DISSIPATIVE MODELS GENERALIZING THE 2D NAVIER-STOKES AND THE SURFACE QUASI-GEOSTROPHIC EQUATIONS

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ABSTRACT. This paper is devoted to the global (in time) regularity problem for a family of active scalar equations with fractional dissipation. Each component of the velocity field u is determined by the active scalar θ through $\mathcal{R}\Lambda^{-1}P(\Lambda)\theta$ where $\mathcal{R}$ denotes a Riesz transform, $\Lambda = (-\Delta)^{1/2}$ and $P(\Lambda)$ represents a family of Fourier multiplier operators. The 2D Navier-Stokes vorticity equations correspond to the special case $P(\Lambda) = I$ while the surface quasi-geostrophic (SQG) equation to $P(\Lambda) = \Lambda$. We obtain the global regularity for a class of equations for which $P(\Lambda)$ and the fractional power of the dissipative Laplacian are required to satisfy an explicit condition. In particular, the active scalar equations with any fractional dissipation and with $P(\Lambda) = (\log(I - \Delta))^{\gamma}$ for any $\gamma > 0$ are globally regular.

1. INTRODUCTION

This paper is devoted to the dissipative active scalar equation

$$(1.1) \qquad \begin{cases} \partial_t \theta + u \cdot \nabla \theta + \kappa(-\Delta)^{\alpha}\theta = 0, & x \in \mathbb{R}^d,\, t > 0, \\ u = (u_j), \quad u_j = \mathcal{R}_l \Lambda^{-1} P(\Lambda)\,\theta, & 1 \le j,\, l \le d, \end{cases}$$

where $\kappa > 0$ and $\alpha > 0$ are parameters, $\theta = \theta(x,t)$ is a scalar function of $x \in \mathbb{R}^d$ and $t \ge 0$, u denotes a velocity field with each of its components u_j $(1 \le j \le d)$ given by a Riesz transform $\mathcal{R}_l$ applied to $\Lambda^{-1}P(\Lambda)\,\theta$. Here the operators $\Lambda = (-\Delta)^{\frac{1}{2}}$, $P(\Lambda)$ and $\mathcal{R}_l$ are defined through their Fourier transforms,

$$\widehat{\Lambda f}(\xi) = |\xi|\widehat{f}(\xi), \quad \widehat{P(\Lambda)f}(\xi) = P(|\xi|)\widehat{f}(\xi), \quad \widehat{\mathcal{R}_l f}(\xi) = \frac{i\,\xi_l}{|\xi|}\,\widehat{f}(\xi),$$

where $1 \le l \le d$ is an integer, $\widehat{f}$ or $\mathcal{F}(f)$ denotes the Fourier transform,

$$\widehat{f}(\xi) = \mathcal{F}(f)(\xi) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{-ix\cdot\xi} f(x)\, dx.$$

We are primarily concerned with the global (in time) regularity issue concerning solutions of (1.1) with a given initial data

$$(1.2) \qquad \theta(x,0) = \theta_0(x), \quad x \in \mathbb{R}^d.$$

A special example of (1.1) is the 2D active scalar equation

$$(1.3) \qquad \begin{cases} \partial_t \theta + u \cdot \nabla \theta + \kappa(-\Delta)^{\alpha}\theta = 0, & x \in \mathbb{R}^2,\, t > 0, \\ u = \nabla^{\perp}\psi \equiv (-\partial_{x_2}\psi, \partial_{x_1}\psi), & \Delta\psi = P(\Lambda)\,\theta \end{cases}$$

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which includes as special cases the 2D Navier-Stokes vorticity equation

$$(1.4) \qquad \begin{cases} \partial_t \omega + u \cdot \nabla \omega - \nu \Delta \omega = 0, \\ u = \nabla^\perp \psi, \quad \Delta \psi = \omega \end{cases}$$

and the dissipative surface quasi-geostrophic (SQG) equation

$$(1.5) \qquad \begin{cases} \partial_t \theta + u \cdot \nabla \theta + \kappa(-\Delta)^\alpha \theta = 0, \\ u = \nabla^\perp \psi, \quad -\Lambda \psi = \theta. \end{cases}$$

There are numerous studies on the Navier-Stokes equations and the global regularity in the 2D case has long been established (see e.g. [22], [38] and [64]). The SQG equation models the dynamics of the potential temperature θ of the 3D quasi-geostrophic equations on the 2D horizontal boundaries and is useful in modeling atmospheric phenomena such as the frontogenesis (see e.g. [25], [65] and [76]). The SQG equation (inviscid or dissipative) is also mathematically important. As detailed in [25], the behavior of its strongly nonlinear solutions are strikingly analogous to that of the potentially singular solutions of the 3D incompressible Navier-Stokes and the Euler equations. The global regularity issue concerning the SQG equation has recently been studied very extensively and many important progress has been made (see e.g. [1], [2], [3], [5], [6], [7], [8], [9], [10], [11], [12], [13], [15], [16], [18], [19], [20], [21], [23], [24], [25], [26], [28], [29], [30], [31], [32], [33], [34], [35], [36], [37], [39], [40], [41], [42], [43], [44], [45], [46], [47], [48], [49], [50], [51], [52], [53], [54], [55], [56], [57], [58], [59], [60], [61], [62], [63], [64], [65], [66], [67], [68], [69], [70], [71], [72], [73], [74], [75], [76], [77], [78], [79], [80], [82], [83], [84], [85], [86], [89], [90], [91], [92], [93], [94], [95], [96], [96], [97], [98], [99], [100], [101], [102], [103], [104], [105], [106]). In particular, the global regularity for the critical case $\alpha = 1/2$ has been successfully established ([7], [60]). The situation in the supercritical case $\alpha < 1/2$ is only partially understood at the time of writing. The results in [28], [29] and [44] imply that any solution of the supercritical SQG equation can develop potential finite time singularity only in the regularity window between L^∞ and C^δ with $\delta < 1 - 2\alpha$. Several very recent preprints on the supercritical case also revealed some very interesting properties of the supercritical dissipation ([3], [36], [57], [84]).

Our goal here is to establish the global regularity of (1.1) for more general operators P. In particular, we are interested in the global regularity of the intermediate equations between the 2D Navier-Stokes equation and the supercritical SQG equation. This paper is a continuation of our previous study on the inviscid counterpart of (1.1) ([14]). The consideration here is restricted to P satisfying the following condition.

Condition 1.1. *The symbol $P = P(|\xi|)$ assumes the following properties:*
   (1) *P is continuous on $\mathbb{R}^d$ and $P \in C^\infty(\mathbb{R}^d \setminus \{0\})$;*
   (2) *P is radially symmetric;*
   (3) *$P = P(|\xi|)$ is nondecreasing in $|\xi|$;*
   (4) *There exist two constants C and C_0 such that*

$$\sup_{2^{-1} \leq |\eta| \leq 2} \left| (I - \Delta_\eta)^n P(2^j |\eta|) \right| \leq C P(C_0 2^j)$$

   for any integer j and $n = 1, 2, \cdots, 1 + \left[\frac{d}{2}\right]$.

We remark that (4) in Condition 1.1 is a very natural condition on symbols of Fourier multiplier operators and is similar to the main condition in the Mihlin-Hörmander Multiplier Theorem (see e.g. [87, p.96]). For notational convenience, we also assume that $P \geq 0$. Some special examples of P are

$$P(\xi) = \left(\log(1 + |\xi|^2)\right)^\gamma \quad \text{with } \gamma \geq 0,$$
$$P(\xi) = \left(\log(1 + \log(1 + |\xi|^2))\right)^\gamma \quad \text{with } \gamma \geq 0,$$
$$P(\xi) = |\xi|^\beta \quad \text{with } \beta \geq 0,$$
$$P(\xi) = (\log(1 + |\xi|^2))^\gamma |\xi|^\beta \quad \text{with } \gamma \geq 0 \text{ and } \beta \geq 0.$$

As in the study of the Navier-Stokes and the Euler equations, the quantity $\|\nabla u\|_{L^\infty}$ plays a crucial role in the global regularity issue. In our previous work on the inviscid counterpart of (1.1), we established bounds for the building blocks $\|\nabla \Delta_j u\|_{L^q}$ and $\|\nabla S_N u\|_{L^q}$ for $1 \leq q \leq \infty$. More precisely, the following theorem is proven in [14].

Theorem 1.2. *Let $u : \mathbb{R}^d \to \mathbb{R}^d$ be a vector field. Assume that u is related to a scalar θ by*

$$(\nabla u)_{jk} = \mathcal{R}_l \mathcal{R}_m P(\Lambda)\, \theta,$$

where $1 \leq j, k, l, m \leq d$, $(\nabla u)_{jk}$ denotes the (j,k)-th entry of ∇u, $\mathcal{R}_l$ denotes the Riesz transform, and P obeys Condition 1.1. Then, for any integers $j \geq 0$ and $N \geq 0$,

$$(1.6) \qquad \|S_N \nabla u\|_{L^p} \leq C_{p,d}\, P(C_0 2^N)\, \|S_N \theta\|_{L^p}, \quad 1 < p < \infty,$$

$$(1.7) \qquad \|\Delta_j \nabla u\|_{L^q} \leq C_d\, P(C_0 2^j)\, \|\Delta_j \theta\|_{L^q}, \quad 1 \leq q \leq \infty,$$

$$(1.8) \qquad \|S_N \nabla u\|_{L^\infty} \leq C_d\, \|\theta\|_{L^1 \cap L^\infty} + C_d\, N\, P(C_0 2^N)\, \|S_{N+1} \theta\|_{L^\infty},$$

where $C_{p,d}$ is a constant depending on p and d only and $C_d s'$ depend on d only.

With the aid of these bounds, we were able to show in [14] that (1.1) with $\kappa = 0$ and $P(\Lambda) = (\log(1 + \log(1 - \Delta)))^\gamma$ for $0 \leq \gamma \leq 1$ has a unique global (in time) solution in the Besov space $B^s_{q,\infty}(\mathbb{R}^d)$ with $d < q \leq \infty$ and $s > 1$. In addition, a regularity criterion is also provided in [14] for (1.1) with $P(\Lambda) = \Lambda^\beta$ for $0 \leq \beta \leq 1$. Our goal here is to extend our study to cover more general operators when we turn on the dissipation. Indeed we are able to establish the global existence and uniqueness for a very general family of symbols. Before stating the result, we introduce the extended Besov spaces. Here $\mathcal{S}'$ denotes the class of tempered distributions and Δ_j with $j \geq -1$ denotes the standard Fourier localization operator. The notation Δ_j, S_N and Besov spaces are now quite standard and can be found in several books and many papers (see e.g. [4], [17], [81], [88]). They can also be found in Appendix A of [14].

Definition 1.3. *Let $s \in \mathbb{R}$ and $1 \leq q, r \leq \infty$. Let $A = \{A_j\}_{j \geq -1}$ with $A_j \geq 0$ be a nondecreasing sequence. The extended Besov space $B^{s,A}_{q,r}$ consists of $f \in \mathcal{S}'(\mathbb{R}^d)$ satisfying*

$$\|f\|_{B^{s,A}_{q,r}} \equiv \left\| 2^{s A_j} \|\Delta_j f\|_{L^q(\mathbb{R}^d)} \right\|_{l^r} < \infty.$$

Obviously, when $A_j = j + 1$, $B^{s,A}_{q,r}$ becomes the standard inhomogeneous Besov space $B^s_{q,r}$. When $A_j = o(j + 1)$ as $j \to \infty$, $B^{s,A}_{q,r}$ is a less regular class than the corresponding Besov space $B^s_{q,r}$; we will refer to these spaces as sub-Besov spaces. When $j = o(A_j)$, $B^{s,A}_{q,r}$, we will refer to the spaces as super-Besov spaces.

With these definitions at our disposal, our main theorem can be stated as follows.

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Theorem 1.4. *Consider the dissipative active scalar equation (1.1) with $\kappa > 0$, $\alpha > 0$ and $P(\xi)$ satisfying Condition 1.1. Let $s > 1$, $2 \le q \le \infty$ and $A = \{A_j\}_{j \ge -1}$ be a nondecreasing sequence with $A_j \ge 0$. Let $\theta_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d) \cap B_{q,\infty}^{s,A}(\mathbb{R}^d)$. Assume either the velocity u is divergence-free or the solution θ is bounded in $L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ for all time. If, there exists a constant C such that for all $j \ge -1$,*

$$\text{(1.9)} \qquad \sum_{k \ge j-1, k \ge -1} \frac{2^{sA_{j-2}} P(2^{k+1})}{2^{sA_k} P(2^{j+1})} < C$$

and

$$\text{(1.10)} \qquad \kappa^{-1} 2^{s(A_j - A_{j-2})} (j+2) P(2^{j+2}) 2^{-2\alpha j} \to 0 \quad as \quad j \to \infty,$$

then (1.1) has a unique global solution θ satisfying

$$\theta \in L^\infty\left([0,\infty); B_{q,\infty}^{s,A}(\mathbb{R}^d)\right).$$

We single out two special consequences of Theorem 1.4. In the case when

$$\text{(1.11)} \qquad P(|\xi|) = \left(\log(I + |\xi|^2)\right)^\gamma, \ \gamma \ge 0 \quad and \quad A_j = (j+1)^b \quad \text{for some } b \le 1,$$

(1.9) is trivially satisfied and the condition in (1.10) reduces to

$$\text{(1.12)} \qquad 2^{s((j+1)^b - j^b)} (j+2)^{1+\gamma} 2^{-2\alpha j} \to 0 \quad as \quad j \to \infty,$$

which is obviously satisfied for any $\alpha > 0$. We thus obtain the following corollary.

Corollary 1.5. *Consider the dissipative Log-Euler equation*

$$\text{(1.13)} \qquad \begin{cases} \partial_t \theta + u \cdot \nabla \theta + \kappa(-\Delta)^\alpha \theta = 0, \\ u = \nabla^\perp \psi, \quad \Delta \psi = (\log(1 - \Delta))^\gamma \theta \end{cases}$$

with $\kappa > 0$, $\alpha > 0$ and $\gamma \ge 0$. Assume that θ_0 satisfies

$$\theta_0 \in Y \equiv L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2) \cap B_{q,\infty}^{s,A}(\mathbb{R}^2)$$

with $s > 1$, $2 \le q \le \infty$ and A given in (1.11). Then (1.13) has a unique global solution θ satisfying

$$\theta \in L^\infty\left([0,\infty); Y\right).$$

The assumption that $A_j = (j+1)^b$ with $b \le 1$ corresponds to the Besov and the sub-Besov spaces. We can also consider the solutions of (1.13) in super-Besov spaces by taking $A_j = (j+1)^b$ for $b > 1$. It is easy to see that (1.12) remains valid if $s b < 2\alpha$. Therefore (1.13) with $2\alpha > s b$ has a global solution in the super-Besov space $B_{q,\infty}^{s,A}$ with $A_j = (j+1)^b$ for $b > 1$.

Another very important special case is when

$$\text{(1.14)} \qquad A_j = j+1, \quad P(\xi) = |\xi|^\beta (\log(1 + |\xi|^2))^\gamma \quad \text{with } \gamma \ge 0 \text{ and } 0 \le \beta < 2\alpha \le 1.$$

Then again (1.9) is obviously satisfied and (1.10) is reduced to

$$2^{s((j+1)^b - j^b)} (j+2)^{1+\gamma} 2^{(\beta - 2\alpha)j} \to 0 \quad as \ j \to \infty,$$

which is clearly true. That is, the following corollary holds.

Corollary 1.6. *Consider the active scalar equation*

(1.15)
$$\begin{cases} \partial_t\theta + u \cdot \nabla\theta + \kappa(-\Delta)^\alpha\theta = 0, \\ u = \nabla^\perp\psi, \quad \Delta\psi = \Lambda^\beta \left(\log(1-\Delta)\right)^\gamma \theta \end{cases}$$

with $\kappa > 0$, $\alpha > 0$, $0 \le \beta < 2\alpha \le 1$ and $\gamma \ge 0$. Assume the initial data $\theta_0 \in Y \equiv L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2) \cap B^{s,A}_{q,\infty}(\mathbb{R}^2)$ with $s > 1$, $2 \le q \le \infty$ and A_j given by (1.14). Then (1.15) has a unique global solution θ satisfying

$$\theta \in L^\infty\left([0,\infty); Y\right).$$

Again we could have studied the global solutions of (1.15) in a super-Besov space $B^{s,A}_{q,\infty}$ with, say $A_j = (j+1)^b$ for $b > 1$. Of course we need to put more restrictions on α. When $\gamma = 0$, (1.15) becomes

(1.16)
$$\begin{cases} \partial_t\theta + u \cdot \nabla\theta + \kappa(-\Delta)^\alpha\theta = 0, \\ u = \nabla^\perp\psi, \quad \Delta\psi = \Lambda^\beta \theta, \end{cases}$$

which we call the generalized SQG equation. Corollary 1.6 does not cover the case when $\beta = 2\alpha$, namely the modified SQG equation. The global regularity of the modified SQG equation with any L^2 initial data has previously been obtained in [23]. In the supercritical case when $\beta > 2\alpha$, the global regularity issue for (1.16) is open. In particular, the global issue for supercritical SQG equation ($\beta = 1$ and $2\alpha < 1$) remains outstandingly open.

Following the ideas in [11] and [25], we approach the global issue of (1.16) in the super case $\beta > 2\alpha$ by considering the geometry of the level curves of its solution. We present a geometric type criterion for the regularity of solutions of (1.16). This sufficient condition controls the regularity of solutions in terms of the space-time integrability of $|\nabla^\perp\theta|$ and the regularity of the direction field $\xi = \nabla^\perp\theta/|\nabla^\perp\theta|$ (unit tangent vector to a level curve of θ).

Theorem 1.7. *Consider (1.16) with $\kappa > 0$, $\alpha > 0$ and $0 \le \beta \le 1$. Let θ be the solution of (1.16) corresponding to the initial data $\theta_0 \in H^m(\mathbb{R}^2)$ with $m > 2$. Let $T > 0$. Suppose there exists $\sigma \in (0,1)$, $q_1 \in (\frac{2}{1+\beta-\sigma}, \infty]$, $p_1 \in (1,\infty]$, $p_2 \in (1, \frac{2}{1+\sigma-\beta})$ and $r_1, r_2 \in [1,\infty]$ such that the followings hold.*

(1.17)
$$\xi \in L^{r_1}(0,T; \dot{\mathcal{F}}^\sigma_{p_1,q}(\mathbb{R}^2)) \quad and \quad \nabla^\perp\theta \in L^{r_2}(0,T; L^{p_2}(\mathbb{R}^2))$$

$$with \qquad \frac{1}{p_1} + \frac{1}{p_2} + \frac{\alpha}{r_1} + \frac{\alpha}{r_2} \le \alpha + \frac{1}{2}(1+\sigma-\beta).$$

Then θ remains in $H^m(\mathbb{R}^2)$ on $[0,T]$. Especially, when $p_1 = r_1 = q = \infty$, (1.17) becomes

$$\xi \in L^\infty(0,T; C^\sigma(\mathbb{R}^2)) \quad and \quad \nabla^\perp\theta \in L^{r_2}(0,T; L^{p_2}(\mathbb{R}^2))$$

$$with \qquad \frac{1}{p_2} + \frac{\alpha}{r_2} \le \alpha + \frac{1}{2}(1+\sigma-\beta).$$

Here $\dot{\mathcal{F}}^s_{p,q}(\mathbb{R}^2)$ denotes a homogeneous Trebiel-Lizorkin type space. For $0 \le s \le 1$, $1 \le p \le \infty$ and $1 \le q \le \infty$, $\dot{\mathcal{F}}^s_{p,q}$ contains functions such that the following semi-norm

is finite,

$$\|f\|_{\dot{\mathcal{F}}_{p,q}^s} = \begin{cases} \left\| \left(\int \dfrac{|f(x+y) - f(x)|^q}{|y|^{n+sq}} \, dy \right)^{\frac{1}{q}} \right\|_{L^p}, & \text{if } q < \infty, \\[4mm] \left\| \sup_{y \neq 0} \dfrac{|f(x+y) - f(x)|}{|y|^s} \right\|_{L^p}, & \text{if } q = \infty \end{cases}$$

We note that if we set $\beta = 1$ in Theorem 1.7, then it reduces to Theorem 1.2 of [11].

The rest of this paper is divided into two sections. Section 2 proves Theorem 1.4 while Section 3 derives the geometric regularity criterion stated in Theorem 1.7.

2. Proof of Theorem 1.4

This section is devoted to the proof of Theorem 1.4, which involves Besov space technique and the bounds stated in Theorem 1.2. In addition, lower bound estimates associated with the fractional dissipation are also used.

Proof of Theorem 1.4. The proof is divided into two main parts. The first part establishes the global (in time) *a priori* bound on solutions of (1.1) while the second part briefly describes the construction of a unique local (in time) solution.

For notational convenience, we write $Y = L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d) \cap B_{q,\infty}^{s,A}(\mathbb{R}^d)$. The first part derives the global bound, for any $T > 0$,

$$(2.1) \qquad \|\theta(\cdot, t)\|_{B_{q,\infty}^{s,A}} \leq C(T, \|\theta_0\|_Y) \quad \text{for} \quad t \leq T$$

and we distinguish between two cases: $q < \infty$ and $q = \infty$. The dissipative term is handled differently in these two cases.

We start with the case when $q < \infty$. When the velocity field u is divergence-free, $\theta_0 \in L^1 \cap L^\infty$ implies the corresponding solution θ of (1.1) satisfies the *a priori* bound

$$(2.2) \qquad \|\theta(\cdot, t)\|_{L^1 \cap L^\infty} \leq \|\theta_0\|_{L^1 \cap L^\infty}, \quad t \geq 0.$$

When u is not divergence-free, (2.2) is assumed. The divergence-free condition is not used in the rest of the proof.

Let $j \geq -1$ be an integer. Applying Δ_j to (1.1) and following a standard decomposition, we have

$$(2.3) \qquad \partial_t \Delta_j \theta + \kappa(-\Delta)^\alpha \Delta_j \theta = J_1 + J_2 + J_3 + J_4 + J_5,$$

where

$$(2.4) \qquad J_1 = -\sum_{|j-k|\leq 2} [\Delta_j, S_{k-1}(u) \cdot \nabla]\Delta_k\theta,$$

$$(2.5) \qquad J_2 = -\sum_{|j-k|\leq 2} (S_{k-1}(u) - S_j(u)) \cdot \nabla\Delta_j\Delta_k\theta,$$

$$(2.6) \qquad J_3 = -S_j(u) \cdot \nabla\Delta_j\theta,$$

$$(2.7) \qquad J_4 = -\sum_{|j-k|\leq 2} \Delta_j(\Delta_k u \cdot \nabla S_{k-1}(\theta)),$$

$$(2.8) \qquad J_5 = -\sum_{k\geq j-1} \Delta_j(\widetilde{\Delta}_k u \cdot \nabla\Delta_k\theta)$$

with $\widetilde{\Delta}_k = \Delta_{k-1} + \Delta_k + \Delta_{k+1}$. We multiply (2.3) by $\Delta_j\theta|\Delta_j\theta|^{q-2}$ and integrate in space. Integrating by parts in the term associated with J_3, we obtain

$$-\int_{\mathbb{R}^d} (S_j(u) \cdot \nabla\Delta_j\theta)\, \Delta_j\theta|\Delta_j\theta|^{q-2}\, dx = \frac{1}{q}\int_{\mathbb{R}^d} (\nabla \cdot S_j u)|\Delta_j\theta|^q\, dx$$
$$= \int_{\mathbb{R}^d} \widetilde{J}_3\, |\Delta_j\theta|^{q-1}\, dx,$$

where $\widetilde{J}_3$ is given by

$$\widetilde{J}_3 = \frac{1}{q}(\nabla \cdot S_j u)|\Delta_j\theta|.$$

Applying Hölder's inequality, we have

$$(2.9) \qquad \frac{1}{q}\frac{d}{dt}\|\Delta_j\theta\|_{L^q}^q + \kappa\int \Delta_j\theta|\Delta_j\theta|^{q-2}(-\Delta)^\alpha\Delta_j\theta\, dx$$
$$\leq \left(\|J_1\|_{L^q} + \|J_2\|_{L^q} + \|\widetilde{J}_3\|_{L^q} + \|J_4\|_{L^q} + \|J_5\|_{L^q} \right) \|\Delta_j\theta\|_{L^q}^{q-1}.$$

For $j \geq 0$, we have the lower bound (see [18] and [98])

$$(2.10) \qquad \int \Delta_j\theta|\Delta_j\theta|^{q-2}(-\Delta)^\alpha\Delta_j\theta \geq C\, 2^{2\alpha j}\, \|\Delta_j\theta\|_{L^q}^q.$$

For $j = -1$, this lower bound is invalid. Still we have

$$(2.11) \qquad \int \Delta_j\theta|\Delta_j\theta|^{q-2}(-\Delta)^\alpha\Delta_j\theta \geq 0.$$

Attention is paid to the case $j \geq 0$ first. Inserting (2.10) in (2.9) leads to

$$\frac{d}{dt}\|\Delta_j\theta\|_{L^q} + \kappa\, 2^{2\alpha j}\, \|\Delta_j\theta\|_{L^q} \leq \|J_1\|_{L^q} + \|J_2\|_{L^q} + \|\widetilde{J}_3\|_{L^q} + \|J_4\|_{L^q} + \|J_5\|_{L^q}.$$

By a standard commutator estimate,

$$\|J_1\|_{L^q} \leq C\sum_{|j-k|\leq 2} \|\nabla S_{k-1}u\|_{L^\infty}\|\Delta_k\theta\|_{L^q}.$$

By Hölder's and Bernstein's inequalities,

$$\|J_2\|_{L^q} \leq C\, \|\nabla\widetilde{\Delta}_j u\|_{L^\infty}\, \|\Delta_j\theta\|_{L^q}.$$

Clearly,

$$\|\widetilde{J_3}\|_{L^q} \le C \, \|\nabla \cdot S_j u\|_{L^\infty} \, \|\Delta_j \theta\|_{L^q}.$$

For J_4 and J_5, we have

$$
\begin{aligned}
\|J_4\|_{L^q} &\le \sum_{|j-k|\le 2} \|\Delta_k u\|_{L^\infty} \, \|\nabla S_{k-1}\theta\|_{L^q}, \\
\|J_5\|_{L^q} &\le \sum_{k\ge j-1} \|\widetilde{\Delta}_k u\|_{L^\infty} \|\Delta_k \nabla\theta\|_{L^q} \\
&\le C \sum_{k\ge j-1} \|\nabla\widetilde{\Delta}_k u\|_{L^\infty} \, \|\Delta_k\theta\|_{L^q}.
\end{aligned}
$$

These terms can be further bounded as follows. By Theorem 1.2,

$$
\begin{aligned}
\|\nabla S_k u\|_{L^\infty} &\le \|\theta_0\|_{L^1\cap L^\infty} + C k \, P(2^{k+1})\|S_{k+1}\theta\|_{L^\infty} \\
&\le \|\theta_0\|_{L^1\cap L^\infty} + C k \, P(2^{k+1})\|\theta_0\|_{L^\infty}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\|J_1\|_{L^q} &\le C \, \|\theta_0\|_{L^1\cap L^\infty} \sum_{|j-k|\le 2} (1 + C k \, P(2^{k+1}))2^{-sA_k} \, 2^{sA_k}\|\Delta_k\theta\|_{L^q} \\
&\le C \, 2^{-sA_j} \, \|\theta_0\|_{L^1\cap L^\infty}\|\theta\|_{B_{q,\infty}^{s,A}} \sum_{|j-k|\le 2} (1 + C k \, P(2^{k+1}))2^{s(A_j-A_k)}.
\end{aligned}
$$

Since A_j is a nondecreasing function of j,

$$(2.12) \qquad\qquad 2^{s(A_j-A_k)} \le 2^{s(A_j-A_{j-2})} \quad \text{for} \quad |k-j| \le 2,$$

where we have adopted the convention that $A_l \equiv 0$ for $l < -1$. Consequently,

$$\|J_1\|_{L^q} \le C \, 2^{-sA_{j-2}} \, \|\theta_0\|_{L^1\cap L^\infty}\|\theta\|_{B_{q,\infty}^{s,A}} \left(1 + (j+2)P(2^{j+2})\right).$$

Clearly, $\|J_2\|_{L^q}$ and $\|J_3\|_{L^q}$ admits the same bound as $\|J_1\|_{L^q}$. By Bernstein's inequality and Theorem 1.2,

$$
\begin{aligned}
\|J_4\|_{L^q} &\le C \sum_{|j-k|\le 2} \|\nabla\Delta_k u\|_{L^q} \, \|S_{k-1}\theta\|_{L^\infty} \\
&\le C \, \|\theta\|_{L^\infty} \sum_{|j-k|\le 2} P(2^{k+1})\|\Delta_k\theta\|_{L^q}.
\end{aligned}
$$

By (2.12), we have

$$\|J_4\|_{L^q} \le C \, 2^{-sA_{j-2}} \, \|\theta_0\|_{L^\infty} \, \|\theta\|_{B_{q,\infty}^{s,A}} \, P(2^{j+2}).$$

By Theorem 1.2,

$$
\begin{aligned}
\|J_5\|_{L^q} &\le C \sum_{k\ge j-1} P(2^{k+1})\|\widetilde{\Delta}_k\theta\|_{L^\infty}\|\Delta_k\theta\|_{L^q} \\
&\le C \, \|\theta_0\|_{L^\infty} \sum_{k\ge j-1} P(2^{k+1})\|\Delta_k\theta\|_{L^q} \\
&\le C \, \|\theta_0\|_{L^\infty} 2^{-sA_{j-2}} \, P(2^{j+1})\|\theta\|_{B_{q,\infty}^{s,A}} \sum_{k\ge j-1} \frac{2^{sA_{j-2}}}{P(2^{j+1})} \frac{P(2^{k+1})}{2^{sA_k}}
\end{aligned}
$$

By (1.9),

$$\|J_5\|_{L^q} \leq C \|\theta_0\|_{L^\infty} 2^{-sA_{j-2}} P(2^{j+1}) \|\theta\|_{B^{s,A}_{q,\infty}}.$$

Collecting all the estimates, we have, for $j \geq 0$,

$$\frac{d}{dt}\|\Delta_j\theta\|_{L^q} + \kappa\, 2^{2\alpha j}\,\|\Delta_j\theta\|_{L^q} \leq C\, 2^{-sA_{j-2}} \|\theta_0\|_{L^1\cap L^\infty}$$
$$\times \|\theta\|_{B^{s,A}_{q,\infty}} \left(1 + (j+2)P(2^{j+2})\right).$$

That is,

$$\frac{d}{dt}\left(e^{\kappa 2^{2\alpha j}t}\|\Delta_j\theta\|_{L^q}\right) \leq C\, e^{\kappa 2^{2\alpha j}t} 2^{-sA_{j-2}} \|\theta_0\|_{L^1\cap L^\infty}\|\theta\|_{B^{s,A}_{q,\infty}} \left(1 + (j+2)P(2^{j+2})\right).$$

Integrating in time and multiplying by $2^{sA_j} \cdot e^{-\kappa 2^{2\alpha j}t}$, we obtain, for $j \geq 0$,

$$(2.13) \qquad 2^{sA_j}\,\|\Delta_j\theta\|_{L^q} \leq 2^{sA_j}\, e^{-\kappa 2^{2\alpha j}t}\|\Delta_j\theta_0\|_{L^q} + K_j,$$

where

$$K_j = C\,\|\theta_0\|_{L^1\cap L^\infty}\left(1 + (j+2)P(2^{j+2})\right) 2^{s(A_j - A_{j-2})} \int_0^t e^{-\kappa 2^{2\alpha j}(t-\tau)}\|\theta(\tau)\|_{B^{s,A}_{q,\infty}}\,d\tau.$$

To further the estimates, we fix $t_0 \leq T$ and let $t \leq t_0$. It is easy to see that K_j admits the upper bound

$$K_j \leq C\,\|\theta_0\|_{L^1\cap L^\infty}\left(1 + (j+2)P(2^{j+2})\right) 2^{s(A_j - A_{j-2})}$$
$$\times \frac{1}{\kappa 2^{2\alpha j}}\left(1 - e^{-\kappa 2^{2\alpha j}t}\right) \sup_{0\leq\tau\leq t_0} \|\theta(\tau)\|_{B^{s,A}_{q,\infty}}.$$

According to (1.10), there exists an integer j_0 such that, for $j \geq j_0$,

$$(2.14) \qquad K_j \leq \frac{1}{2} \sup_{0\leq\tau\leq t_0} \|\theta(\tau)\|_{B^{s,A}_{q,\infty}}.$$

For $0 \leq j \leq j_0$,

$$(2.15) \quad K_j \leq C\,\|\theta_0\|_{L^1\cap L^\infty}\left(1 + (j_0+2)P(2^{j_0+2})\right) \max_{0\leq j\leq j_0} 2^{s(A_j - A_{j-2})} \int_0^t \|\theta(\tau)\|_{B^{s,A}_{q,\infty}}\,d\tau.$$

We now turn to the case when $j = -1$. By combining (2.3) and (2.11) and estimating $\|J_1\|_{L^q}$ through $\|J_5\|_{L^q}$ in an similar fashion as for the case $j \geq 0$, we obtain

$$(2.16) \qquad \|\Delta_{-1}\theta(t)\|_{L^q} \leq \|\Delta_{-1}\theta(0)\|_{L^q} + C\,\|\theta_0\|_{L^1\cap L^\infty} \int_0^t \|\theta(\tau)\|_{B^{s,A}_{q,\infty}}\,d\tau.$$

Putting (2.13) and (2.16) together, we find, for any $j \geq -1$,

$$(2.17) \qquad 2^{sA_j}\,\|\Delta_j\theta\|_{L^q} \leq \|\theta_0\|_{B^{s,A}_{q,\infty}} + K_j,$$

where K_j obeys the bound (2.14) for $j \geq j_0$ and the bound in (2.15) for $-1 \leq j < j_0$. Applying $\sup_{j\geq -1}$ to (2.17) and using the simple fact that

$$\sup_{j\geq -1} K_j \leq \sup_{j\geq j_0} K_j + \sup_{-1\leq j < j_0} K_j,$$

we obtain

$$\|\theta(t)\|_{B^{s,A}_{q,\infty}} \leq \|\theta_0\|_{B^{s,A}_{q,\infty}} + \frac{1}{2}\sup_{0\leq\tau\leq t_0}\|\theta(\tau)\|_{B^{s,A}_{q,\infty}} + C(\theta_0, j_0)\int_0^t \|\theta(\tau)\|_{B^{s,A}_{q,\infty}}\,d\tau,$$

where

$$C(\theta_0, j_0) = C \, \|\theta_0\|_{L^1 \cap L^\infty} \left(1 + (j_0 + 2)P(2^{j_0+2})\right) \max_{0 \le j \le j_0} 2^{s(A_j - A_{j-2})}.$$

Now taking supremum over $t \in [0, t_0]$, we obtain

$$\sup_{0 \le \tau \le t_0} \|\theta(\tau)\|_{B^{s,A}_{q,\infty}} \le 2 \, \|\theta_0\|_{B^{s,A}_{q,\infty}} + C(\theta_0, j_0) \int_0^{t_0} \|\theta(\tau)\|_{B^{s,A}_{q,\infty}} \, d\tau,$$

Gronwall's inequality then implies (2.1) for any $t \le t_0 \le T$. This finishes the case when $q < \infty$.

We now turn to the case when $q = \infty$. For $j \ge 0$, applying Δ_j yields

$$\partial_t \Delta_j \theta + S_j u \cdot \nabla(\Delta_j \theta) + \kappa(-\Delta)^\alpha \Delta_j \theta = J_1 + J_2 + J_4 + J_5$$

where J_1, J_2, J_4 and J_5 are as defined in (2.4), (2.5), (2.7) and (2.8), respectively. According to Lemma 2.1 below, we have

(2.18) $\partial_t \|\Delta_j \theta\|_{L^\infty} + C \, 2^{2\alpha j} \|\Delta_j \theta\|_{L^\infty} \le \|J_1\|_{L^\infty} + \|J_2\|_{L^\infty} + \|J_4\|_{L^\infty} + \|J_5\|_{L^\infty}.$

The terms on the right can be estimated similarly as in the case when $q < \infty$. For $j = -1$, (2.18) is replaced by

$$\partial_t \|\Delta_{-1} \theta\|_{L^\infty} \le \|J_1\|_{L^\infty} + \|J_2\|_{L^\infty} + \|J_4\|_{L^\infty} + \|J_5\|_{L^\infty}.$$

The rest of the proof for this case is then very similar to the case $q < \infty$ and we thus omit further details.

We briefly describe the construction of a local solution of (1.1) and prove its uniqueness. The solution is constructed through the method of successive approximation. More precisely, we consider a successive approximation sequence $\{\theta^{(n)}\}$ satisfying

(2.19)
$$\begin{cases} \theta^{(1)} = S_2 \theta_0, \\[2mm] u^{(n)} = (u_j^{(n)}), \quad u_j^{(n)} = \mathcal{R}_l \Lambda^{-1} P(\Lambda) \theta^{(n)}, \\[2mm] \partial_t \theta^{(n+1)} + u^{(n)} \cdot \nabla \theta^{(n+1)} + \kappa(-\Delta)^\alpha \theta^{(n+1)} = 0, \\[2mm] \theta^{(n+1)}(x, 0) = S_{n+2} \theta_0 \end{cases}$$

and show that $\{\theta^{(n)}\}$ converges to a solution of (1.1). It suffices to prove the following properties of $\{\theta^{(n)}\}$:

i) There exists $T_1 > 0$ such that $\theta^{(n)}$ is bounded uniformly in $B^{s,A}_{q,\infty}$ for any $t \in [0, T_1]$, namely

$$\|\theta^{(n)}(\cdot, t)\|_{B^{s,A}_{q,\infty}} \le C_1 \|\theta_0\|_Y, \quad t \in [0, T_1],$$

where C_1 is a constant independent of n.

ii) There exists $T_2 > 0$ such that $\eta^{(n+1)} = \theta^{(n+1)} - \theta^{(n)}$ is a Cauchy sequence in $B^{s-1,A}_{q,\infty}$,

$$\|\eta^{(n)}(\cdot, t)\|_{B^{s-1,A}_{q,\infty}} \le C_2 \, 2^{-n}, \quad t \in [0, T_2],$$

where C_2 is independent of n and depends on T_2 and $\|\theta_0\|_Y$ only.

Since the essential ingredients in the proof of i) and ii) have appeared in proving the *a priori* bound, we omit the details. The uniqueness can be established by estimating the difference of any two solutions in $B_{q,\infty}^{s-1,A}$. A similar argument as in the proof of ii) would yield the desired result. This completes the proof of Theorem 1.4. □

We have used the following lemma in the proof of Theorem 1.4. It is obtained in [90].

Lemma 2.1. *Let $j \geq 0$ be an integer. Let θ, u and f be smooth functions solving the equation*

$$\partial_t \Delta_j \theta + u \cdot \nabla \Delta_j \theta + \kappa(-\Delta)^\alpha \Delta_j \theta = f,$$

where $\kappa > 0$ is a parameter. Assume that $\Delta_j \theta$ vanishes at infinity. Then, there exists a constant C independent of θ, u, f and j such that

$$\partial_t \|\Delta_j \theta\|_{L^\infty} + C \, 2^{2\alpha j} \|\Delta_j \theta\|_{L^\infty} \leq \|f\|_{L^\infty}.$$

3. GEOMETRIC REGULARITY CRITERION

In this section we prove Theorem 1.7. For this we recall the following Serrin type of criterion, which is proved for $\beta = 1$ in [11, Theorem 1.1], and obviously holds true for our case of $\beta \in [0,1]$.

Theorem 3.1. *Let $\theta(x,t)$ be a solution of (1.16) with initial data $\theta_0 \in H^m(\mathbb{R}^2)$ with $m > 2$. Let $T > 0$. If there are indices p, r with $\frac{1}{\alpha} < p < \infty$ and $1 < r < \infty$ respectively such that*

$$(3.1) \qquad \nabla^\perp \theta \in L^r(0, T; L^p(\mathbb{R}^2)) \quad \text{with} \quad \frac{1}{p} + \frac{\alpha}{r} \leq \alpha,$$

then θ remains in $H^m(\mathbb{R}^2)$ on $[0,T]$.

Proof of Theorem 1.7. Since the proof is similar to that of Theorem 1.2 in [11], we will be brief here mostly pointing out the essential changes. Let p be an integer of the form $p = 2^k$, where k is a positive integer, and satisfy

$$(3.2) \qquad \frac{1}{\alpha} \leq p < \infty.$$

We take operation of $\nabla^\perp$ on (1.16), and take $L^2(\mathbb{R}^2)$ inner product of it by $\nabla^\perp \theta(x,t)|\nabla^\perp \theta(x,t)|^{p-2}$, and then substituting $u = -\nabla^\perp \Lambda^{-2+\beta} \theta$ into it, we have

$$\frac{1}{p}\frac{d}{dt}\|\nabla^\perp \theta(t)\|_{L^p}^p + \kappa \int (\Lambda^{2\alpha} \nabla^\perp \theta) \cdot \nabla^\perp \theta |\nabla^\perp \theta|^{p-2} dx$$

$$= \int (\nabla^\perp \theta \cdot \nabla) u \cdot \nabla^\perp \theta |\nabla^\perp \theta|^{p-2} dx$$

$$= \int \int [\nabla \theta(x,t) \cdot \hat{y}][\nabla^\perp \theta(x+y,t) \cdot \nabla \theta(x,t)] \frac{dy}{|y|^{1+\beta}} |\nabla^\perp \theta(x,t)|^{p-2} dx$$

$$(3.3) \qquad := I,$$

where the integral with respect to y in the right hand side is in the sense of principal value. The dissipation term can be estimated

$$\kappa \int (\Lambda^{2\alpha}\nabla^\perp\theta) \cdot \nabla^\perp\theta|\nabla^\perp\theta|^{p-2}dx \geq \frac{\kappa}{p}\int \left|\Lambda^\alpha|\nabla^\perp\theta|^{\frac{p}{2}}\right|^2 dx$$

$$(3.4) \qquad\qquad \geq \frac{\kappa C_\alpha}{p}\left(\int |\nabla^\perp\theta|^{\frac{p}{1-\alpha}}dx\right)^{1-\alpha} = \frac{\kappa C_\alpha}{p}\|\nabla^\perp\theta\|^p_{L^{\frac{p}{1-\alpha}}},$$

where we used Lemma 2.4 of [31] and the embedding $L^2_\alpha(\mathbb{R}^2) \hookrightarrow L^{\frac{2}{1-\alpha}}(\mathbb{R}^2)$. Next, we estimate I as follows.

$$I = \int\int(\xi^\perp(x,t)\cdot\hat{y})[\xi(x+y,t)\cdot\xi^\perp(x,t)]|\nabla^\perp\theta(x+y,t)|\frac{dy}{|y|^{1+\beta}}|\nabla^\perp\theta(x,t)|^p dx$$

$$= \int\int(\xi^\perp(x,t)\cdot\hat{y})[\xi(x+y,t)-\xi(x,t)]\cdot\xi^\perp(x,t)|\nabla^\perp\theta(x+y,t)|\frac{dy}{|y|^{1+\beta}}|\nabla^\perp\theta(x,t)|^p dx$$

$$\leq \int\int|\xi(x+y,t)-\xi(x,t)||\nabla^\perp\theta(x+y,t)|\frac{dy}{|y|^{\frac{2+(\beta-1+s)q}{q}+\frac{2-sq'}{q'}}}|\nabla^\perp\theta(x,t)|^p dx$$

$$\leq \int\left(\int\frac{|\xi(x+y,t)-\xi(x,t)|^q}{|y|^{2+(\beta-1+s)q}}dy\right)^{\frac{1}{q}}\left(\int\frac{|\nabla^\perp\theta(x+y,t)|^{q'}}{|y|^{2-sq'}}dy\right)^{\frac{1}{q'}}|\nabla^\perp\theta|^p dx$$

$$\leq \|\xi\|_{\dot{\mathcal{F}}^\sigma_{p_1,q}}\left\|\{I_{sq'}(|\nabla^\perp\theta|^{q'})\}^{\frac{1}{q'}}\right\|_{L^{\tilde{p}_2}}\|\nabla^\perp\theta\|^p_{L^{p_3}},$$

where we used the fact $\xi(x,t)\cdot\xi^\perp(x,t) = 0$ in the second equality, and Hölder's inequality in the second and the third inequalities with the exponents satisfying

$$(3.5) \qquad\qquad \frac{1}{p_1}+\frac{1}{\tilde{p}_2}+\frac{p}{p_3} = 1, \qquad \frac{1}{q}+\frac{1}{q'} = 1,$$

and $I_a(\cdot)$, $0 < a < 2$, is the operator defined by the Riesz potential. We also set

$$(3.6) \qquad\qquad\qquad \sigma = \beta - 1 + s$$

in the last inequality. After this, we apply Hardy-Littlewood-Sobolev inequality and Young's inequality to estimate I, which is similar to the proof of Theorem 1.2 of [11], and deduce

$$(3.7) \qquad \frac{d}{dt}\|\nabla^\perp\theta(t)\|^p_{L^p}+\frac{\kappa C_\alpha}{2}\|\nabla^\perp\theta(t)\|^p_{L^{\frac{p}{1-\alpha}}} \leq C\|\xi(t)\|^Q_{\dot{\mathcal{F}}^\sigma_{p_1,q}}\|\nabla^\perp\theta(t)\|^Q_{L^{p_2}}\|\nabla^\perp\theta(t)\|^p_{L^p},$$

where we set

$$(3.8) \qquad\qquad\qquad Q = \frac{2\alpha p_1 p_2}{(2\alpha+s)p_1 p_2 - 2p_1 - 2p_2},$$

which need to satisfy

$$(3.9) \qquad\qquad\qquad \frac{1}{r_1}+\frac{1}{r_2} \leq \frac{1}{Q}.$$

We note that (3.9) is equivalent to

$$\frac{1}{p_1}+\frac{1}{p_2}+\frac{\alpha}{r_1}+\frac{\alpha}{r_2} \leq \alpha+\frac{1}{2}(1+\sigma-\beta)$$

after substituting Q and s from (3.8) and (3.6) respectively into (3.9). Since

$$\int_0^T \|\xi(t)\|_{\dot{\mathcal{F}}^\sigma_{p_1,q}}^Q \|\nabla^\perp \theta(t)\|_{L^{p_2}}^Q dt \le \left(\int_0^T \|\xi(t)\|_{\dot{\mathcal{F}}^\sigma_{p_1,q}}^{r_1} dt \right)^{\frac{Q}{r_1}} \left(\int_0^T \|\nabla^\perp \theta(t)\|_{L^{p_2}}^{r_2} dt \right)^{\frac{Q}{r_2}} < \infty$$

by our hypothesis, The inequality (3.7) leads us to

$$\int_0^T \|\nabla^\perp \theta\|_{L^{\frac{p}{1-\alpha}}}^p dt < \infty.$$

Now applying Theorem 3.1, we conclude the proof. $\qquad\square$

Acknowledgements

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