Topologies as points within a Stone space: lattice theory meets topology.

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Abstract

For a non-empty set X, the collection $Top(X)$ of all topologies on X sits inside the Boolean lattice $\mathcal{P}(\mathcal{P}(X))$ (when ordered by set-theoretic inclusion) which in turn can be naturally identified with the Stone space $2^{\mathcal{P}(X)}$. Via this identification then, $Top(X)$ naturally inherits the subspace topology from $2^{\mathcal{P}(X)}$. Extending ideas of Frink (1942), we apply lattice-theoretic methods to establish an equivalence between the topological closures of sublattices of $2^{\mathcal{P}(X)}$ and their (completely distributive) completions. We exploit this equivalence when searching for countably infinite compact subsets within $Top(X)$ and in crystalizing the Borel complexity of $Top(X)$. We exhibit infinite compact subsets of $Top(X)$ including, in particular, copies of the Stone-Čech and one-point compactifications of discrete spaces.

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1 Introduction

For a non-empty set X, the collection $Top(X)$ of all topologies on X sits inside the Boolean lattice $\mathcal{P}(\mathcal{P}(X))$ (when ordered by set-theoretic inclusion) which in turn can be naturally identified with the Stone space $2^{\mathcal{P}(X)}$. Via this identification then, $Top(X)$ naturally inherits the subspace topology from $2^{\mathcal{P}(X)}$ (see [1]), a subspace about which little is known. Frink [3] showed that endowing the lattice $\mathcal{P}(\mathcal{P}(X))$ with either the *interval* or the *order* topology yields the same space as $2^{\mathcal{P}(X)}$. In the same paper, Frink also proved that a lattice is complete if and only if it is compact in its interval topology.

These ideas enable us to apply lattice-theoretic techniques in the investigation of an object, whose individual elements provide a rich source of topological inquiry and knowledge. Just as the Stone-Čech compactification $\beta\mathbb{N}$ for discrete space $\mathbb{N}$ is homeomorphic to the subspace of all ultrafilters on $\mathbb{N}$, so the question of how all topologies on a fixed infinite set X behave collectively as a natural subspace of $2^{\mathcal{P}(X)}$ is interesting and yet is little explored. We establish that complete sublattices of $\mathcal{P}(\mathcal{P}(X))$ provide a rich supply of compact subsets within $2^{\mathcal{P}(X)}$. It is then possible to find infinite compact subsets of $Top(X)$ by purely lattice-theoretic means and to gain further insight into the topological complexity of $Top(X)$.

The first section of this paper focuses on extending the aforementioned results by describing the equivalence between the topological closures of sublattices of $2^{\mathcal{P}(X)}$ and their (completely distributive) completions. We exploit such an equivalence when searching for countably infinite compact subsets within $Top(X)$ and in crystalizing the Borel complexity of $Top(X)$. The last section is devoted to describing other infinite compact subsets of $Top(X)$ including, in particular, copies of the Stone-Čech and one-point compactifications of discrete spaces.

2 Preliminaries

For convenience and unless otherwise indicated, $2^{\mathcal{P}(X)}$ shall denote the usual Boolean algebra for an infinite set X equipped with the subset inclusion order ($\subseteq$) in addition to the usual product space, where 2 is the discrete space. We reserve the use of the symbol $\subset$ for cases of proper or strict containment only. The topology of any subset P of $2^{\mathcal{P}(X)}$ then is simply the usual subspace topology on P (which we shall denote where necessary by P_Π) while $\overline{P}$ will denote the topological closure of P in the space $2^{\mathcal{P}(X)}$.

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For a sublattice P of $2^{\mathcal{P}(X)}$, we denote by $\hat{P}$ its lattice-theoretic completion. Thus $\hat{P} = \bigcap\{L \subseteq 2^{\mathcal{P}(X)} \mid P \subseteq L$, L a sublattice of $2^{\mathcal{P}(X)}$ and $L = \hat{L}\}$. Of course, finite sublattices are trivially complete and, in general, the *free completion* of a lattice does not exist (see [2], [7], [8]). That said, the *completely distributive completion* of any partial order exists and is unique up to isomorphism [6]. Moreover, for a partial order P:

$$x \in \hat{P} \text{ if and only if } x = \bigwedge \bigvee S \text{ for all } S \subseteq P \text{ so that } x \leq \bigvee S.$$

Since $2^{\mathcal{P}(X)}$ (as well as any sublattice thereof) is completely distributive we have an explicit description of each element in the lattice-theoretic completion of any sublattice of $2^{\mathcal{P}(X)}$.

Definition 2.1. Let $(P, \leq)$ be a poset with $p \in P$. We define $p^{\downarrow} = \{x \in P \mid x \leq p\}$, $p^{\uparrow} = \{x \in P \mid p \leq x\}$, $p_{\downarrow} = p^{\downarrow} \setminus \{p\}$ and $p_{\uparrow} = p^{\uparrow} \setminus \{p\}$.

Definition 2.2. We shall adopt the following notation:

(i) If P is a lattice, and $S \subseteq P$, then we denote by $< S >_L$ the sublattice of P generated by closing off S under finite meets and finite joins.

(ii) If $\mathcal{S} \subseteq \mathcal{P}(X)$, then $< \mathcal{S} >_T$ denotes the topology σ on X generated by closing off $\mathcal{S}$ under finite intersections and arbitrary unions. Thus $\mathcal{S} \cup \{\emptyset, X\}$ is a subbase for σ.

Definition 2.3. Given any $S \subseteq 2^{\mathcal{P}(X)}$ we let $R_S = \{a \in 2^{\mathcal{P}(X)} \mid \forall b \in S$, either $b \subseteq a$ or $a \subseteq b\}$ and refer to it as the *set of relations of* S.

3 Completeness and compactness of sublattices

Lemma 3.1. Let P be a sublattice of $2^{\mathcal{P}(X)}$, let $S \subseteq P$ and let $x = \bigvee S$. If $x \notin P$, then x is a limit point of P.

Proof. Let $\bigcap A_i^+ \cap \bigcap B_j^-$ be an arbitrary open neighbourhood of x. Then for each of the finitely many i, there is $s_i \in S$ such that $A_i \in s_i$; furthermore $B_j \notin s_i$ for each j and each i. Thus $\bigvee_i s_i \in \bigcap A_i^+ \cap \bigcap B_j^- \cap P$ and clearly $\bigvee_i s_i \neq x$.

□

Recall that the *interval topology* on a poset P is the one generated by $\{x^{\uparrow} \mid x \in P\} \cup \{x^{\downarrow} \mid x \in P\} \cup \{P, \emptyset\}$ as a subbase for the closed sets; we denote it by $P_<$. The *order topology* P_O on a lattice P is defined in terms of Moore-Smith convergence. A filter $\mathcal{F}$ of subsets from P is said to Moore-Smith-converge to a point $l \in P$ whenever

$$\bigwedge_{F \in \mathcal{F}} \bigvee F = l = \bigvee_{F \in \mathcal{F}} \bigwedge F.$$

We then take $F \subseteq P$ to be closed if and only if any convergent filter that contains F converges to a point in F. For a lattice P, $P_< \subseteq P_O$ [3].

Lemma 3.2. Let P be a sublattice of $2^{\mathcal{P}(X)}$. Then $P_< \subseteq P_\Pi \subseteq P_O$ and all three topologies coincide when P is a complete sublattice of $2^{\mathcal{P}(X)}$. Moreover, all three topologies on P are compact if and only if P is complete.

Proof. The first inequality is true since for any $x \in P$, we have that

$$x^{\uparrow} \cap P = \bigcap_{A \in x} (A^+ \cap P).$$

A similar argument holds for $x^{\downarrow} \cap P$. For the second inequality and without loss of generality, take any subbasic closed set A^+ and let $\mathcal{F}$ be a convergent filter in P containing $A^+ \cap P$; this forces $\bigwedge(A^+ \cap P)$ to exist in P. Since P is a sublattice of $2^{\mathcal{P}(X)}$ then

$$\bigvee_{F \in \mathcal{F}} \bigwedge F = \bigcup_{F \in \mathcal{F}} \bigcap F$$

and $A \in \bigcup_{F \in \mathcal{F}} \bigcap F$. Consequently, $\bigcup_{F \in \mathcal{F}} \bigcap F \in A^+ \cap P$.

Next, if P is complete then for any $A \subseteq X$, it follows that $\bigcap(A^+ \cap P) \in P$ and $\bigcup(A^- \cap P) \in P$. In turn, $(A^+ \cap P) = (\bigcap(A^+ \cap P))^{\uparrow}$, $(A^- \cap P) = (\bigcup(A^- \cap P))^{\downarrow}$ and $P_< = P_\Pi$. Since $P_<$ is T_2 then P_O must be compact

Hausdorff [4] and $P_O = P_<$. The last claim is true since Frink [3] shows that a complete lattice is compact in its interval topology if and only if it is complete.

□

Not only is a sublattice P compact in $2^{\mathcal{P}(X)}$ precisely when P is complete but also the closure of P within $2^{\mathcal{P}(X)}$ is indeed its lattice theoretic completion:

Theorem 3.3. Given an infinite sublattice P of $2^{\mathcal{P}(X)}$ and $x \in 2^{\mathcal{P}(X)}$,

(i) x is a limit point of P only if x can be expressed in the form $\bigwedge_{j\in J} \bigvee_{i\in I_j} x_{i,j}$, where $x \neq x_{i,j}$ for each i, j and $\{x_{i,j} \mid i \in I_j\}_{j\in J}$ are infinite subsets of P.

(ii) $\overline{P} = \hat{P}$; that is, the topological closure of P in $2^{\mathcal{P}(X)}$ coincides with its lattice-theoretic completion.

Proof. (i) Let x be a limit point of P and assume, without loss of generality, that x is infinite. If $x = \mathcal{P}(X)$ then since all neighbourhoods of x meet P, we must have $x = \bigvee(x_\downarrow \cap P)$ and we are done. Otherwise fix any $B \notin x$ and notice that $\forall A \in x$ we have that $A^+ \cap B^- \cap P \neq \emptyset$. Furthermore $|A^+ \cap B^- \cap P| \geq \aleph_0$ since otherwise we can find (in Hausdorff $2^{\mathcal{P}(X)}$) a neighbourhood of x which is disjoint from P. Holding B fixed, it is simple to see that $x \subseteq \bigvee_{A\in x} (\bigvee(A^+ \cap B^- \cap P))$ while $B \notin \bigvee_{A\in x} (\bigvee(A^+ \cap B^- \cap P))$. To this end, we must only intersect all such suprema for each $B \notin x$ and we have the required form. In symbols:

$$x = \bigwedge_{B\notin x} \bigvee_{A\in x} \left(\bigvee(A^+ \cap B^- \cap P)\right).$$

For (ii) we must only notice that (i) ⇒ (ii). Indeed, take any $x \in \hat{P}$. If $x \in P$, we are done. Otherwise, if $x = \bigvee S$, for $S \subseteq P$ then Lemma 3.1 applies. The last possibility is for $x = \bigwedge_{k\in K} \bigvee S_k$ where $S_k \subseteq P$ and $x \subset \bigvee S_k$. Take a basic open set $\bigcap A_i^+ \cap \bigcap B_j^-$ about x and observe that for all i and for all $k \in K$, $A_i^+ \cap S_k \neq \emptyset$. As for the B_j^-, we know that for any j we can find a k_j for which $B_j \notin \bigvee S_{k_j}$. Hence, for each j take a finite collection of elements from its corresponding $< S_{k_j} >_L$ (i.e. the one for which $B_j \notin \bigvee S_{k_j}$) so that the join of such a collection contains all A_i. Taking the intersection of all such collections for each j we have an element of P that is contained in the aforementioned basic open set and thus x is a limit point of P.

□

Thus for example, a chain Ω in $2^{\mathcal{P}(X)}$ is by default a sublattice of $2^{\mathcal{P}(X)}$ and so its closure $\overline{\Omega}$ in $2^{\mathcal{P}(X)}$ is its lattice-theoretic completion, which is again a chain. In fact, observe that

$$\hat{\Omega}(=\overline{\Omega}) = \left\{\bigcap \Omega\right\} \cup \left\{\bigcup \Omega\right\} \cup \Omega \cup \left\{\bigcup(b_\downarrow \cap \Omega) \mid b \in R_\Omega\right\} \cup \left\{\bigcap(b_\uparrow \cap \Omega) \mid b \in R_\Omega\right\}.$$

Remark 3.4. Let (X, σ) be any topological space containing a convergent sequence $(x_n)_{n\in\omega}$ where $x_n \to x_\omega$. That $x_n \to x_\omega$ is equivalent to demanding that any open set containing x_ω must contain all but finitely many points from (x_n). Notice that the same is true for ω in the ordinal space $\omega + 1$ (with the order topology) . Moreover, any natural number in $\omega + 1$ is isolated and hence ω is a discrete subspace of $\omega + 1$. Thus $\omega + 1$, as an indexing set for any convergent sequence with its limit $\{x_n, x_\omega\}$, sets up a natural and continuous mapping $\phi : \omega+1 \to \{x_n, x_\omega\}_{n\in\omega}$ (where $n \to x_n$) whereupon compactness naturally transfers. With that in mind, let $\Omega = \{a_1, a_2, \ldots\}$ be a well-ordered chain in $2^{\mathcal{P}(X)}$ with α as its indexing ordinal. If $\beta \in \alpha$ is a limit ordinal, then any open set about β contains infinitely many ordinals below β. Notice that this might not be the case with a_β, for if $a_\beta \neq \bigcup(a_\beta)_\downarrow$ then a_β can be separated from $(a_\beta)_\downarrow$ by means of open sets. Thus, the bijection $h : \alpha \to \Omega$ for which $\beta \mapsto a_\beta$ is clearly open: h is a homeomorphism if and only if for any limit ordinal $\beta \in \alpha$ we have $a_\beta = \bigcup(a_\beta)_\downarrow$.

4 For X infinite, $Top(X)$ is neither a G_δ nor an F_σ set

Lemma 4.1. For any $\{A_i \mid i \in \omega\} \subseteq \mathcal{P}(X)$, $\bigcap_{i\in\omega} A_i^+$ contains a sublattice of $\mathcal{P}(X)$ that is not join complete; that is, $(\bigcap_{i\in\omega} A_i^+) \cap (LatB(X) \setminus Top(X)) \neq \emptyset$.

Proof. Consider $< \{A_i \mid i \in \omega\} >_L$ and suppose that it is join complete (otherwise, we are done). Notice that its countable cardinality demands that only finitely many of the A_is can be singletons. Since X is infinite, we may choose a countable infinite collection of singletons $\mathcal{S} = \{\{p\} \mid p \in X\}$ from $\mathcal{P}(X)$ and generate a lattice

$K =< \{A_i\} \cup \mathcal{S} >_L$. Then K cannot be join complete for there are uncountably many subsets of $\cup\mathcal{S}$ (i.e. joins of $\mathcal{S}$) and only $\aleph_0$ many elements in K.

□

Theorem 4.2. $Top(X)$ is not a G_δ set.

Proof. Suppose that $Top(X) = \bigcap_{k\in\omega} \mathcal{O}_k$, where

$$\mathcal{O}_k = \bigcup_{\alpha\in\beta_k} \left(\left(\bigcap_{i_\alpha \le n_\alpha} A^+_{i_\alpha} \right) \cap \left(\bigcap_{j_\alpha \le m_\alpha} B^-_{j_\alpha} \right) \right).$$

Now, the discrete topology $\mathcal{D}$ on X must be in this intersection of open sets. Thus for each $k \in \omega$, it must belong to at least one basic open set of the form $(\bigcap_{i_\alpha\le n_\alpha} A^+_{i_\alpha}) \cap (\bigcap_{j_\alpha\le m_\alpha} B^-_{j_\alpha})$ and since $\mathcal{D}$ contains all sets, then no subbasic open set can be of the form B^-. That is, $\mathcal{D} \in \bigcap_{k\in\omega} A^+_k$ after some renumeration of the As. Applying Lemma 4.1 to $\bigcap_{k\in\omega} A^+_k$ we can find a sublattice of $\mathcal{P}(X)$ that belongs to $\bigcap_{k\in\omega} A^+_k$ and that is not join complete - a contradiction.

□

In fact, Lemma 4.1 proves something much stronger. Define recursively:

$$\begin{aligned}
G^0_\delta &:= \{\text{all } G_\delta \text{ sets}\} \\
G^0_{\delta\sigma} &:= \{\text{all countable unions of } G_\delta \text{ sets}\} \\
G^\beta_\delta &:= \{\text{all countable intersections of } G^{\beta-1}_{\delta\sigma} \text{ sets}\} && \text{(for } \beta \text{ a successor ordinal)} \\
G^\beta_{\delta\sigma} &:= \{\text{all countable unions of } G^\beta_\delta \text{ sets}\} && \text{(for } \beta \text{ a successor ordinal)} \\
G^\gamma_\delta &:= \bigcup_{\beta\in\gamma} G^\beta_\delta && \text{(for } \gamma \text{ limit ordinal)} \\
G^\gamma_{\delta\sigma} &:= \bigcup_{\beta\in\gamma} G^\beta_{\delta\sigma} && \text{(for } \gamma \text{ limit ordinal)}
\end{aligned}$$

Take for any $n \in \mathbb{N}$ a set G_n, say, from G^n_δ and assume that $Top(X) = \bigcap_{n\in\omega} G_n$. Since $\mathcal{D} \in \bigcap_n G_n$ then for each $n \in \mathbb{N}$ we can find a countable collection $\{A_{i_n}\}_{i\in\omega}$ of subsets of X corresponding to each G_n so that

$$\mathcal{D} \in \bigcap_{i_n} A^+_{i_n} \subset G_n$$

and consequently

$$\mathcal{D} \in \bigcap_{n\in\omega} \bigcap_{i_n} A^+_{i_n} \subset \bigcap_{n\in\omega} G_n.$$

By Lemma 4.1, there is a lattice that is not join complete belonging to $\bigcap_{n\in\omega}\bigcap_{i_n} A^+_{i_n}$. Hence, $Top(X) \neq \bigcap_{n\in\omega} G_n$. Notice that the same is true for any countable limit ordinal. That is, for any $\beta \in \omega_1$ so that $\mathcal{D} \in G \in G^\beta_\delta$ it is possible to extract a countable collection of open sets A_i $(i \in \omega)$ so that $\mathcal{D} \in \bigcap_{i\in\omega} A^+_i \subseteq G$, in which case Lemma 4.1 completes the proof.

Corollary 4.3. $Top(X) \notin G^\beta_\delta$ for $\beta \in \omega_1$.

In other words, it is not possible to generate (in the sense of Borel sets) $Top(X)$ by means of open sets. If $2^{\mathcal{P}(X)}$ was metrizable (which it is not) then the above corollary would suffice to show that $Top(X)$ is not a Borel set.

Corollary 4.4. $Top(X)$ is not Čech complete.

Proof. We showed above that any countable intersection of open sets from $2^{\mathcal{P}(X)}$ containing the discrete topology on X contains an element of $LatB(X) \setminus Top(X)$. Hence, $Top(X)$ is not a G_δ set in $LatB(X)$.

□

Theorem 4.5. $Top(X)$ is not an F_σ set.

Proof. If $Top(X)$ is an F_σ set, then it must be of the form

$$Top(X) = \bigcup_{k\in\omega} \mathcal{C}_k$$

where each $\mathcal{C}_k$ is a closed set. We will show by contradiction that at least one such closed set must contain a sequence of topologies whose limit is not a topology. Since the limit of any sequence must be present in the closure of the sequence, then the aforementioned closed set will contain an element that is not a topology. We prove the above for $|X| = \aleph_0$ and note that the same is true for any X with $|X| \geq \aleph_0$.

Let $k : [0,1] \to \mathcal{P}(\mathbb{N})$ be an injective order morphism so that $\forall a \in [0,1]$, $\bigcup_{b<a} k(b) = k(a)$ and $k(1) \neq \mathbb{N}$. That is, $k([0,1])$ is a dense and uncountable linear order in $\mathcal{P}(\mathbb{N})$ where $a < b \Rightarrow k(a) \subset k(b)$. Next, for any a define $\tau_a = \mathcal{P}(k(a)) \cup \{\mathbb{N}\}$. Notice that $\forall a \in [0,1]$, $\tau_a \in Top(\mathbb{N})$ and $\bigcup_{b<a} \tau_b \notin Top(\mathbb{N})$ (since $k(a) \notin \bigcup_{b<a} \tau_b$), and $\{\tau_a\}_{a\in[0,1]}$ is an uncountable dense linear order in $Top(\mathbb{N})$. If $Top(\mathbb{N}) = \bigcup_{k\in\omega} \mathcal{C}_k$ where each $\mathcal{C}_k$ is closed then there must exist one set $\mathcal{C}$ from $\{\mathcal{C}_k\}_{k\in\omega}$ which contains an uncountable set $D \subset \{\tau_a\}_{a\in[0,1]}$ for which $\mu(D) > 0$ (non-zero measure). We immediately get that D must contain a densely ordered subset that in turn contains a strictly increasing sequence, call it S. By Theorem 3.3, $\bigcup S \in \hat{S}$ but $\bigcup S \notin Top(\mathbb{N})$. To this end we have $\bigcup S \in \mathcal{C}$, a contradiction.

□

Corollary 4.6. For $\beta \in \omega_1$, the following are equivalent:

(a) $Top(X) \in G_\delta^\beta$.

(b) $Top(X)$ is a G_δ set.

(c) $Top(X)$ is an F_σ set.

(d) $Top(X)$ is Čech complete.

(e) X is finite.

5 Compact infinite subsets of $Top(X)$

In this section, we provide examples of compact infinite subsets of $Top(X)$. Note in particular that any countable chain of topologies must converge to its union which may not itself be a topology. For example, consider the nested sequence of finite topologies $\{\tau_i \mid \tau_i = \mathcal{P}(\{x_0, x_1 \ldots, x_i\}) \cup \{X, \emptyset\}\}$, where $\{x_0, x_1, \ldots\}$ is a countable infinite subset of X. Then $\tau_n \to \bigcup \tau_i$ but notice that $\bigcup \tau_i$ fails to be a topology as $\{x_0, x_1, \ldots\}$ does not belong to any τ_i. Of course, $LatB(X)$, as a compactification of $Top(X)$, will contain all such limits.

Example 5.1. For simplicity, take a countable infinite subset C of X. Enumerate $C = \{a_0, a_1, a_2, \ldots\}$ and create a sequence in $\mathcal{P}(C)$ as follows: $C_0 = \{a_0\}$, $C_1 = \{a_0, a_1\}$, ..., $C_\omega = C$. Now, for any $n \in \omega$ let $\tau_n = \{C_m \mid m \leq n\} \cup \{X, \emptyset\}$ and $\tau_\omega = \{C_m \mid m \in \omega\} \cup \{C, X, \emptyset\}$; then $(\tau_n)_{n\in\omega}$ converges to (non-Hausdorff) τ_ω in $Top(X)$. Indeed, let $B = \bigcap A_i^+ \cap \bigcap B_j^-$ be a basic open set containing τ_ω. Then no $B_j = C_m$ for any $m \in \omega$ and any A_i must be either $\emptyset$, X, C or a C_n, for some $n \in \omega$. Since there are only finitely many A_i then there exists an $m \in \omega$ for which all $A_i \in \tau_m$.

In view of the above example, we can construct a convergent sequence of compact non-Hausdorff topologies, whose limit is both compact and Hausdorff.

Example 5.2. Let $[a,b] \subset \mathbb{R}$, and define any strictly increasing sequence $\{x_n \mid a < x_n < b\}_{n\in\omega}$ whose limit is b. Next, let $\mathcal{N}_b = \{(c,b] \mid a \leq c \leq b\}$, $\mathcal{N}_a(x) = \{[a,c) \mid c \leq x\}$ and

$$\begin{aligned}
\tau_0 &=< \{[a,b)\} \cup \mathcal{N}_b \cup \mathcal{N}_a(x_0) \cup \{\emptyset\} >_T \\
\tau_1 &=< \{[a,b)\} \cup \mathcal{N}_b \cup \mathcal{N}_a(x_1) \cup \{\emptyset\} >_T \\
&\vdots \\
\tau_\omega &= \bigcup_{i\in\omega} \tau_i
\end{aligned}$$

By design, $\tau_n \ \to \tau_\omega$. Observe also that τ_ω is the usual Euclidean topology on $[a,b]$ and so is compact and Hausdorff. Indeed, given any $c \in [a,b]$ we must only check that $(c,b] \in \tau_\omega$. Since $x_n \to b$ then there exists a $k \in \omega$ so that $x_k > c$, hence $(c,b] \in \tau_k$. To show that each τ_n $(n \in \omega)$ is compact, take an open cover $\mathcal{C}$ of $[a,b]$ from τ_k $(k \in \omega)$. Notice that since a must be covered then $[a,c_1) \in \mathcal{C}$ for some $c_1 \leq x_k$. Since x_k must also be covered by some element in $\mathcal{C}$ then, for some $c_2 \leq x_k$, either (c_2,b) or $(c_2,b]$ belong to $\mathcal{C}$. If $c_1 > c_2$ then we're done. Otherwise, notice that $\tau_k \restriction [c_1,c_2]$ is the usual topology on $\mathbb{R}$ restricted to $[c_1,c_2]$ (which yields a compact space). Finally, no τ_n is Hausdorff (since b can't be separated from all points in $[a,b]$).

The above example confirms that the collection of compact non-T_2 topologies on a set X fails to be closed given the existence of a countable chain of compact non-T_2 topologies whose union (and topological limit)is a T_2 and compact topology.

Even though any compact topology is contained in a maximal compact topology [5] it is possible to construct strictly increasing sequences of compact topologies whose limits are not compact. Consider the half-open half-closed interval $[a,b)$ equipped with the (convergent) sequence of topologies τ_i as in the previous Example, with $\mathcal{N}_b = \{(c,b) \mid a \leq c\}$ modified accordingly. It is clear that every topology, with the exception of τ_ω, is compact.

Nested sequences are not the only type of compact infinite subsets of $Top(X)$. Recall that an *atom* in $Top(X)$ is a topology of the form $\{\emptyset, A, X\}$ where A is a nonempty and proper subset of X. Consider then the following theorem where $\mathcal{I}$ denotes the trivial topology on X.

Theorem 5.3. Let $\mathcal{T}$ be an infinite collection of atoms in $Top(X)$. Then $\overline{\mathcal{T}} = \mathcal{T} \cup \{\mathcal{I}\}$.

Proof. Let $\mathcal{A} \subset \mathcal{P}(X)$ so that $\emptyset \notin \mathcal{A}$, $\tau_A = \{X, \emptyset, A\}$ and $T_\mathcal{A} = \{\tau_A \mid A \in \mathcal{A}\}$. Consider the following closed set

$$\mathcal{C} = X^+ \cap \emptyset^+ \cap \left(\bigcap_{D \in \mathcal{P}(X) \setminus \mathcal{A}} D^- \right) \cap \left(\bigcap_{B,C \in \mathcal{A}} (B^- \cup C^-) \right)$$

where, of course, $B \neq C$. Then $T_\mathcal{A} \subseteq \mathcal{C}$ and $\mathcal{I} \in \mathcal{C}$. Any family that contains any element from $\mathcal{P}(X) \setminus \mathcal{A}$ can't belong to $\mathcal{C}$ and any family (and topology) that contains elements from $\mathcal{A}$ can contain at most one. Thus $T_\mathcal{A} \cup \{\mathcal{I}\} = \mathcal{C}$. Finally, any neighbourhood of $\mathcal{I}$ must intersect $T_\mathcal{A}$ and the result follows. □

Corollary 5.4. Let $\mathcal{T}$ be any infinite collection of atoms in $Top(X)$. Then $\mathcal{T} \cup \{\mathcal{I}\}$ is the one-point compactification of $\mathcal{T}$ in $2^{\mathcal{P}(X)}$.

Given two topologies in $Top(X)$ we say that they are disjoint provided their intersection is the trivial topology $\mathcal{I}$ on X.

Theorem 5.5. Let $\mathcal{T}$ be an infinite collection of pairwise disjoint topologies on X. Then $\mathcal{T} \cup \{\mathcal{I}\} = \overline{\mathcal{T}}$ is the one-point compactification of $\mathcal{T}$ in $2^{\mathcal{P}(X)}$.

Proof. In the following expression, σ and ρ denote topologies in $\mathcal{T}$ while A and B denote certain nonempty and proper open subsets of X. We claim that

$$\mathcal{C} = X^+ \cap \emptyset^+ \cap \left(\bigcap_{D \notin \cup \mathcal{T}} D^- \right) \cap \left(\bigcap_{\substack{(A,B) \in (\sigma,\rho) \\ \sigma \neq \rho}} (A^- \cup B^-) \right) \cap \left(\bigcap_{\substack{A \neq B \\ A,B \in \sigma}} (A^+ \cup B^-) \right) = \mathcal{T} \cup \{\mathcal{I}\}.$$

Clearly $\mathcal{C}$ is closed and $\mathcal{T} \cup \{\mathcal{I}\} \subseteq \mathcal{C}$. Let $x \in 2^{\mathcal{P}(X)} \setminus \mathcal{T}$ such that $\mathcal{I} \subset x$. If $x \nsubseteq \cup \mathcal{T}$ then there must exist $O \in x$ so that $O \notin \cup \mathcal{T}$; since $x \notin O^-$ then $x \notin \mathcal{C}$. If $x \subseteq \cup \mathcal{T}$ then it is either contained in a topology from $\mathcal{T}$ or not. In the former case, take $\rho \in \mathcal{T}$ where $x \subset \rho$, $U \in \rho \setminus x$ and $V \in \rho \cap x$ where V is neither empty nor X. Then x fails to belong to $U^+ \cup V^-$. Otherwise we are guaranteed a pair of distinct topologies, ρ and σ, in $\mathcal{T}$ for which there exists $V \in x \cap \rho$ and $U \in x \cap \sigma$ and neither open set is equal to X or $\emptyset$. Hence, $x \notin V^- \cup U^-$ and we have that $\mathcal{C} = \mathcal{T} \cup \{\mathcal{I}\}$. Finally, any neighbourhood of $\mathcal{I}$ must have a cofinite intersection with $\mathcal{T}$ and the result follows. □

Corollary 5.6. For any (infinite) discrete space Y, where $|Y| \leq 2^{|X|}$, $Top(X)$ contains a copy of its one-point compactification.

Proof. There are $2^{|X|}$ atoms in $Top(X)$ and all are disjoint from each other. □

6 $\beta\mathbb{N}$ in $Top(\mathbb{N})$

An *ultratopology* $\mathcal{T}$ on a set X is of the form $\mathcal{T} = \mathcal{P}(X \setminus \{x\}) \cup \mathcal{U}$, where $\mathcal{U}$ is an ultrafilter on X and $\{x\} \notin \mathcal{U}$. We shall use the notation $\mathcal{T}_\mathcal{U}$ for such an ultratopology when we wish to identify the associated ultrafilter $\mathcal{U}$. We denote by $Ult(X)$ the set of all ultratopologies on X. Whenever $\mathcal{U}$ is a principal (non-principal) ultrafilter, $\mathcal{T}$ is called a principal (non-principal) ultratopology. Denote by TYPE(x) the set $\{\mathcal{F} \mid \mathcal{F}$ is an ultrafilter and $\{x\} \notin \mathcal{F}\}$ and by TYPE[x] the set $\{\mathcal{T} \in Top(X) \mid \mathcal{T}$ is an ultratopology and $\{x\} \notin \mathcal{T}\}$. Note that {TYPE[$x$]: $x \in X$} is a partition of $Ult(X)$.

Given X, define $\mathcal{U}_X$ to be the set of all ultrafilters on X and for any $\mathcal{F} \in \mathcal{U}_X$ and $A \subset X$ let

$$\mathcal{F}_A = \mathcal{F} \restriction (X \setminus A) = \{N \cap (X \setminus A) \mid N \in \mathcal{F}\}.$$

In other words, $\mathcal{F}_A$ is the *trace* of $\mathcal{F}$ on $X \setminus A$. Whenever $A = \{a\}$ we let $\mathcal{F}_{\{a\}} = \mathcal{F}_a$.

Lemma 6.1. Let $n \in \mathbb{N}$ and denote $\mathbb{N}' = \mathbb{N} \setminus \{n\}$ then

(i) $\forall \mathcal{F} \in$ TYPE(n), $\mathcal{F} = \mathcal{F}_n \cup \{M \cup \{n\} \mid M \in \mathcal{F}_n\}$,

(ii) $\forall \mathcal{F} \in$ TYPE(n), $\mathcal{F}_n \in \mathcal{U}_{\mathbb{N}'}$ and

(iii) $\mathcal{F}, \mathcal{G} \in$ TYPE(n) so that $\mathcal{G} \neq \mathcal{F}$ implies that $\mathcal{F}_n \neq \mathcal{G}_n$.

Proof. (i) We must only show that $\mathcal{F}_n \subset \mathcal{F}$. Indeed, if $\mathcal{F}_n \subset \mathcal{F}$ then given any $A \in \mathcal{F}_n$ we have that $A \cup \{n\} \in \mathcal{F}$ since $\mathcal{F}$ is a filter. So let $A \in \mathcal{F}_n$ and notice that either $A \in \mathcal{F}$ or $A \cup \{n\} \in \mathcal{F}$. Since the former case is trivial assume that $A \cup \{n\} \in \mathcal{F}$. Notice that since $\mathcal{F} \in$ TYPE(n) then $(\mathbb{N}' \setminus A) \cup \{n\} \notin \mathcal{F}$ and thus $A = \mathbb{N} \setminus ((\mathbb{N}' \setminus A) \cup \{n\}) \in \mathcal{F}$.

(ii) Take $A \in \mathcal{F}_n$. If $A \subset B \subseteq \mathbb{N}'$, then $B \in \mathcal{F}$ and consequently $B \in \mathcal{F}_n$. Next, let $A, B \in \mathcal{F}_n$ and notice that $A, B \in F$ and $A \cap B \in \mathcal{F}$. Thus, $A \cap B \in \mathcal{F}_n$. Lastly, let $A \subset \mathbb{N}'$ and notice that either A or its complement in $\mathbb{N}$ belong to $\mathcal{F}$. In the former case we are done so assume $\mathbb{N} \setminus A \in \mathcal{F}$. To this end, we must only notice that $\mathbb{N}' \setminus A = (\mathbb{N} \setminus A) \cap \mathbb{N}' \in \mathcal{F}_n$.

(iii) This follows directly from (i).

□

Theorem 6.2. For any $n \in \mathbb{N}$ the mapping $\mathfrak{F}$: TYPE(n)$\rightarrow \mathcal{U}_{\mathbb{N}'}$ for which $\mathcal{F} \mapsto \mathcal{F}_n$ is a bijection.

Proof. Lemma 6.1 part (ii) tells us that the range is well-defined. Also, for any filter $\mathcal{H} \in \mathcal{U}_{\mathbb{N}'}$ it is easy to see that $\mathcal{F}_\mathcal{H} = \mathcal{H} \cup \{M \cup \{n\} \mid M \in \mathcal{H}\} \in$ TYPE(n) and that $\mathcal{F}_\mathcal{H} \restriction \mathbb{N}' = \mathcal{H}$. Lastly, from part (iii) of Lemma 6.1 we get injectivity.

□

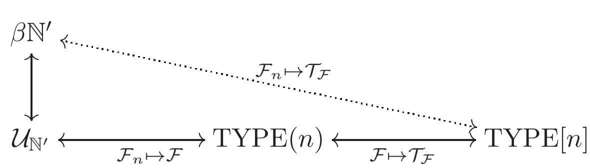
Theorem 6.3. For any $n \in \mathbb{N}$, TYPE[n] is homeomorphic to $\beta\mathbb{N}$.

Proof. Since TYPE(n) can be bijected with $\mathcal{U}_{\mathbb{N}'}$ (by $\mathcal{F} \mapsto \mathcal{F}_n$) then the same is true of TYPE[n]. That is, for any $\mathcal{T}_\mathcal{F} \in$ TYPE[n] we canonically map $\mathcal{T}_\mathcal{F} \mapsto \mathcal{F}_n$ so that $\mathcal{T}_\mathcal{F} = \mathcal{P}(\mathbb{N}') \cup \mathcal{F}$. Recall that a subbase for $\beta\mathbb{N}'$ is comprised of sets of the form $A' = \{\mathcal{F} \in \mathcal{U}_{\mathbb{N}'} \mid A \in \mathcal{F}\}$ for all $A \in \mathcal{P}(\mathbb{N}')$. We claim that for any $A \subseteq \mathbb{N}'$, $A' \mapsto (A \cup \{n\})^+ \cap$ TYPE[n]. Indeed, if $A \in \mathcal{F}_n$ for some $\mathcal{F} \in$ TYPE(n) then by Lemma 6.1 $A \cup \{n\} \in \mathcal{F}$ and $\mathcal{T}_\mathcal{F} \in (A \cup \{n\})^+ \cap$ TYPE[n].

Similarly, for any $A \in \mathcal{P}(\mathbb{N}')$, $A^+ \cap$ TYPE[n]= TYPE[n] which bijects to $\mathcal{U}_{\mathbb{N}'}$. If $A \subseteq \mathbb{N}$ with $n \in A$ then for any ultratopology $\mathcal{T}_\mathcal{F}$ in $A^+ \cap$ TYPE[n] it must be the case that $A \setminus \{n\} \in \mathcal{F}_n$. Consequently, $A^+ \cap$ TYPE[n] $\mapsto (A \setminus \{n\})'$.

□

In a nutshell, we have the following diagram of the above claim:



Corollary 6.4. The F_σ set $Ult(\mathbb{N}) = \bigcup_{n\in\mathbb{N}} \text{TYPE}[n]$ is not compact.

Proof. Consider the following open cover of $Ult(\mathbb{N})$:

$$\bigcup_{n\in\mathbb{N}} \{n\}^-.$$

Note that $\forall n \in \mathbb{N}$, $\{n\}^- \cap Ult(\mathbb{N}) = \text{TYPE}[n]$ and so no finite subcollection of the above cover can cover $Ult(\mathbb{N})$. □

In particular, the discrete topology on $\mathbb{N}$ is a limit point of $Ult(\mathbb{N})$. Lemma 6.1 and Theorem 6.3 can be extended to any infinite set X. That said, for $|X| > \aleph_0$, $Ult(X)$ is not an F_σ set.

Theorem 6.5. For Y a discrete space with $|Y| \leq |X|$, $Top(X)$ contains a copy of βY.

Proof. The proof is trivial for $|Y| = |X|$. Otherwise, take a copy of βY within $Top(Y)$ and an injection $i : Y \to X$. Then $\forall \rho \in Top(Y)$, $\rho_X = \{A \subset X \mid i^{-1}(A) \in \rho\} \cup \{X\}$ is a topology on X. Moreover, $\{\rho_X \mid \rho \in \beta Y\}$ is a homeomorphic copy of βY in $Top(X)$. □

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