 dvdx$$

and so

$$||g_1^\varepsilon||_{L^2_{F^{-1}}(\mathbb{R}^{2n})} \leq C(\varepsilon^{\frac{\alpha}{2}}||\rho^\varepsilon||_{L^2(\mathbb{R}^n)} + (\varepsilon^{\frac{\alpha}{2}}+\varepsilon)||\nabla_x\rho^\varepsilon||_{L^2(\mathbb{R}^n)})$$

which gives the first inequality in Lemma 3.7.

Differentiating (51) with respect to t (note that only ρ^ε depends on t) and proceeding similarly, we deduce:

$$\begin{aligned} ||\partial_t g_1^\varepsilon||_{L^2F^{-1}(\mathbb{R}^{2n})} &\leq C(\varepsilon^{\frac{\alpha}{2}}||\partial_t\rho^\varepsilon||_{L^2(\mathbb{R}^n)} + (\varepsilon^{\frac{\alpha}{2}}+\varepsilon)||\partial_t\nabla_x\rho^\varepsilon||_{L^2(\mathbb{R}^n)}) \\ &\leq C(\varepsilon^{\frac{\alpha}{2}}||\rho^\varepsilon||_{H^2(\mathbb{R}^n)} + (\varepsilon^{\frac{\alpha}{2}}+\varepsilon)||\nabla_x\rho^\varepsilon||_{H^2(\mathbb{R}^n)}). \end{aligned}$$

□

Proof of Lemma 3.8. We recall that (35) gives

$$\begin{aligned} g_2^\varepsilon &= Q^{-1}\big(-Q^+(g_1^\varepsilon) - \varepsilon^\alpha\partial_t\rho^\varepsilon F\big) \\ &= Q^{-1}\big(-Q^+(\mathcal{T}_\varepsilon^{-1}(\varepsilon v\cdot\nabla_x\rho^\varepsilon)F) + \varepsilon^\alpha L^\varepsilon(\rho^\varepsilon)F\big) \end{aligned}$$

and so Corollary 3.4 yields

$$||g_2^\varepsilon/F||_{L^\infty_v(\mathbb{R}^n)} \leq C||-Q^+(\mathcal{T}_\varepsilon^{-1}(\varepsilon v\cdot\nabla_x\rho^\varepsilon)F)/F + \varepsilon^\alpha L^\varepsilon(\rho^\varepsilon)||_{L^\infty_v(\mathbb{R}^n)}$$

and so

$$\begin{aligned} ||g_2^\varepsilon/F||_{L^2_x(\mathbb{R}^n;L^\infty_v(\mathbb{R}^n))} &\leq C||Q^+(\mathcal{T}_\varepsilon^{-1}(\varepsilon v\cdot\nabla_x\rho^\varepsilon)F)/F||_{L^2_x(\mathbb{R}^n;L^\infty_v(\mathbb{R}^n))} + \varepsilon^\alpha||L^\varepsilon(\rho^\varepsilon)||_{L^2(\mathbb{R}^n)} \\ &\leq C||Q^+(\mathcal{T}_\varepsilon^{1}(\varepsilon v\cdot\nabla_x\rho^\varepsilon)F)/F||_{L^2_x(\mathbb{R}^n;L^\infty_v(\mathbb{R}^n))} + \varepsilon^\alpha||\rho^\varepsilon||_{H^2(\mathbb{R}^n)} \end{aligned}$$

Furthermore, Proposition 3.5 implies

$$||Q^+(\mathcal{T}_\varepsilon^{-1}(\varepsilon v\cdot\nabla_x\rho^\varepsilon)F)/F||_{L^2_x(\mathbb{R}^n;L^\infty_v(\mathbb{R}^n))}\leq C(\varepsilon+\varepsilon^\alpha)||\rho^\varepsilon||_{H^1(\mathbb{R}^n)}$$

which yields the first estimate.

Next, we write

$$\partial_t g_2^\varepsilon \quad = \quad Q^{-1}\big(-Q^+(\mathcal{T}_\varepsilon^{-1}(\varepsilon v\cdot\nabla_x\partial_t\rho^\varepsilon)F)+\varepsilon^\alpha L^\varepsilon(\partial_t\rho^\varepsilon)F\big)$$

and so the same argument as above yields

$$||\partial_t g_2^\varepsilon||_{L^\infty(0,\infty;L^2_{F^{-1}}(\mathbb{R}^{2n}))}\leq C(\varepsilon^\alpha+\varepsilon)||\partial_t\rho^\varepsilon||_{L^\infty(0,\infty;H^2(\mathbb{R}^n))}.$$

Since $\partial_t\rho^\varepsilon=-L^\varepsilon(\rho^\varepsilon)$, we have

$$||\partial_t\rho^\varepsilon||_{L^\infty(0,\infty;H^2(\mathbb{R}^n))}=||\rho^\varepsilon||_{L^\infty(0,\infty;H^4(\mathbb{R}^n))}$$

and the second estimate follows. □

Proof of Lemma 3.9. We note that

$$||\mathcal{T}_\varepsilon^{-1}h||_{L^2_{F^{-1}}(\mathbb{R}^{2n})}\leq C||h||_{L^2_{F^{-1}}(\mathbb{R}^{2n})}$$

for any h in $L^2_{F^{-1}}(\mathbb{R}^{2n})$. Thus, noticing that g_3^ε, satisfies

$$(\nu+\varepsilon v\cdot\nabla_x)g_3^\varepsilon=\varepsilon v\cdot\nabla_x g_2^\varepsilon \quad\text{or}\quad (\nu+\varepsilon v\cdot\nabla_x)(g_3^\varepsilon-g_2^\varepsilon)=\nu g_2^\varepsilon$$

we get:

$$||g_3^\varepsilon||_{L^2_{F^{-1}}(\mathbb{R}^{2n})}\leq C||g_2^\varepsilon||_{L^2_{F^{-1}}(\mathbb{R}^{2n})}\leq C(\varepsilon^\alpha+\varepsilon)||\rho^\varepsilon||_{H^2(\mathbb{R}^n)}$$

and a similar argument gives the bound on $\partial_t g_3^\varepsilon$.

It remains to estimate $Q^+(g_3^\varepsilon)$. Proposition 3.5 implies

$$\left\|\frac{Q^+(g_3^\varepsilon)}{F}\right\|_{L^2_x(\mathbb{R}^n;L^\infty_v(\mathbb{R}^n))}\leq C\varepsilon^\alpha\left\|\frac{g_2^\varepsilon}{F}\right\|_{L^2_x(\mathbb{R}^n;L^\infty_v(\mathbb{R}^n))}+C(\varepsilon^\alpha+\varepsilon)\left\|\frac{\nabla_x g_2^\varepsilon}{F}\right\|_{L^2_x(\mathbb{R}^n;L^\infty_v(\mathbb{R}^n))},\tag{52}$$

and Lemma 3.8 yields

$$||g_2^\varepsilon/F||_{L^2_x(\mathbb{R}^n;L^\infty_v(\mathbb{R}^n))}\leq C(\varepsilon^\alpha+\varepsilon)||\rho^\varepsilon||_{H^2(\mathbb{R}^n)}.$$

Furthermore, Lemma 3.10 completes the proof of Lemma 3.9, since it implies

$$\left\|\frac{Q^+(g_3^\varepsilon)}{F}\right\|_{L^2_x(\mathbb{R}^n;L^\infty_v(\mathbb{R}^n))}\leq C[\varepsilon^\alpha(\varepsilon^\alpha+\varepsilon)+(\varepsilon^\alpha+\varepsilon)^2+\varepsilon^{\alpha/2}(\varepsilon^\alpha+\varepsilon)]||\rho^\varepsilon||_{H^2(\mathbb{R}^n)}.$$

□

Proof of Lemma 3.10. First, taking the derivative in the equation satisfied by g_2^ε, we obtain:

$$Q(\nabla_x g_2^\varepsilon)=Q_x(g_2^\varepsilon)-Q_x^+(g_1^\varepsilon)-Q^+(\nabla_x g_1^\varepsilon)-\varepsilon^\alpha\nabla_x(L^\varepsilon(\rho^\varepsilon))F.\tag{53}$$

where Q_x denotes the operator defined as Q with $\nabla_x\sigma$ instead of σ. We need to derive some bound on all the terms in the right hand side of (53).

First, it is easy to check that

$$\left\| \frac{Q_x(g_2^\varepsilon)}{F} \right\|_{L^2_x(\mathbb{R}^n;L^\infty_v(\mathbb{R}^n))} \leq C||g_2^\varepsilon/F||_{L^2_x(\mathbb{R}^n;L^\infty_v(\mathbb{R}^n))}$$

and

$$\left\| \frac{Q_x^+(g_1^\varepsilon)}{F} \right\|_{L^2_x(\mathbb{R}^n;L^\infty_v(\mathbb{R}^n))} \leq C||g_1^\varepsilon||_{L^2_{F^{-1}}(\mathbb{R}^{2n})}.$$

Furthermore, $\nabla_x g_1^\varepsilon$ satisfies

$$(\nu(x,v) + \varepsilon v \cdot \nabla_x)\nabla_x g_1^\varepsilon = -\nabla_x \nu g_1^\varepsilon - \varepsilon v \cdot \nabla_x(\nabla_x \rho^\varepsilon F)$$

and so

$$\begin{aligned} Q^+(\nabla_x g_1^\varepsilon) &= Q^+(\mathcal{T}_\varepsilon^{-1}(-\varepsilon v \cdot \nabla_x \nabla_x \rho^\varepsilon F)) \\ &\quad -Q^+(\mathcal{T}_\varepsilon^{-1}(\nabla_x \nu g_1^\varepsilon)). \end{aligned}$$

Proposition 3.5 implies

$$||Q^+(\mathcal{T}_\varepsilon^{-1}(-\varepsilon v \cdot \nabla_x \nabla_x \rho^\varepsilon F))/F||_{L^2_x(\mathbb{R}^n;L^\infty_v(\mathbb{R}^n))} \leq C(\varepsilon^\alpha + \varepsilon)||\rho^\varepsilon||_{H^2(\mathbb{R}^n)}$$

while we clearly have

$$\begin{aligned} ||Q^+(\mathcal{T}_\varepsilon^{-1}(\nabla_x \nu g_1^\varepsilon))/F||_{L^2_x(\mathbb{R}^n;L^\infty_v(\mathbb{R}^n))} &\leq C||\mathcal{T}_\varepsilon^{-1}(\nabla_x \nu g_1^\varepsilon)||_{L^2_{F^{-1}}(\mathbb{R}^{2n})} \\ &\leq C||\nabla_x \nu g_1^\varepsilon||_{L^2_{F^{-1}}(\mathbb{R}^{2n})} \\ &\leq C||g_1^\varepsilon||_{L^2_{F^{-1}}(\mathbb{R}^{2n})} \\ &\leq C\varepsilon^{\alpha/2}||\rho^\varepsilon||_{H^1(\mathbb{R}^n)}. \end{aligned}$$

We deduce

$$\left\| \frac{Q^+(\nabla_x g_1^\varepsilon)}{F} \right\|_{L^2_x(\mathbb{R}^n;L^\infty(\mathbb{R}^n))} \leq C(\varepsilon^\alpha + \varepsilon)||\rho^\varepsilon||_{H^2(\mathbb{R}^n)} + C\varepsilon^{\alpha/2}||\rho^\varepsilon||_{H^1(\mathbb{R}^n)}.$$

Finally,

$$\begin{aligned} \varepsilon^\alpha \nabla_x(L^\varepsilon(\rho^\varepsilon)) &= -\int_{\mathbb{R}^n} \nabla_x(Q^+(g_1^\varepsilon))\,dv \\ &= -\int_{\mathbb{R}^n} Q^+(\nabla_x g_1^\varepsilon)\,dv - \int_{\mathbb{R}^n} Q_x^+(g_1^\varepsilon)\,dv \end{aligned}$$

and so

$$\begin{aligned} ||\varepsilon^\alpha \nabla_x(L^\varepsilon(\rho^\varepsilon))||_{L^2(\mathbb{R}^n)} &\leq \left\| \frac{Q^+(\nabla_x g_1^\varepsilon)}{F} \right\|_{L^2_x(\mathbb{R}^n;L^\infty_v(\mathbb{R}^n))} + \left\| \frac{Q_x^+(g_1^\varepsilon)}{F} \right\|_{L^2_x(\mathbb{R}^n;L^\infty_v(\mathbb{R}^n))} \\ &\leq C(\varepsilon^\alpha + \varepsilon)||\rho^\varepsilon||_{H^2(\mathbb{R}^n)} + C\varepsilon^{\alpha/2}||\rho^\varepsilon||_{H^1(\mathbb{R}^n)}. \end{aligned}$$

Putting the pieces together, (53) implies (together with Corollary 3.4) that

$$\left\| \frac{\nabla_x g_2^\varepsilon}{F} \right\|_{L^2_x L^\infty_v} \leq C(\varepsilon^\alpha + \varepsilon)||\rho^\varepsilon||_{H^2} + C\varepsilon^{\alpha/2}||\rho^\varepsilon||_{H^1(\mathbb{R}^n)}.$$

□

3.5 Passage to the limit in the fractional diffusion equation

Proposition 3.6 implies

$$||f^\varepsilon - \rho^\varepsilon F(v)||_{L^\infty(0,\infty;L^2_{F^{-1}}(\mathbb{R}^n\times\mathbb{R}^n))} \longrightarrow 0.$$

To conclude the proof of Theorem 1.1, it thus only remains to show the following proposition:

Proposition 3.11. *The solution ρ^ε of (45) converges strongly in $L^\infty(0,T;L^2(\mathbb{R}^n))$ to $\rho_0(t,x)$ solution of (13). More precisely:*

$$||\rho^\varepsilon - \rho_0||_{L^\infty(0,T;L^2(\mathbb{R}^n))} \leq CT\varepsilon^{1-\alpha/2}||\rho_0||_{L^\infty(0,\infty;H^2(\mathbb{R}^n))}.$$

First, we recall the following simpler result (which is proved in [13]):

Proposition 3.12. *The solution ρ^ε of (45) converges weak* in $L^\infty(0,\infty;H^k(\mathbb{R}^n))$ to ρ_0 solution of (13).*

Proof. Let $\phi(t,x)$ be a test function in $\mathcal{D}([0,\infty),\mathbb{R}^n)$, the weak formulation of the equation (45) gives

$$\int_0^\infty \int_{\mathbb{R}^n} \rho^\varepsilon \left(\partial_t\phi - L^\varepsilon(\phi)\right) dtdx = \int_{\mathbb{R}^n} \phi(0,x)\rho_{in}(0,x)dx.$$

Furthermore, it is proved in [13] that for any smooth function ϕ, $L^\varepsilon(\phi)$ converges to $L(\phi)$ uniformly in x and t. Together with the bounds on ρ^ε given by Proposition 3.2, this implies Proposition 3.12. Note also that by lower semicontinuity of the norm with respect to the weak convergence, we get:

$$||\rho_0||_{L^\infty(0,\infty;H^k(B_R))} \leq ||\rho^\varepsilon||_{L^\infty(0,\infty;H^k(\mathbb{R}^n))}.$$

□

Proof of Proposition 3.11. In order to show the strong convergence, we note that

$$\partial_t(\rho^\varepsilon - \rho_0) + L^\varepsilon(\rho^\varepsilon - \rho_0) = L^\varepsilon(\rho_0) - L^0(\rho_0)$$

and so

$$\frac{d}{dt}\int |\rho^\varepsilon - \rho_0|^2\,dx \leq \left(\int |L^\varepsilon(\rho_0) - L^0(\rho_0)|^2\,dx\right)^{1/2}\left(\int |\rho^\varepsilon - \rho_0|^2\,dx\right)^{1/2}.$$

We deduce

$$\begin{aligned}\left(\int |\rho^\varepsilon(t) - \rho_0(t)|^2\,dx\right)^{1/2} &\leq \int_0^T \left(\int |L^\varepsilon(\rho_0) - L^0(\rho_0)|^2\,dx\right)^{1/2} dt \\ &\leq T||L^\varepsilon(\rho_0) - L^0(\rho_0)||_{L^\infty(0,\infty;L^2(\mathbb{R}^n))}\end{aligned}$$

A computation similar to that of the proof of Lemma 3.1 now gives

$$\int_{\mathbb{R}^n} |L^\varepsilon(\rho_0) - L(\rho_0)|^2 dx \leq \int_{\mathbb{R}^n} |I_2^\varepsilon(\rho_0) + I_3^\varepsilon(\rho_0) - L(\rho_0)|^2 dx + C\varepsilon^{2-\alpha}||\rho_0||^2_{L^\infty(0,\infty;H^2(\mathbb{R}^n))}.$$

where

$$\begin{aligned}I_2^\varepsilon(\rho_0) &= \int_{\varepsilon\leq|w|\leq 1} \nu(x,w/\varepsilon)\int_0^{+\infty}\left[e^{-\int_0^z \nu(x-ws,w/\varepsilon)\,ds} - e^{-\nu(x,w/\varepsilon)z}\right] w\cdot\nabla_x\rho_0(x-wz)\,\frac{1}{|w|^{n+\alpha}}\,dz\,dw \\ &\quad + \int_{\varepsilon\leq|w|\leq 1}\int_0^{+\infty} e^{-\nu(x,w/\varepsilon)z} w\cdot D_x^2\rho_0(x-wz)\cdot w\,dz\frac{1}{|w|^{n+\alpha}}\,dw\end{aligned}$$

and

$$I_3^\varepsilon(\rho_0) = -\int_{|w|\geq 1}\int_0^{+\infty} e^{-\int_0^z \nu(x-ws,w/\varepsilon)\,ds}\nu(x,w/\varepsilon)\,\nu(x-wz,v)\left[\rho_0(x-wz)-\rho_0(x)\right]\frac{1}{|w|^{n+\alpha}}\,dz\,dw.$$

But

$$\int_{\mathbb{R}^n}|I_2^\varepsilon(\rho_0)+I_3^\varepsilon(\rho_0)-L(\rho_0)|^2dx$$

$$\leq \int_{\mathbb{R}^n}\left(\int_0^\infty\int_{|w|<\varepsilon}\nu(x,w/\varepsilon)\frac{[e^{-\int_0^z \nu(x-ws,w/\varepsilon)\,ds}-e^{-\nu(x,w/\varepsilon)z}]}{|w|^{n+\alpha}}\,w\cdot\nabla_x\rho_0(x-wz)dz\,dw\right)^2dx$$

$$+\int_{\mathbb{R}^n}\left(\int_0^\infty\int_{|w|<\varepsilon}e^{-\nu(x,w/\varepsilon)z}w\cdot D_x^2\rho_0(x-wz)\cdot w\,dz\frac{1}{|w|^{n+\alpha}}\,dw\right)^2dx.$$

and so

$$\begin{aligned}\int_{\mathbb{R}^n}|I_2^\varepsilon(\rho_0)+I_3^\varepsilon(\rho_0)-L(\rho_0)|^2dx &\leq C\int_{|w|<\varepsilon}|w|^{2-\alpha-n}dw||\rho_0||^2_{L^\infty(0,\infty;H^2(\mathbb{R}^n))}\\ &\leq C\varepsilon^{2-\alpha}||\rho_0||^2_{L^\infty(0,\infty;H^2(\mathbb{R}^n))}\end{aligned}$$

which allows to conclude.

□

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