

**Nonlinear Dynamics and Chaos:
Where do we go from here?**

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Chapter 1

Bifurcation and Degenerate Decomposition in Multiple Time Scale Dynamical Systems

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1.1 Introduction

In keeping with the spirit of the Colston conference on Nonlinear Dynamics and Chaos, this chapter emphasizes ideas more than details, describing my vision of how the bifurcation theory of multiple time scale systems will unfold. Multiple time scale dynamical systems are rife with complicated phenomena. The subject has a complicated history that interweaves three different viewpoints: nonstandard analysis, classical asymptotics and geometric singular perturbation theory. My perspective is decidedly geometric but draws upon asymptotic analysis, recognizing the fundamental contributions first expressed in the language of nonstandard analysis. The success of dynamical systems theory in elucidating patterns of bifurcation in generic systems with a single time scale motivates the goal here, namely to extend this bifurcation theory to systems with two time scales. There are substantial obstacles to realizing this objective, both theoretical and computational. Consequently, the final shape that the theory will take is still fuzzy.

It may seem strange to talk about computational barriers to a mathematical theory, so I give some explanation for this. Much of the progress

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in dynamical systems theory throughout its history has been inspired by close examination of examples. This has been the case even before the widespread use of computers to study dynamical systems. As an early instance of an example involving multiple time scales, the concept of **relaxation oscillations** was introduced by van der Pol in the context of the equation that bears his name. Though the analysis in his paper is restricted to this single system, there is a clear statement that the idea of relaxation oscillation applies to a broad class of multiple time scale system. In recent decades, computer studies of models have been one of the foundations for the creation of new theory. Computer simulation and analysis enables extensive studies of model systems. Work on such systems as quadratic maps, the Henon mapping, the standard map, the Lorenz system and Duffing's equation has been crucial in developing intuition about general properties of dynamical systems. General theory has been based upon what has been learned from these examples, in some cases incorporating computer proofs to establish facts that are not readily proved otherwise. In the realm of multiple time scale systems, the strategy of working from computer studies has been less prevalent than in other areas of dynamical systems theory. This is due partly to the failure of "standard" numerical integration algorithms to resolve important qualitative properties of multiple time scale systems. This failure is not only a consequence of the long times required to resolve the fast time scales in a system with two time scales, but also to the extreme instability and disparity of scales in the phase and parameter spaces that we encounter when studying these systems. Asymptotic analysis of local phenomena in multiple time scale systems demonstrates that the theory is subtle, frequently involving quantities that scale exponentially with the ratio of time scales. Consequently, our understanding of the phenomenology of multiple time scale systems is not yet at the level of our understanding of other important classes of dynamical systems. For example, much less is known about bifurcations of the forced van der Pol equation than the bifurcations of the Lorenz system, even though the van der Pol system has a much older heritage, including the monumental work of Cartwright and Littlewood that established the existence of chaotic dynamics in dissipative dynamical systems. Thus, I suggest that the development of better computational methods for studying multiple time scale systems is needed to help create more mathematical theory.

Bifurcation theory of dynamical systems classifies bifurcations by codimension and describes their unfoldings. The primary objects of interest are families of C^r vector fields that depend upon k parameters. The theory examines dynamical phenomena that are **persistent**, meaning that C^s perturbations of the family exhibit the same phenomenon. If a phenomenon occurs persistently in a k parameter family but not in $k - 1$ parameter families, then we say that it has **codimension k** . This definition generalizes the usage of codimension in singularity theory but is imprecise in

this setting. Since formal definitions of the concept of codimension are complicated, I shall focus upon phenomenology here. We want to have a good understanding of a rich set of examples before attempting to formulate a comprehensive, rigorous theory. In the simplest cases, codimension k bifurcations occur on a manifold of codimension k in the space of C^r vector fields. Equivalently, in these cases there will be a set of k **defining equations** for the bifurcations.

An unfolding of a codimension k bifurcation is a k parameter family that contains the codimension k bifurcation in a persistent way. The term unfolding also refers to a description of the dynamics within such a k parameter family. The usefulness of bifurcation theory is due in part to the structural stability properties of unfoldings. However, those results come after our main task here, determining the unfoldings themselves. The local structure and existence of normal forms for multiple time scale systems has been studied both using asymptotic [10, 20] and geometric methods [1, 14]. Apart from the analysis of canards in the van der Pol model [7], little attention has been given to bifurcation in this analysis. We emphasize here periodic orbits, especially **relaxation oscillations** in which the periodic motions include segments that alternately evolve on the slow and fast time scales. Bifurcations of relaxation oscillations add a new set of issues to those that have been addressed for multiple time scale systems. Theory formulated for systems with a single time scale is still valid, but there are enormous distortions in the unfoldings of systems with two time scales compared to those of single time scale systems. As we remarked above, this distortion can be sufficiently extreme that software designed for single time scale systems is unable to compute important qualitative aspects of these unfoldings. Instead of tackling the bifurcations of relaxation oscillations directly, we concentrate on the **slow-fast decomposition** of trajectories. Analyzing this decomposition in terms of transversality properties enables us to gain insight into the bifurcations.

1.2 Definitions and Background

We study **slow-fast** systems in this paper that are written in the form

$$\begin{aligned}\varepsilon \dot{x} &= f(x, y) \\ \dot{y} &= g(x, y)\end{aligned}\tag{1.1}$$

or

$$\begin{aligned}x' &= f(x, y) \\ y' &= \varepsilon g(x, y)\end{aligned}\tag{1.2}$$

Here $x \in R^m$ are the fast variables and $y \in R^n$ are the slow variables. In the limit $\varepsilon = 0$, system (1.1) becomes a system of **differential algebraic**

equations (called the **reduced** system) on the slow time scale and system (1.2) becomes a parametrized family of differential equations (called the **layer equations**) on the fast time scale. For fixed y , the equations $\dot{x} = f(x, y)$ are a **fast subsystem**. We limit our attention in this paper to systems in which the attractors of the fast subsystems are all equilibria. In this case, we say that the slow-fast system has no rapid oscillations. The **critical manifold** of the slow-fast system is the set of solutions of $f(x, y) = 0$ in R^{m+n} .

In slow-fast systems with no rapid oscillations, trajectories approach an ε -neighborhood of the critical manifold on the fast time scale, and then move inside this neighborhood on the slow time scale. This leads us to expect that as $\varepsilon \rightarrow 0$ the limit of a trajectory with a specified initial condition will be a finite union of trajectories of fast subsystems and curves that lie on the critical manifold. The curves on the critical manifold are themselves trajectories of a system of differential equations called the **slow flow** of the system. This image is made more explicit by several general theorems about the dynamics of slow-fast systems near their critical manifolds. We recall some of these results here, omitting details.

The first result is that the critical manifolds of generic slow-fast systems are indeed manifolds. This is a corollary of the Transversality Theorem, stating that the regular values of a mapping $f : R^{m+n} \rightarrow R^m$ form a set of second Baire category. For generic systems (1.1), zero is a regular value of f and its inverse image for f is a manifold of dimension n . However, note that generic one parameter families of slow-fast systems may encounter parameters for which there are singularities of the critical manifold since the rank deficient linear maps $A : R^{m+n} \rightarrow R^m$ form a subset of codimension $n + 1$.

Projection π_s of the critical manifold onto the space of slow variables plays a special role in the theory. At regular points of π_s , we can solve the equation $f(x, y) = 0$ implicitly to obtain a locally defined function $x(y)$ whose graph is an open region of the critical manifold. The equation $\dot{y} = g(x(y), y)$ then defines a vector field on the critical manifold that is called the **slow flow** of the slow-fast system. While the slow-fast system is not tangent to the critical manifold, the slow flow often yields an approximation to the flow on a nearby invariant manifold, called a **slow manifold** (or true slow manifold). Existence of the slow manifold and convergence of trajectories of the slow-fast system on the slow manifold to those of the slow flow on the critical manifold was proved by Tikhonov [20] in the case of attracting slow manifolds and by Fenichel [8] in the hyperbolic case. The slow manifolds are not unique, but rather consist of a collection of slow manifolds that lie within a distance that is $O(e^{-c/\varepsilon})$ from one another. The distance between the slow manifolds and critical manifold is $O(\varepsilon)$.

At singular points of the projection π_s of the critical manifold, we cannot expect the slow flow to be defined. Consider the following system

with one slow and one fast variable that is a standard example of a slow-fast system with a fold in its critical manifold:

$$\begin{aligned}\varepsilon\dot{x} &= y + x^2 \\ \dot{y} &= 1\end{aligned}\tag{1.3}$$

At the fold point $(x, y) = (0, 0)$, the equation $\dot{y} = 1$ is incompatible with the algebraic equation $y = -x^2$ obtained by setting $\varepsilon = 0$. This is annoying, but doesn't seem like a big obstacle to understanding the reduced system. However, when the critical manifold of a slow-fast system is two dimensional, new phenomena occur that prompt further analysis. This is illustrated by the system

$$\begin{aligned}\varepsilon\dot{x} &= y + x^2 \\ \dot{y} &= ax + bz \\ \dot{z} &= 1\end{aligned}\tag{1.4}$$

Here the critical manifold is defined by $y = x^2$ and the fold curve is defined by $x = y = 0$. At points where $bz < 0$, the slow flow is pointing toward the fold curve on the two sheets of the slow manifold. Where $bz > 0$, the slow flow is pointing away from the fold curve on the two sheets of the slow manifold. At the origin, there is a transition. To study this transition more carefully, we rescale the slow flow so that it has a continuous extension across the fold curve. We can use (x, z) as a regular system of coordinates near the fold. We differentiate the algebraic expression $y + x^2$ appearing in the reduced system of (1.4) to obtain $\dot{y} = -2x\dot{x}$ for motion of the reduced system on the critical manifold. We then replace \dot{y} by $-2x\dot{x}$ in the slow flow equations. Rescaling the resulting equations by $-2x$ yields the system

$$\begin{aligned}\dot{x} &= ax + bz \\ \dot{z} &= -2x\end{aligned}\tag{1.5}$$

This rescaled slow flow is clearly defined across the fold curve. It has the same trajectories as the slow flow except that the direction of the trajectories is reversed in the half plane $x > 0$. The origin is an equilibrium point of this system, called a **folded equilibrium**. Folded equilibria occur at the transition for motion in the slow flow toward a fold to motion away from the fold. Folded equilibria in systems with two dimensional slow manifolds are classified by the type of equilibrium in the rescaled system, so that we speak of folded nodes, foci and saddles.

In the case of folds for the system (1.3) with one slow and one fast variable, the geometry of the slow-fast system for $\varepsilon > 0$ is evident: when trajectories reach the fold, they jump along a fast segment. With the scaling of variables $x = \varepsilon^{(1/3)}X$; $y = \varepsilon^{(2/3)}Y$; $t = \varepsilon^{(2/3)}T$, ε disappears

from the system 1.3. Thus, the geometry of the flows for different values of ε are all qualitatively the same, with features that scale as indicated by the coordinate transformation. Solutions of the equivalent **Riccati** equation $\dot{x} = t + x^2$ have been studied in terms of special functions [20]. There is a single solution that remains close to the parabola $y + x^2 = 0$ with $x > 0$ for all time $t < 0$. Figure 1.1 shows a phase portrait of the flow past the origin in system (1.3).

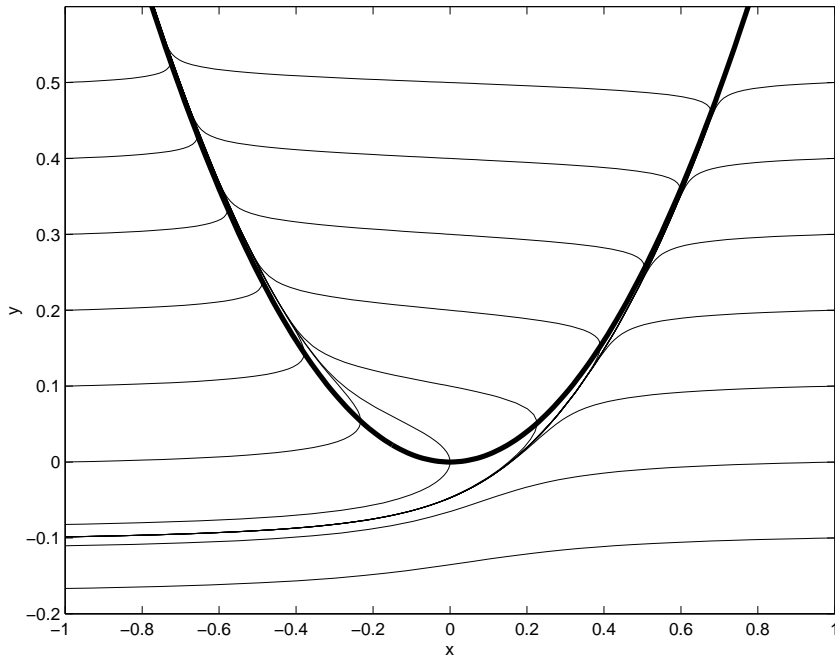


Figure 1.1. The flow in a neighborhood of the origin for the system (1.3). The critical manifold is the parabola drawn with a thick line.

With two slow variables, the geometry of the solutions of (1.4) that pass near a folded singularity is much more complicated than that of system (1.3). In the case of a folded saddle, Benoit [3] proved that there are two algebraic solutions of (1.4) that remain close to the critical manifold for all time. These solutions have a hyperbolic structure and hence stable and unstable manifolds that divide the phase space into regions with similar asymptotic properties. It is noteworthy that one of the algebraic solutions passes from the stable sheet of the slow manifold to the unstable sheet. There are solutions in the unstable manifold of this trajectory that flow along the unstable sheet of the slow manifold and then leave the slow manifold, with a fast segment that either ends back at the stable sheet of

the slow manifold or goes to infinity in finite time. Solutions that flow along the unstable sheet for a finite period on the slow time scale are called **canards**. Typically, the regions of initial conditions that give rise to canards are **exponentially thin**; i.e. of width $O(e^{-c/\varepsilon})$ for a suitable positive constant c that is independent of ε . The reason for this behavior is the fast instability of the slow manifold. This instability is also an impediment to numerical computation of the canards. Deviations from the slow manifold are amplified exponentially on the fast time scale, so that even tiny perturbations on the order of round-off errors are quickly magnified into large separations of numerically computed trajectories from the canards.

Part of our message in this lecture is that canards play a central role in the bifurcations of relaxation oscillations. In the singular limit, changes in generic families of periodic orbits appear to be discontinuous. In many of these situations, canards form the “connecting glue” that spans these discontinuities. There are several mechanisms that lead to the formation of periodic orbits in generic one parameter families of slow-fast systems. In the next two sections, we discuss some of these, illustrating how the slow-fast decompositions of trajectories gives a starting point for the analysis of periodic orbits with canards. We end this section with comments about two open problems.

The characterization of canards that are associated with folded nodes in systems with two dimensional critical manifolds has not carried out fully. Benoit [3] proved the existence of canards near folded nodes, but he was unable to determine fully what the set of canard trajectories are in this case. Folded nodes present another problem for the general theory of relaxation oscillations in addition to the question of characterizing the canards. On the critical manifold, there is an entire open region of trajectories that flow to the folded node under the slow flow. Since the subsequent evolution of these trajectories from the folded node may involve canards beginning at the folded node, it is possible that there may be open regions of relaxation oscillations possessing canards in systems with two slow variables that do not collapse in the singular limit. I do not know of any examples that have been studied carefully to determine the properties of the trajectories that pass through regions with folded nodes. Relaxation oscillations passing through a folded node occur in the forced van der Pol system (described in Section §5 below) but the parameter regions and scale on which the relevant dynamics occur may make this a difficult example to analyze numerically.

I am unaware of any systematic studies of the slow-fast dynamics in systems with $n > 2$ slow variables. The critical manifolds of such systems are n -dimensional and the folds of their critical manifolds are $(n - 1)$ -dimensional. Using singularity theory, we can introduce coordinates so that a regular fold is given by the equations $y_1 + x_1^2 = 0$; $x_i = 0$, $1 < i \leq m$. Following the argument described above, we use coordinates (x_1, y_2, \dots, y_n) near the fold and use the relation $\dot{y}_1 = 2x_1\dot{x}_1$ to obtain the slow flow in

this coordinate system. Rescaling the system by the factor $2x_1$, we obtain a rescaled vector field on the critical manifold that has equilibria along the $(n - 2)$ -dimensional submanifold defined by $x_1 = g_1(0, y) = 0$, where g_1 is the right hand side of the equation for y_1 in these coordinates. What happens near the folds of these generic slow-fast systems? I believe that little is known about such systems.

1.3 Slow-fast Decompositions

Let γ_ε be a continuous family of trajectories in R^{m+n} for system (1.1) defined for $\varepsilon \geq 0$ small. For $\varepsilon = 0$, we understand that a trajectory is a union of trajectories of fast subsystems and the slow flow. A **slow-fast decomposition** of γ is defined by a partition of γ that depends continuously on ε with the property that γ_0 is partitioned into segments that lie in the critical manifold and segments that lie in its complement. To discuss the decomposition, we introduce some more terminology. First, we will continue to use the term trajectory to refer to a family of the type described above: a trajectory of a slow-fast system is a continuous family γ_ε that is (1) smooth for each $\varepsilon > 0$ and (2) solves the slow-fast equations when $\varepsilon > 0$. The **reduced trajectory** is the restriction of a trajectory to $\varepsilon = 0$. We partition a reduced trajectory into **slow segments** that lie on the critical manifold and **fast segments** on which the limiting differential-algebraic equation does not vanish. Without additional assumptions, there is little reason to believe that trajectories with finite slow-fast decompositions will exist. We approach this issue by seeking to characterize the settings in which we find stable relaxation oscillations whose slow-fast decomposition is stable under perturbation.

When a trajectory arrives at a fold that terminates a slow segment, the fold point is a non-hyperbolic equilibrium point of the fast subsystem. The fast segment originating at this equilibrium will lie in its (weak) unstable manifold. The only circumstance in which we expect there to be a unique trajectory of the fast subsystem with this equilibrium point as its α -limit set is the case in which the fold point is a nondegenerate saddle-node with spectrum in the (closed) left half plane. (I am unaware of any systematic study of examples of relaxation oscillations in which the fold points have higher dimensional unstable manifolds.) So we confine ourselves to this setting and make three assumptions about the relaxation oscillations that will be at the beginning of a family:

- (i) The ends of the slow segments are fold points of the critical manifold with eigenvalues for the fast subsystem that are non-positive.
- (ii) The ends of the fast segments are regular points of a stable sheet of the critical manifold. and
- (iii) The slow flow is transverse to the umbra of the fold.

We explain the third assumption in more detail. The first implies that the fold points near the trajectory are saddle-nodes of their fast subsystems with one dimensional weakly unstable manifolds. From each of these fold points, there is a unique trajectory with the fold point as its α -limit set. We call the ω -limit sets of these points the **umbra** of the fold. The umbra is a codimension one submanifold of the critical manifold since we assume that the ω -limit sets of all trajectories are equilibrium points of the fast subsystems. If the slow flow is tangent to the fold, we shall say that there is an umbral tangency. We shall call a relaxation oscillation that satisfies properties (1-3) above a **simple** relaxation oscillation. The following theorem is an immediate consequence of results of Levinson [16] analyzing the properties of trajectories that jump at regular fold points.

Theorem: Hyperbolic periodic orbits that are simple relaxation oscillations have slow-fast decompositions that vary continuously with perturbation.

For trajectories that satisfy the first two assumptions above, we can augment the slow flow of system (1.1) to obtain a **hybrid** system that represents the limit behavior of trajectories as $\varepsilon = 0$. Hybrid dynamical systems [2] are ones in which there are

- a discrete set of bounded manifolds and flows on these manifolds, and
- mappings from the manifold boundaries into the manifolds

Here the transition maps from the manifold boundaries will be the projection of a fold along the fast subsystem to its umbra. So the hybrid system that we obtain consists of the stable sheets of the critical manifold with its slow flow together with the maps of regular folds on the boundary to their umbra. The definition of this hybrid system will break down at folded equilibria, at trajectories whose fast segments do not end in the interior of a stable sheet of the critical manifold, and at boundary points more complicated than a fold. Return maps for this hybrid system need not be locally invertible in the neighborhood of trajectories with umbral tangencies. In Section §5, we illustrate these ideas with the forced van der Pol system as a model example.

1.4 Degenerate Decomposition and Bifurcation

This section examines generic mechanisms by which a one parameter family of stable, simple relaxation oscillations can reach a parameter value where its slow-fast decomposition becomes degenerate. We examine the slow-fast decomposition of the reduced periodic orbit and use transversality theory to determine the types of degeneracies that will persist with smooth perturbations of a one parameter family. Persistence implies that the reduced orbits have at most one degeneracy in their slow-fast decomposition. Degeneracies that may occur include the following:

- (i) A fast segment ends at a regular fold point. (There are two cases that differ as to whether the slow flow approaches or leaves the fold near this point.)
- (ii) A slow segment ends at a folded saddle.
- (iii) A fast segment encounters a saddle point of the fast subsystem.
- (iv) There is a point of Hopf bifurcation at a fold.
- (v) A slow segment ends at a cusp.
- (vi) The reduced system has a quadratic umbral tangency.

This list is incomplete, not taking account of reduced trajectories that contain folded nodes. We leave as an open question the formulation of a classification of codimension one degenerate decompositions that contains all of the degeneracies that occur in generic families of simple relaxation oscillations. For each of the cases on the list, we want to determine what the presence of a degenerate decomposition implies about nearby canards and bifurcations and to analyze asymptotic properties of the families of periodic orbits related to the degenerate decomposition. There are several levels on which this analysis can be performed. The least precise level is to introduce specific models for portions of the flow and base the analysis upon these models. The most elementary descriptions of homoclinic “Silnikov” bifurcation are a well known example of this type of analysis [13]. Our ultimate goal is to give a rigorous analysis. In the case of Hopf bifurcation at a fold, Dumortier and Roussarie [7] have given a thorough analysis of one version of the van der Pol system.

Kathleen Hoffman, Warren Weckesser and I have begun to formulate analyses for some of the cases on the above list. I give here the barest sketch of the case in which there is a fast segment that begins at a fold and ends at a fold with slow trajectories flowing away from the fold. This type of degenerate decomposition occurs in a model of reciprocal inhibition of a pair of neurons studied by Guckenheimer et al. [11].

Figure 1.2 is a three dimensional plot of the slow manifold and two periodic orbits of the family we now describe. We work with a system that has 2 slow variables and 1 fast variable. We assume that the critical manifold, shown shaded in the figure, has three non-intersecting fold curves and is independent of the parameter in the family. The fold curves are labeled 1,2,3 and plotted as dotted lines. The fast variable is x ; the axes are labeled in the figure. The projections of the first and third fold curves onto the $y - z$ plane intersect transversally. Cross-sections of the critical manifold orthogonal to the z axis are quartic curves with two local maxima (on folds 1 and 3) and a local minimum (on fold 2). The fast flow is assumed to be in the negative x direction for points above the critical manifold and in the positive x direction for points below the critical manifold. Folds 2 and 3 are in-folds with trajectories of the slow flow moving into the folds; fold 1 is an out-fold with trajectories of the slow flow leaving the fold. The

fold curves partition the critical manifold into four sheets, that are labeled a, b, c, d with increasing x . Sheets b and d of the critical manifold are stable, sheets a and c are unstable. The slow flow on sheets a and b is downwards, the slow flow on sheet d is upwards, and the slow flow on sheet c changes direction from downwards along fold 2 to upwards along fold 3.

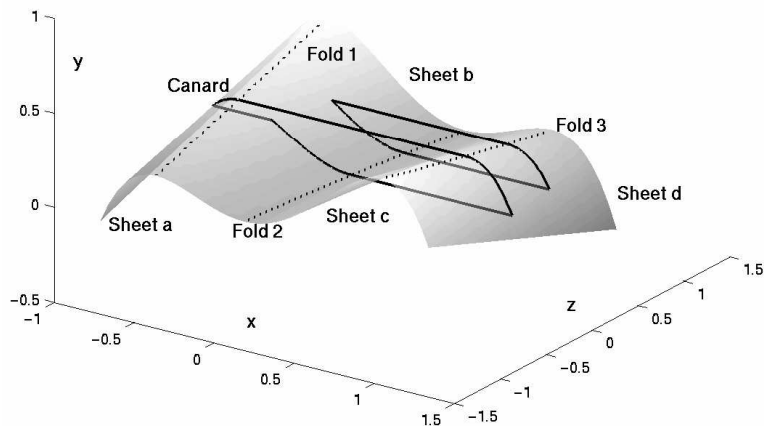


Figure 1.2. Schematic representation of a system having a periodic orbit with a degenerate slow-fast decomposition of the type in-fold \rightarrow out-fold.

Assume that at an initial value of the parameter, there is a stable periodic orbit with two slow segments on sheets b and d and two fast segments, the first from fold 3 to sheet b and the second from fold 2 to sheet d . In the figure, this is the smaller closed curve drawn with solid lines. The orbit is drawn so that the slow flow is parallel to the $x - y$ plane. Assume further that, with the changing parameter, the orbits move in the direction of decreasing z . At a critical value of the parameter, the fast segment beginning on fold 3 hits fold 1 instead of the interior of sheet b . This is the location of a degenerate decomposition. The continuation of the family from the degenerate decomposition will have periodic orbits with canards lying on sheet a . The longer closed curve in the figure shows one of these. The canard that is shown terminates at a fast segment that connects sheet a with sheet b . In the periodic orbit, the canard is followed by a slow segment along sheet b , a fast segment from fold 2 to sheet d , a

slow segment on sheet d and finally the fast segment from fold 3 to fold 1.

Using models for different portions of the flow, we describe approximations of the return maps for the canard periodic orbits in this family. The return map is a composition of transition maps along slow and fast segments. Introduce uniformizing coordinates near fold 1 at the end of the fast segment from fold 3 to fold 1:

$$\begin{aligned}\varepsilon\dot{x} &= y - x^2 \\ \dot{y} &= 1 \\ \dot{z} &= x\end{aligned}\tag{1.6}$$

(Here, (x, y, z) are different coordinates from those depicted in the figure, but x is still the direction of fast motion.) In these coordinates, model the trajectories originating at fold 3 as the plane parametrized by (v, u, au) . This set of trajectories has umbra on sheet b along the curve $(x, y, z) = (\sqrt{u}, u, au)$, $u > 0$. In (x, z) coordinates on the critical manifold, $z = x^2/a$ on the curve of incoming trajectories from fold 3. The slow trajectories of the model system (1.6) lie along curves $z = \frac{2}{3}x^3 + c$ for varying c . For points (\sqrt{u}, au) , $c(u) = au - \frac{2}{3}u^{3/2}$. On the critical manifold, these points reach $x = 1$ along the slow flow with $z = \frac{2}{3} + au - u^{3/2}$. The leading order term is regular, but the map is not C^2 . The orbit with a degenerate slow-fast decomposition may still be stable.

The canards for this family occur along sheet a , extending the slow flow trajectory with $z = \frac{2}{3}x^3$. If a canard jumps back to sheet b with $x = -x_d > -1$, then the value of z along the jump is $z_d = -\frac{2}{3}x_d^3$ and the value of c is $c_d = -\frac{4}{3}x_d^3$. This arrival point on sheet b reaches $x = 1$ with $z = \frac{2}{3} - \frac{2}{3}x_d^3$. Depending upon the sign of a , the intersections of the trajectories initiated by these canards overlaps the set of trajectories originating at $(x, y, z) = (\sqrt{u}, u, au)$, $u > 0$ or is disjoint from this set of trajectories.

Beyond the “scaling region” of the model equations (1.6), trajectories that remain near the canard to reach $x = x_d$ come from an “exponentially thin slab” along sheet a of the critical manifold that flows across fold 3. Let $y = y_c$ give the intersection of the infinite canards of the model system (1.6) with $x = 1$. (This is a set parallel to the z axis since the normal form equations for \dot{x} and \dot{y} are independent of z .) Denoting $t_j = O(1)$ the time of a jump from the canard starting near fold curve 1, the difference of its y coordinate from y_c is $y_p \approx \exp(-L/\varepsilon)$ where L is given by an integral of the fast eigenvalue along the canard trajectory to the terminal time of the canard. The phase space coordinates of the end point of the canard are estimated by $y_j = y_c + y_p - \frac{\varepsilon}{L} \ln y_p$, $x_j = \sqrt{y_j}$ and $z_c = (-\frac{\varepsilon}{L} \ln y_p)^{3/2}$. The derivative of (x_j, y_j, z_j) with respect to y_p has order $\varepsilon/y_p = \varepsilon \exp(-L/\varepsilon)$ whose magnitude is large. Thus the formation of the canards leads to violent stretching of the return map for the cycles.

Assuming that the orbit with a degenerate decomposition is stable, there are now four cases to analyze. These are determined by the sign of a and orientation of the reduced return map for the periodic orbit with degenerate decomposition. One case yields a monotone return map with a single saddle-node and one case yields a single period doubling. One of the two remaining cases yields a period doubling cascade and chaotic invariant sets. The final case begins with a stable, orientation reversing return map at the degenerate orbit but then changes orientation as the canards develop and there is a saddle-node bifurcation. In all of these cases, the bifurcations of the original periodic orbit occur as the canards begin to develop, not with the $O(\varepsilon)$ separation found in the case of Hopf bifurcation and the canard formation in the van der Pol system [7].

This sketch of how relaxation oscillations bifurcate close to a degenerate decomposition with a fast segment joining two folds is far from a complete or rigorous analysis. Filling in details of that analysis is one of the places that we go from here. Some of the other cases on the list of degenerate decompositions have been studied already, but there is hardly a coherent theory of codimension one bifurcation of stable relaxation oscillations analogous to what we know about bifurcations of systems with a single time scale. The cases of periodic orbits passing through a folded saddle and umbral tangencies occur in the forced van der Pol system, described in the next section. Our discussion of this example gives the flavor of how the study of the reduced systems and the degeneracies in the slow-fast decompositions of their trajectories gives new insight into the dynamics of multiple time scale dynamical systems.

1.5 The Forced van der Pol Equation

The forced van der Pol equation can be written in the form

$$\begin{aligned}\varepsilon\dot{x} &= y + x - \frac{x^3}{3} \\ \dot{y} &= -x + a \sin(2\pi\theta) \\ \dot{\theta} &= \omega\end{aligned}\tag{1.7}$$

Here θ is a cyclic variable on the circle S^1 that is normalized to have length 1 and we restrict our attention to the regime in which $\varepsilon > 0$ is small. This system was introduced and studied by van der Pol in the 1920's. He introduced the term relaxation oscillations to describe the solutions of the unforced system [25]. A generalization of the van der Pol equation, the Fitzhugh-Nagumo system [22] has been widely studied as a model of nerve impulses. The literature on the dynamics of the forced van der Pol system is dominated by the work of Cartwright and Littlewood during the period approximately 1940-55 [4, 5, 17, 18]. Recently, Kathleen Hoffman,

Warren Weckesser and I have begun to study this system further, using the conceptual framework described in previous sections [12]. In particular, we are interested in characterizing the bifurcations of periodic orbits and other invariant sets that take place in this system. Apart from a couple of numerical studies [9, 19] at values of ε that are large enough to make tracking canards feasible with initial value solvers, little has been done in this direction. Earlier work has been based primarily upon analysis of the return map to a section of constant θ . In contrast, we focus upon the slow-fast decomposition of trajectories. These are readily determined since the fold curves are given by $x = \pm 1$, $y = \mp 2/3$ and the fast subsystems flow parallel to the x -axis. Fast segments beginning at the fold with $x = \pm 1$ return to the critical manifold at $x = \mp 2$. In our view, the slow flow provides a scaffolding that will enable a comprehensive understanding of the dynamics for the system.

The rescaled slow flow on the critical manifold $y = x^3/3 - x$ has a global representation in (x, θ) coordinates as

$$\begin{aligned} x' &= -x + a \sin(2\pi\theta) \\ \theta' &= \omega(x^2 - 1) \end{aligned} \tag{1.8}$$

The equilibrium points of this system can be easily determined and its phase portraits produced numerically. Figure 1.5 shows a representative phase portrait for parameter values $a = 1.5$, $\omega = 1$. When $0 < a < 1$, the slow flow has no folded singularities. When $a < 1$, there are four folded singularities, two on each circle of the fold curve. The limits of the simplest relaxation oscillations in the system correspond to closed curves obtained from initial conditions that begin on the circle $x = 2$, flow to $x = 1$, connect to $x = -2$ along a segment $\theta = \text{const}$, flow from there to $x = -1$ and then connect to the initial point with another segment $\theta = \text{const}$. (There are also simple relaxation oscillations for some parameter values that have more than two slow segments and more than two fast segments.) All of these relaxation oscillations are stable in the fast directions of the flow; some are also stable along the slow manifold and some are unstable along the slow manifold.

We investigate the simple relaxation oscillations in more detail by constructing the return map to $x = 2$ for the reduced system. The analysis is made easier by exploiting the symmetry $S(x, y, \theta) = (-x, -y, \theta + 0.5)$ of the slow-fast system and its restriction to the critical manifold and rescaled slow flow. The half return map H is defined by following the slow flow from $x = 2$ to $x = 1$, jumping from $x = 1$ to $x = -2$ and applying the by symmetry. If $(2, \theta_0)$ flows to $(1, \theta_1)$ under the slow flow, then $H(\theta_0) = \theta_1 + 0.5 \pmod{1}$. When $a > 1$, the map H is discontinuous at points lying in the stable manifold of the folded saddle. The limit values of H at the two sides of the discontinuity are $\theta_s + 0.5$ and $\theta_u + 0.5$ where the folded saddle has

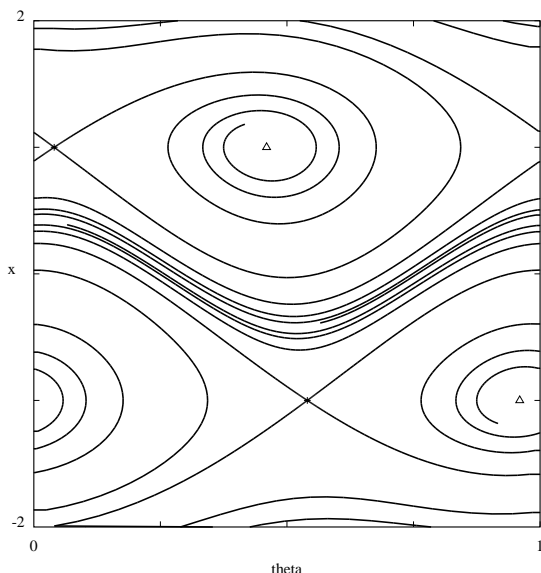


Figure 1.3. The phase portrait of the rescaled slow equation for the forced van der Pol equation with $a = 4$ and $\omega = 1.55$. The circles $x = \pm 1$ and $x = \pm 2$ are drawn together with stable and unstable manifolds of the folded saddles. The stable manifolds have negative slope at the saddle points and the unstable manifolds have positive slope.

(x, θ) coordinates $(1, \theta_s)$ and $(1, \theta_u)$ is the (first) intersection of the unstable manifold of the folded saddle with $x = 1$. The map H has one local maximum and one local minimum when $a > 2$. These occur at the points of umbral tangency $(2, \frac{1}{2\pi} \sin^{-1}(2/a))$ and $(2, 0.5 - \frac{1}{2\pi} \sin^{-1}(2/a))$ where $\dot{x} = 0$ in the slow flow.

Fixed points of H correspond to trajectories with two fast and two slow segments, the fast and slow segments each forming a symmetric pair. Our investigations of these fixed points show two primary types of bifurcations: saddle-nodes and “homoclinic” bifurcations where the fixed points approach a discontinuity of H . The saddle-node bifurcations do not involve degenerate decomposition of periodic orbits in a direct way. They correspond to saddle-nodes of periodic orbits in which a stable and unstable periodic orbit of the reduced system coalesce. The second type of bifurcation occurs when the periodic orbits in the slow flow approach heteroclinic orbits that connect the two folded saddles with two connecting segments that are related by the symmetry. There are two forms of heteroclinic orbit that differ as to whether the connecting orbits contain a segment that follows the unstable manifolds of the folded saddles or not. The periodic

orbits that contain segments lying close to the unstable manifolds of the folded saddle occur to the right of the stable manifolds of the folded saddle. There are additional distinctions that can be made that reflect the types of branches of the half return map that contain points from the homoclinic orbits.

Analysis of the half return map H does not address directly the issues of canards and bifurcations associated to degenerate decompositions that we discussed in previous sections. The discovery of chaotic dynamics in the forced van der Pol system by Cartwright and Littlewood was a seminal event in the history of dynamical systems [4]. Their discovery was based upon results of Birkhoff in the case that there are two fixed points of H whose orbits in the reduced system have different periods. The subsequent analysis of Littlewood [17, 18] makes it evident that the chaotic trajectories involve canards.

We have begun to extend our analysis of the reduced system and its half return map to take account of the limiting behavior of canard solutions in the forced van der Pol system. Figure 1.4 shows a plot ($a = 4$, $\omega = 1.55$) that allows us to determine the limiting position of canards as $\varepsilon \rightarrow 0$ and the subsequent trajectories to their next return(s) with the circle $x = 2$. Trajectories that reach the folded saddle at $x = 1$ along its stable manifold (dashed line) can continue past the folded saddle along the stable manifold (dotted line) to smaller values of x . (Recall that our rescaling of the slow flow reversed time in the strip $|x| < 1$.) In the three dimensional phase space, the canard trajectories follow the unstable sheet of the critical manifold, from which they can then jump to one of the two stable sheets of the critical manifold. The jumps are parallel to the x axis, and their images on the two stable sheets are also drawn as dotted lines. These images act like the umbra of fold curves in that they are the initial points for slow segments of the trajectories that begin on the canards. The canards that jump up to $x > 1$ have trajectories that next reach the circle $x = 1$ in the interval between θ_u (at the end point of the dash-dotted unstable manifold of the folded saddle) and the intersection point of the stable manifold with $x = 1$. From $x = 1$, these trajectories jump to $x = -2$. One of them lands in the stable manifold of the lower folded saddle (dashed line in the lower half of the figure). The extension of the trajectory jumping from the unstable manifold of the (upper) folded saddle is shown as a dash-dotted line. The canards that jump down to $x < -1$ initiate slow segments that flow to $x = -1$ and then jump back to $x = 2$. One of these trajectories lands in the stable manifold of the upper folded saddle at its third intersection with the circle $x = 2$. The trajectories at the endpoints of the umbra from the canards that jump down are plotted as solid lines.

Figure 1.5 plots the value of θ at returns of the canards described in the previous paragraph to $x = 2$. The dotted line is at a (integer translated) value of the third intersection of the stable manifold of the

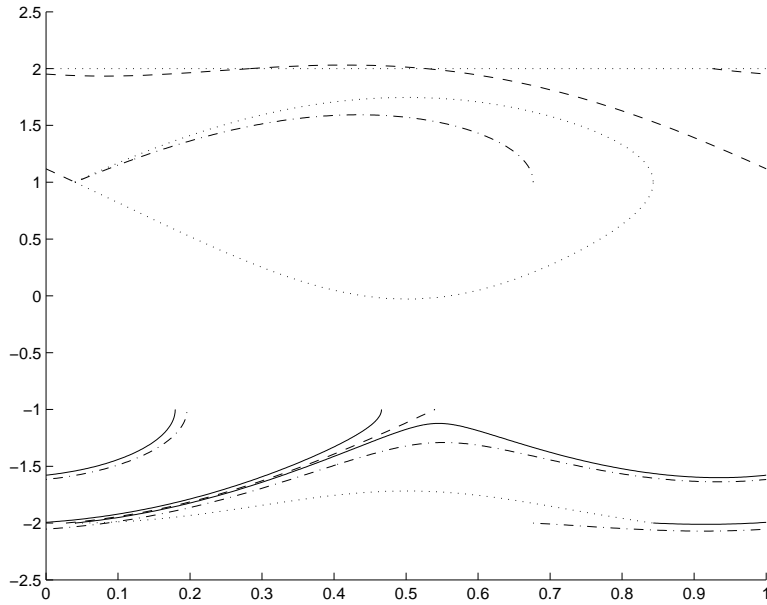


Figure 1.4. Information about canards in the forced van der Pol equation with $a = 4$ and $\omega = 1.55$. Canard trajectories at the upper folded saddle follow the dotted stable manifold down to the right and jump vertically up or down to one of the two other dotted curves. The unstable manifold of the upper folded saddle and its prolongation after jumping from $x = 1$ to $x = -2$ is drawn as a dash-dotted curve. Branches of both folded saddles are drawn as dashed curves. The solid curves in the lower half of the figure begin at the ends of the canard umbra shown as the lower dotted curve.

folded saddle with $x = 2$. Its intersection with the solid line comes from a canard that jumps down, flows to $x = -1$ and then jumps to a point in the stable manifold of the upper folded saddle. The vertical segment of the graph comes from the canard that jumps up, flows to $x = 1$ and then jumps to $x = -2$, landing in the stable manifold of the lower folded saddle. The points in this region will follow canards symmetric to the ones described above. In particular, there will be one canard that jumps down, flows to $x = -1$ and then jumps to $x = 2$, landing in a point of the stable manifold of the upper saddle. This description of the evolution of canards leads to the identification of the solenoid [23] (suspension of a horseshoe) discovered by Cartwright and Littlewood. A cross-section to the stable manifold of the folded saddle will first flow along the slow manifold becoming compressed in the transverse direction to the slow manifold. A small portion will then follow canards and the evolution described above,

the transverse direction to the vector field inside the slow manifold being stretched along the canards. Two portions of these canards will return to the upper sheet of the slow manifold, passing through the original cross-section to the stable manifold of the folded saddle. The return map of this cross-section appears to have a horseshoe. We end with a conjecture and a question about this description of chaotic dynamics in the forced van der Pol equation:

Conjecture: For the parameter values $a = 4$, $\omega = 1.55$ and $\varepsilon > 0$ sufficiently small, the nonwandering set of the forced van der Pol equation (1.7) consists precisely of two stable periodic orbits and a hyperbolic solenoid that is the suspension of a full shift map on two symbols.

Question: What are the bifurcations that lead to the creation and destruction of the chaotic trajectories in the forced van der Pol system?

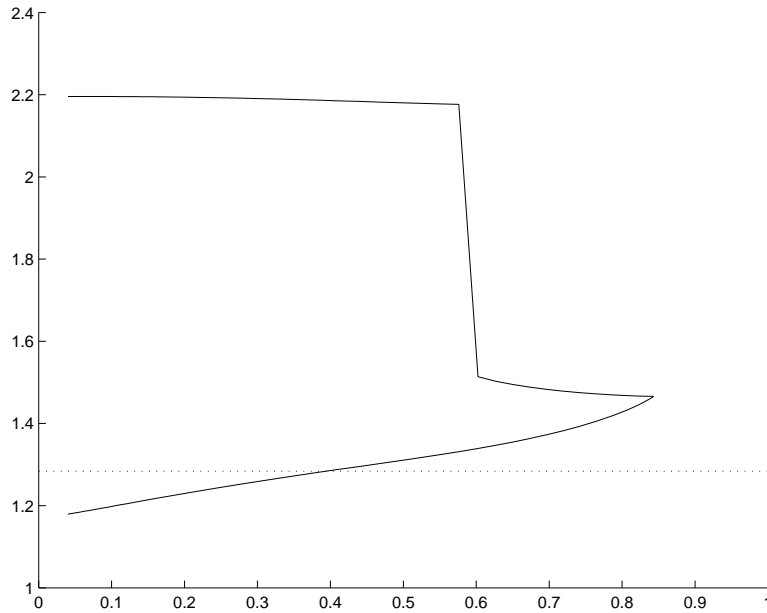


Figure 1.5. The θ coordinate of returns of canard trajectories to the circle $x = 2$ are plotted, vs. their initial θ coordinate. The steep vertical segment is an artifact: there is a discontinuity where the trajectories contain segments of the stable manifold of the lower folded saddle. The dotted line gives the θ coordinate for an intersection of the stable manifold of the upper folded saddle with $x = 2$. The extreme point at the right of the curve corresponds to the “maximal” canard, the trajectory that follows the stable manifold of the folded saddle all the way to its intersection with $x = 1$ before jumping to $x = -2$.

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