On $(N(k), \xi)$-semi-Riemannian manifolds: Semisymmetries

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Abstract. $(N(k), \xi)$-semi-Riemannian manifolds are defined. Examples and properties of $(N(k), \xi)$-semi-Riemannian manifolds are given. Some relations involving $\mathcal{T}_a$-curvature tensor in $(N(k), \xi)$-semi-Riemannian manifolds are proved. ξ-$\mathcal{T}_a$-flat $(N(k), \xi)$-semi-Riemannian manifolds are defined. It is proved that if M is an n-dimensional ξ-$\mathcal{T}_a$-flat $(N(k), \xi)$-semi-Riemannian manifold, then it is η-Einstein under an algebraic condition. We prove that a semi-Riemannian manifold, which is T-recurrent or T-symmetric, is always T-semisymmetric, where T is any tensor of type $(1,3)$. $(\mathcal{T}_a, \mathcal{T}_b)$-semisymmetric semi-Riemannian manifold is defined and studied. The results for $\mathcal{T}_a$-semisymmetric, $\mathcal{T}_a$-symmetric, $\mathcal{T}_a$-recurrent $(N(k), \xi)$-semi-Riemannian manifolds are obtained. The definition of $(\mathcal{T}_a, S_{\mathcal{T}_b})$-semisymmetric semi-Riemannian manifold is given. $(\mathcal{T}_a, S_{\mathcal{T}_b})$-semisymmetric $(N(k), \xi)$-semi-Riemannian manifolds are classified. Some results for $\mathcal{T}_a$-Ricci-semisymmetric $(N(k), \xi)$-semi-Riemannian manifolds are obtained.

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1 Introduction

Let (M, g) be an n-dimensional semi-Riemannian manifold and $\mathfrak{X}(M)$ the Lie algebra of vector fields in M. Throughout the paper we assume that $X, Y, Z, U, V, W \in \mathfrak{X}(M)$, unless specifically stated otherwise.

A semi-Riemannian manifold M is said to be flat if $R(X,Y)Z = 0$. It is said to be ξ-flat if $R(X,Y)\xi = 0$, where ξ is a non-null unit vector field in M. The condition of ξ-flatness is weaker than the condition of flatness. In 2006, De and Biswas [7] studied the ξ-conformally flat contact metric manifolds with $\xi \in N(k)$. They proved that a contact metric manifold with $\xi \in N(k)$ is ξ-conformally flat if and only if it is η-Einstein manifold. Recently, in 2010, Dwivedi and Kim [12] proved that a Sasakian manifold is ξ-conharmonically flat if and only if it is η-Einstein.

A semi-Riemannian manifold M is said to be semisymmetric [45] if it satisfies $R(X,Y) \cdot R = 0$, where $R(X,Y)$ acts as a derivation on R. Semisymmetric manifold is a generalization of manifold of constant curvature and symmetric manifold ($\nabla R = 0$). A semi-Riemannian manifold is said to be recurrent [53] if it satisfies $\nabla R = \alpha \otimes R$, where α is 1-form. In 1972, Takagi [46] gave an example of Riemannian manifolds satisfying $R(X,Y) \cdot R = 0$ but not $\nabla R = 0$.

A semi-Riemannian manifold M is said to be Ricci-semisymmetric [10] if its Ricci tensor S satisfies $R(X,Y) \cdot S = 0$, where $R(X,Y)$ acts as a derivation on S. Ricci-semisymmetric manifold is a generalization of manifold of constant curvature, Einstein manifold, Ricci symmetric manifold, symmetric manifold and semisymmetric manifold.

Ricci-semisymmetric manifolds are studied by Adati and Miyazawa [2], Hong et al. [16], Pandey and Verma [34], Perrone [35] and Tripathi et al. [51]. After this Özgür [31] studied the Weyl Ricci-semisymmetric manifold. Hong, Özgür and Tripathi ([16], [33]) studied the concircular Ricci-semisymmetric manifold.

The paper is organized as follows. In Section 2, we give the definition of $\mathcal{T}$-curvature tensor. In Section 3, we define $(N(k), \xi)$-semi-Riemannian manifolds. Examples and properties of $(N(k), \xi)$-semi-Riemannian manifolds are given. $N(k)$-contact metric manifold, (ε)-Sasakian, Sasakian, Kenmotsu, (ε)-para-Sasakian and para-Sasakian manifolds are examples of $(N(k), \xi)$-semi-Riemannian manifolds. We obtain the relations for $\mathcal{T}_a$-curvature tensor in $(N(k),\xi)$-semi-Riemannian manifold. In Section 4, the definition of ξ-$\mathcal{T}_a$-flat $(N(k),\xi)$-semi-Riemannian manifold is given. Necessary conditions for a ξ-$\mathcal{T}_a$-flat $(N(k),\xi)$-semi-Riemannian manifold are mentioned. It is proved that an n-dimensional ξ-$\mathcal{T}_a$-flat $(N(k),\xi)$-semi-Riemannian manifold is η-Einstein under an algebraic condition. The necessary and sufficient condition for an n-dimensional $(N(k),\xi)$-semi-Riemannian manifold to be ξ-$\mathcal{T}_a$-flat is obtained, where $\mathcal{T}_a$-curvature tensor is one of the quasi-conformal curvature tensor, conformal curvature tensor, conharmonic curvature tensor, $\mathcal{M}$-projective curvature tensor or $\mathcal{W}_2$-curvature tensor. In Section 5, the definition of T-recurrent, T-symmetric and T-semisymmetric semi-Riemannian manifolds are given, where T is any tensor of type $(1,3)$. It is proved that if a semi-Riemannian manifold is T-recurrent or T-symmetric, then it is always T-semisymmetric. In Section 6, $(\mathcal{T}_a,\mathcal{T}_b)$-semisymmetric semi-Riemannian manifolds are defined and classified. It is proved that $\mathcal{T}_a$-semisymmetric $(N(k),\xi)$-semi-Riemannian manifolds are either η-Einstein or Einstein and manifold of constant curvature. $\mathcal{T}_a$-semisymmetric $(N(k),\xi)$-semi-Riemannian manifold is proved to be $\mathcal{T}_a$-flat under certain condition. A $\mathcal{T}_a$-semisymmetric $(N(k),\xi)$-semi-Riemannian manifold is $\mathcal{T}_a$-conservative under some condition. It is also proved that if an $(N(k),\xi)$-semi-Riemannian manifold is of constant curvature, then it is $\mathcal{T}_a$-semisymmetric under some algebraic conditions. In the last section, the definition of $(\mathcal{T}_a,S_{\mathcal{T}_b})$-semisymmetric semi-Riemannian manifold is given. $(\mathcal{T}_a,S_{\mathcal{T}_b})$-semisymmetric $(N(k),\xi)$-semi-Riemannian manifolds are classified. The results for $\mathcal{T}_a$-Ricci-semisymmetric $(N(k),\xi)$-semi-Riemannian manifolds are obtained. An n-dimensional $(R,S_{\mathcal{T}_a})$-semisymmetric $(N(k),\xi)$-semi-Riemannian manifold is Einstein under an algebraic condition. An Einstein manifold is $(R,S_{\mathcal{T}_a})$-semisymmetric. If M is an Einstein manifold such that $\mathcal{T}_a\in\{R,\mathcal{C}_*,\mathcal{C},\mathcal{L},\mathcal{V},\mathcal{M},\mathcal{W}_0,\mathcal{W}_0^*,\mathcal{W}_3\}$, then it is $\mathcal{T}_a$-Ricci-semisymmetric.

2 $\mathcal{T}$-curvature tensor

Definition 2.1 In an n-dimensional semi-Riemannian manifold (M,g), $\mathcal{T}$*-curvature tensor* [52] is a tensor of type $(1,3)$, which is defined by

$$\begin{aligned}\mathcal{T}(X,Y)Z &= a_0\,R(X,Y)Z \\ &+ a_1\,S(Y,Z)X + a_2\,S(X,Z)Y + a_3\,S(X,Y)Z \\ &+ a_4\,g(Y,Z)QX + a_5\,g(X,Z)QY + a_6\,g(X,Y)QZ \\ &+ a_7\,r\,(g(Y,Z)X - g(X,Z)Y),\end{aligned}\tag{2.1}$$

where $a_0,\dots,a_7$ are real numbers; and R, S, Q and r are the curvature tensor, the Ricci tensor, the Ricci operator and the scalar curvature respectively.

In particular, the $\mathcal{T}$-curvature tensor is reduced to

1. the *curvature tensor* R if $$a_0=1,\quad a_1=\cdots=a_7=0,$$
2. the *quasi-conformal curvature tensor* $\mathcal{C}_*$ [56] if $$a_1=-a_2=a_4=-a_5,\quad a_3=a_6=0,\quad a_7=-\frac{1}{n}\left(\frac{a_0}{n-1}+2a_1\right),$$
3. the *conformal curvature tensor* $\mathcal{C}$ [13, p. 90] if $$a_0=1,\quad a_1=-a_2=a_4=-a_5=-\frac{1}{n-2},\quad a_3=a_6=0,\quad a_7=\frac{1}{(n-1)(n-2)},$$
4. the *conharmonic curvature tensor* $\mathcal{L}$ [14] if $$a_0=1,\quad a_1=-a_2=a_4=-a_5=-\frac{1}{n-2},\quad a_3=a_6=0,\quad a_7=0,$$
5. the *concircular curvature tensor* $\mathcal{V}$ ([54], [55, p. 87]) if $$a_0=1,\quad a_1=a_2=a_3=a_4=a_5=a_6=0,\quad a_7=-\frac{1}{n(n-1)},$$
6. the *pseudo-projective curvature tensor* $\mathcal{P}_*$ [39] if $$a_1=-a_2,\quad a_3=a_4=a_5=a_6=0,\quad a_7=-\frac{1}{n}\left(\frac{a_0}{n-1}+a_1\right),$$

7. the *projective curvature tensor* $\mathcal{P}$ [55, p. 84] if
$$a_0 = 1, \quad a_1 = -a_2 = -\frac{1}{(n-1)}, \quad a_3 = a_4 = a_5 = a_6 = a_7 = 0,$$

8. the $\mathcal{M}$-*projective curvature tensor* [37] if
$$a_0 = 1, \quad a_1 = -a_2 = a_4 = -a_5 = -\frac{1}{2(n-1)}, \quad a_3 = a_6 = a_7 = 0,$$

9. the $\mathcal{W}_0$-*curvature tensor* [37, Eq. (1.4)] if
$$a_0 = 1, \quad a_1 = -a_5 = -\frac{1}{(n-1)}, \quad a_2 = a_3 = a_4 = a_6 = a_7 = 0,$$

10. the $\mathcal{W}_0^*$-*curvature tensor* [37, Eq. (2.1)] if
$$a_0 = 1, \quad a_1 = -a_5 = \frac{1}{(n-1)}, \quad a_2 = a_3 = a_4 = a_6 = a_7 = 0,$$

11. the $\mathcal{W}_1$-*curvature tensor* [37] if
$$a_0 = 1, \quad a_1 = -a_2 = \frac{1}{(n-1)}, \quad a_3 = a_4 = a_5 = a_6 = a_7 = 0,$$

12. the $\mathcal{W}_1^*$-*curvature tensor* [37] if
$$a_0 = 1, \quad a_1 = -a_2 = -\frac{1}{(n-1)}, \quad a_3 = a_4 = a_5 = a_6 = a_7 = 0,$$

13. the $\mathcal{W}_2$-*curvature tensor* [36] if
$$a_0 = 1, \quad a_4 = -a_5 = -\frac{1}{(n-1)}, \quad a_1 = a_2 = a_3 = a_6 = a_7 = 0,$$

14. the $\mathcal{W}_3$-*curvature tensor* [37] if
$$a_0 = 1, \quad a_2 = -a_4 = -\frac{1}{(n-1)}, \quad a_1 = a_3 = a_5 = a_6 = a_7 = 0,$$

15. the $\mathcal{W}_4$-*curvature tensor* [37] if
$$a_0 = 1, \quad a_5 = -a_6 = \frac{1}{(n-1)}, \quad a_1 = a_2 = a_3 = a_4 = a_7 = 0,$$

16. the $\mathcal{W}_5$-*curvature tensor* [38] if
$$a_0 = 1, \quad a_2 = -a_5 = -\frac{1}{(n-1)}, \quad a_1 = a_3 = a_4 = a_6 = a_7 = 0,$$

17. the $\mathcal{W}_6$-*curvature tensor* [38] if
$$a_0 = 1, \quad a_1 = -a_6 = -\frac{1}{(n-1)}, \quad a_2 = a_3 = a_4 = a_5 = a_7 = 0,$$

18. the $\mathcal{W}_7$-*curvature tensor* [38] if
$$a_0 = 1, \quad a_1 = -a_4 = -\frac{1}{(n-1)}, \quad a_2 = a_3 = a_5 = a_6 = a_7 = 0,$$

19. the $\mathcal{W}_8$-*curvature tensor* [38] if
$$a_0 = 1, \quad a_1 = -a_3 = -\frac{1}{(n-1)}, \quad a_2 = a_4 = a_5 = a_6 = a_7 = 0,$$

20. the $\mathcal{W}_9$-*curvature tensor* [38] if
$$a_0 = 1, \quad a_3 = -a_4 = \frac{1}{(n-1)}, \quad a_1 = a_2 = a_5 = a_6 = a_7 = 0.$$

Denoting

$$\mathcal{T}(X,Y,Z,V) = g(\mathcal{T}(X,Y)Z,V),$$

we write the curvature tensor $\mathcal{T}$ in its $(0,4)$ form as follows.

$$\begin{aligned}
\mathcal{T}(X,Y,Z,V) &= a_0\,R(X,Y,Z,V) \\
&\quad + a_1\,S(Y,Z)\,g(X,V) + a_2\,S(X,Z)\,g(Y,V) \\
&\quad + a_3\,S(X,Y)\,g(Z,V) + a_4\,S(X,V)\,g(Y,Z) \\
&\quad + a_5\,S(Y,V)\,g(X,Z) + a_6\,S(Z,V)\,g(X,Y) \\
&\quad + a_7\,r\,(g(Y,Z)\,g(X,V) - g(X,Z)\,g(Y,V)). \qquad (2.2)
\end{aligned}$$

In a semi-Riemannian manifold (M,g), let $\{e_i\}$, $i = 1,\ldots,n$ be a local orthonormal basis, define

$$(\operatorname{div}\mathcal{T})(X,Y,Z) = \sum_{i=1}^{n}\varepsilon_i g((\nabla_{e_i}\mathcal{T})(X,Y)Z,e_i),$$

where $\varepsilon_i = g(e_i,e_i)$. Then

$$\begin{aligned}
(\operatorname{div}\mathcal{T})(X,Y,Z) &= (a_0+a_1)(\nabla_X S)(Y,Z) + (-a_0+a_2)(\nabla_Y S)(X,Z) \\
&\quad + a_3(\nabla_Z S)(X,Y) + \left(\frac{a_4}{2} + a_7\right)(\nabla_X r)g(Y,Z) \\
&\quad + \left(\frac{a_5}{2} - a_7\right)(\nabla_Y r)g(X,Z) + \frac{a_6}{2}(\nabla_Z r)g(X,Y), \qquad (2.3)
\end{aligned}$$

$$\begin{aligned}
S_{\mathcal{T}}(X,Y) &= (a_0 + na_1 + a_2 + a_3 + a_5 + a_6)S(X,Y) \\
&\quad + (a_4 + (n-1)a_7)r\,g(X,Y). \qquad (2.4)
\end{aligned}$$

Definition 2.2 An n-dimensional semi-Riemannian manifold is said to be $\mathcal{T}$-conservative [52] if $\operatorname{div}\mathcal{T} = 0$.

Notation 2.3 *We will call $\mathcal{T}$-curvature tensor as $\mathcal{T}_a$-curvature tensor, whenever it is necessary. If $a_0,\ldots,a_7$ are replaced by $b_0,\ldots,b_7$ in the definition of $\mathcal{T}$-curvature tensor, then we will call $\mathcal{T}$-curvature tensor as $\mathcal{T}_b$-curvature tensor.*

3 $(N(k),\xi)$-semi-Riemannian manifolds

Let (M,g) be an n-dimensional semi-Riemannian manifold [30] equipped with a semi-Riemannian metric g. If index$(g) = 1$ then g is a Lorentzian metric and (M,g) a Lorentzian manifold [3]. If g is positive definite then g is an usual Riemannian metric and (M,g) a Riemannian manifold.

The *k-nullity distribution* [48] of (M,g) for a real number k is the distribution

$$N(k) : p \mapsto N_p(k) = \{Z \in T_pM : R(X,Y)Z = k(g(Y,Z)X - g(X,Z)Y)\}.$$

Let ξ be a non-null unit vector field in (M,g) and η its associated 1-form. Thus

$$g(\xi,\xi) = \varepsilon,$$

where $\varepsilon = 1$ or -1 according as ξ is spacelike or timelike, and

$$\eta(X) = \varepsilon g(X,\xi), \qquad \eta(\xi) = 1. \qquad (3.1)$$

Definition 3.1 An *$(N(k),\xi)$-semi-Riemannian manifold* consists of a semi-Riemannian manifold (M,g), a k-nullity distribution $N(k)$ on (M,g) and a non-null unit vector field ξ in (M,g) belonging to $N(k)$.

Now, we intend to give some examples of $(N(k),\xi)$-semi-Riemannian manifolds. For this purpose we collect some definitions from the geometry of almost contact manifolds and almost paracontact manifolds as follows:

Almost contact manifolds

Let M be a smooth manifold of dimension $n = 2m+1$. Let φ, ξ and η be tensor fields of type $(1,1)$, $(1,0)$ and $(0,1)$, respectively. If φ, ξ and η satisfy the conditions

$$\varphi^2 = -I + \eta \otimes \xi, \tag{3.2}$$

$$\eta(\xi) = 1, \tag{3.3}$$

where I denotes the identity transformation, then M is said to have an almost contact structure (φ, ξ, η). A manifold M alongwith an almost contact structure is called an *almost contact manifold* [4]. Let g be a semi-Riemannian metric on M such that

$$g(\varphi X, \varphi Y) = g(X, Y) - \varepsilon\eta(X)\eta(Y), \tag{3.4}$$

where $\varepsilon = \pm 1$. Then (M, g) is an (ε)-*almost contact metric manifold* [11] equipped with an (ε)-*almost contact metric structure* $(\varphi, \xi, \eta, g, \varepsilon)$. In particular, if the metric g is positive definite, then an (ε)-almost contact metric manifold is the usual *almost contact metric manifold* [4].

From (3.4), it follows that

$$g(X, \varphi Y) = -g(\varphi X, Y) \tag{3.5}$$

and

$$g(X, \xi) = \varepsilon\eta(X). \tag{3.6}$$

From (3.3) and (3.6), we have

$$g(\xi, \xi) = \varepsilon. \tag{3.7}$$

In an (ε)-almost contact metric manifold, the fundamental 2-form Φ is defined by

$$\Phi(X, Y) = g(X, \varphi Y). \tag{3.8}$$

An (ε)-almost contact metric manifold with $\Phi = d\eta$ is an (ε)-*contact metric manifold* [47]. For $\varepsilon = 1$ and g Riemannian, M is the usual *contact metric manifold* [4]. A contact metric manifold with $\xi \in N(k)$, is called a $N(k)$-*contact metric manifold* [5].

An (ε)-almost contact metric structure $(\varphi, \xi, \eta, g, \varepsilon)$ is called an (ε)-*Sasakian structure* if

$$(\nabla_X \varphi) Y = g(X, Y)\xi - \varepsilon\eta(Y) X,$$

where ∇ is Levi-Civita connection with respect to the metric g. A manifold endowed with an (ε)-Sasakian structure is called an (ε)-*Sasakian manifold* [47]. For $\varepsilon = 1$ and g Riemannian, M is the usual *Sasakian manifold* [41, 4].

An almost contact metric manifold is a *Kenmotsu manifold* [18] if

$$(\nabla_X \varphi) Y = g(\varphi X, Y)\xi - \eta(Y)\varphi X. \tag{3.9}$$

By (3.9), we have

$$\nabla_X \xi = X - \eta(X)\xi. \tag{3.10}$$

Almost paracontact manifolds

Let M be an n-dimensional *almost paracontact manifold* [42] equipped with an *almost paracontact structure* (φ, ξ, η), where φ, ξ and η are tensor fields of type $(1,1)$, $(1,0)$ and $(0,1)$, respectively; and satisfy the conditions

$$\varphi^2 = I - \eta \otimes \xi, \tag{3.11}$$

$$\eta(\xi) = 1. \tag{3.12}$$

Let g be a semi-Riemannian metric on M such that

$$g(\varphi X, \varphi Y) = g(X, Y) - \varepsilon\eta(X)\eta(Y), \tag{3.13}$$

where $\varepsilon = \pm 1$. Then (M, g) is an (ε)-*almost paracontact metric manifold* equipped with an (ε)-*almost paracontact metric structure* $(\varphi, \xi, \eta, g, \varepsilon)$. In particular, if index$(g) = 1$, then an (ε)-almost paracontact metric manifold is said to be a *Lorentzian almost paracontact manifold*. In particular, if the metric g is positive definite, then an (ε)-almost paracontact metric manifold is the usual *almost paracontact metric manifold* [42].

The equation (3.13) is equivalent to

$$g(X, \varphi Y) = g(\varphi X, Y) \tag{3.14}$$

along with

$$g(X, \xi) = \varepsilon\eta(X). \tag{3.15}$$

From (3.12) and (3.15), we have

$$g(\xi, \xi) = \varepsilon. \tag{3.16}$$

An (ε)-almost paracontact metric structure is called an (ε)-*para-Sasakian structure* [51] if

$$(\nabla_X \varphi) Y = -g(\varphi X, \varphi Y)\xi - \varepsilon\eta(Y)\varphi^2 X, \tag{3.17}$$

where ∇ is Levi-Civita connection with respect to the metric g. A manifold endowed with an (ε)-para-Sasakian structure is called an (ε)-*para-Sasakian manifold* [51]. For $\varepsilon = 1$ and g Riemannian, M is the usual para-Sasakian manifold [42]. For $\varepsilon = -1$, g Lorentzian and ξ replaced by $-\xi$, M becomes a Lorentzian para-Sasakian manifold [22].

Example 3.2 The following are some well known examples of $(N(k), \xi)$-semi-Riemannian manifolds:

1. An $N(k)$-contact metric manifold [5] is an $(N(k), \xi)$-Riemannian manifold.
2. A Sasakian manifold [41] is an $(N(1), \xi)$-Riemannian manifold.
3. A Kenmotsu manifold [18] is an $(N(-1), \xi)$-Riemannian manifold.
4. An (ε)-Sasakian manifold [47] an $(N(\varepsilon), \xi)$-semi-Riemannian manifold.
5. A para-Sasakian manifold [42] is an $(N(-1), \xi)$-Riemannian manifold.
6. An (ε)-para-Sasakian manifold [51] is an $(N(-\varepsilon), \xi)$-semi-Riemannian manifold.

In an n-dimensional $(N(k), \xi)$-semi-Riemannian manifold (M, g), it is easy to verify that

$$R(X, Y)\xi = \varepsilon k(\eta(Y)X - \eta(X)Y), \tag{3.18}$$

$$R(\xi, X)Y = \varepsilon k(\varepsilon g(X, Y)\xi - \eta(Y)X), \tag{3.19}$$

$$R(\xi, X)\xi = \varepsilon k(\eta(X)\xi - X), \tag{3.20}$$

$$R(X, Y, Z, \xi) = \varepsilon k(\eta(X) g(Y, Z) - \eta(Y) g(X, Z)), \tag{3.21}$$

$$\eta(R(X, Y) Z) = k(\eta(X) g(Y, Z) - \eta(Y) g(X, Z)), \tag{3.22}$$

$$S(X, \xi) = \varepsilon k(n-1)\eta(X), \tag{3.23}$$

$$Q\xi = k(n-1)\xi, \tag{3.24}$$

$$S(\xi, \xi) = \varepsilon k(n-1), \tag{3.25}$$

$$\eta(QX) = \varepsilon g(QX, \xi) = \varepsilon S(X, \xi) = k(n-1)\eta(X). \tag{3.26}$$

Moreover, define

$$S^\ell(X, Y) = g(Q^\ell X, Y) = S(Q^{\ell-1}X, Y), \tag{3.27}$$

where $\ell = 0, 1, 2, \ldots$ and $S^0 = g$. Using (3.26) in (3.27), we get

$$S^\ell(X, \xi) = \varepsilon k^\ell (n-1)^\ell \eta(X). \tag{3.28}$$

Now, we state the following Lemma without proof.

Lemma 3.3 *Let M be an n-dimensional $(N(k),\xi)$-semi-Riemannian manifold. Then*

$$\begin{aligned}\mathcal{T}_a(X,Y)\xi &= (-\varepsilon k a_0 + \varepsilon k(n-1)a_2 - \varepsilon a_7\, r)\eta(X)Y \\ &\quad + (\varepsilon k a_0 + \varepsilon k(n-1)a_1 + \varepsilon a_7\, r)\,\eta(Y)X \\ &\quad + a_3\, S(X,Y)\xi + \varepsilon a_4\, \eta(Y)QX \\ &\quad + \varepsilon a_5 \eta(X)QY + k(n-1)a_6 g(X,Y)\xi, \end{aligned} \tag{3.29}$$

$$\begin{aligned}\mathcal{T}_a(\xi,X)\xi &= (-\varepsilon k a_0 + \varepsilon k(n-1)a_2 - \varepsilon a_7\, r)X + \varepsilon a_5\, QX \\ &\quad + \{\varepsilon k a_0 + \varepsilon k(n-1)a_1 + \varepsilon k(n-1)a_3 \\ &\qquad + \varepsilon k(n-1)a_4 + \varepsilon k(n-1)a_6 + \varepsilon a_7\, r\}\, \eta(X)\xi, \end{aligned} \tag{3.30}$$

$$\begin{aligned}\mathcal{T}_a(\xi,Y)Z &= (k a_0 + k(n-1)a_4 + a_7 r) g\,(Y,Z)\,\xi \\ &\quad + a_1\, S\,(Y,Z)\,\xi + \varepsilon k(n-1)a_3 \eta(Y)Z \\ &\quad + \varepsilon a_5\, \eta(Z)QY + \varepsilon a_6\, \eta(Y)QZ \\ &\quad + (-\varepsilon k a_0 + \varepsilon k(n-1)a_2 - \varepsilon a_7\, r)\eta(Z)Y, \end{aligned} \tag{3.31}$$

$$\begin{aligned}\eta(\mathcal{T}_a\,(X,Y)\xi) &= \varepsilon k(n-1)(a_1+a_2+a_4+a_5)\eta(X)\eta(Y) \\ &\quad + a_3\, S(X,Y) + k(n-1)a_6 g(X,Y), \end{aligned} \tag{3.32}$$

$$\begin{aligned}\mathcal{T}_a\,(X,Y,\xi,V) &= (-\varepsilon k a_0 + \varepsilon k(n-1)a_2 - \varepsilon a_7\, r)\eta(X)g(Y,V) \\ &\quad + (\varepsilon k a_0 + \varepsilon k(n-1)a_1 + \varepsilon a_7\, r)\,\eta(Y)g(X,V) \\ &\quad + \varepsilon a_3\, S(X,Y)\eta(V) + \varepsilon a_4\, \eta(Y)S(X,V) \\ &\quad + \varepsilon a_5\, \eta(X)S(Y,V) + \varepsilon k(n-1)a_6\, g(X,Y)\eta(V), \end{aligned} \tag{3.33}$$

$$\begin{aligned}\mathcal{T}_a(X,\xi)\xi &= \{-\varepsilon k a_0 + \varepsilon k(n-1)a_2 + \varepsilon k(n-1)a_3 \\ &\qquad + \varepsilon k(n-1)a_5 + \varepsilon k(n-1)a_6 - \varepsilon a_7\, r\}\, \eta(X)\xi \\ &\quad + (\varepsilon k a_0 + \varepsilon k(n-1)a_1 + \varepsilon a_7\, r)\, X + \varepsilon a_4\, QX, \end{aligned} \tag{3.34}$$

$$\begin{aligned}S_{\mathcal{T}_a}(X,\xi) &= \{\varepsilon k(n-1)(a_0 + n a_1 + a_2 + a_3 + a_5 + a_6) \\ &\quad + \varepsilon r(a_4 + (n-1)a_7)\}\, \eta(X), \end{aligned} \tag{3.35}$$

$$\begin{aligned}S_{\mathcal{T}_a}(\xi,\xi) &= \varepsilon k(n-1)(a_0 + n a_1 + a_2 + a_3 + a_5 + a_6) \\ &\quad + \varepsilon r(a_4 + (n-1)a_7). \end{aligned} \tag{3.36}$$

Remark 3.4 The relations (3.18) – (3.36) are true for

1. a $N(k)$-contact metric manifold [5] ($\varepsilon = 1$),
2. a Sasakian manifold [41] ($k = 1$, $\varepsilon = 1$),
3. a Kenmotsu manifold [18] ($k = -1$, $\varepsilon = 1$),
4. an (ε)-Sasakian manifold [47] ($k = \varepsilon$, $\varepsilon k = 1$),
5. a para-Sasakian manifold [42] ($k = -1$, $\varepsilon = 1$), and
6. an (ε)-para-Sasakian manifold [51] ($k = -\varepsilon$, $\varepsilon k = -1$).

Even, all the relations and results of this paper will be true for the above six cases.

4 ξ-$\mathcal{T}_a$-flat $(N(k),\xi)$-semi-Riemannian manifolds

Definition 4.1 An n-dimensional $(N(k),\xi)$-semi-Riemannian manifold (M,g) is said to be ξ-$\mathcal{T}_a$-flat if it satisfies

$$\mathcal{T}_a(X,Y)\xi = 0.$$

In particular, if $\mathcal{T}_a$ is equal to R, $\mathcal{C}_*$, $\mathcal{C}$, $\mathcal{L}$, $\mathcal{V}$, $\mathcal{P}_*$, $\mathcal{P}$, $\mathcal{M}$, $\mathcal{W}_0$, $\mathcal{W}_0^*$, $\mathcal{W}_1$, $\mathcal{W}_1^*$, $\mathcal{W}_2$, $\mathcal{W}_3$, $\mathcal{W}_4$, $\mathcal{W}_5$, $\mathcal{W}_6$, $\mathcal{W}_7$, $\mathcal{W}_8$, $\mathcal{W}_9$, then it becomes ξ-flat, ξ-quasi-conformally flat, ξ-conformally flat, ξ-conharmonically flat, ξ-concircularly flat, ξ-pseudo-projectively flat, ξ-projectively flat, ξ-$\mathcal{M}$-flat, ξ-$\mathcal{W}_0$-flat, ξ-$\mathcal{W}_0^*$-flat, ξ-$\mathcal{W}_1$-flat, ξ-$\mathcal{W}_1^*$-flat, ξ-$\mathcal{W}_2$-flat, ξ-$\mathcal{W}_3$-flat, ξ-$\mathcal{W}_4$-flat, ξ-$\mathcal{W}_5$-flat, ξ-$\mathcal{W}_6$-flat, ξ-$\mathcal{W}_7$-flat, ξ-$\mathcal{W}_8$-flat, ξ-$\mathcal{W}_9$-flat, respectively.

Theorem 4.2 *Let M be an n-dimensional ξ-$\mathcal{T}_a$-flat $(N(k),\xi)$-semi-Riemannian manifold.*

1. *If $a_4 \neq 0$ and $a_4 + (n-1)a_7 \neq 0$, then*

$$S = D_2\, g + D_3 \eta \otimes \eta, \tag{4.1}$$

where

$$D_2 = -\frac{ka_0 + k(n-1)a_1 + a_7 r}{a_4} \tag{4.2}$$

and

$$D_3 = \frac{ka_0 - (n-1)k(a_2 + a_3 + a_5 + a_6) + a_7 r}{\varepsilon a_4}. \tag{4.3}$$

Therefore it is an η-Einstein manifold and

$$r = -\frac{k(n-1)\,(a_0 + na_1 + a_2 + a_3 + a_5 + a_6)}{a_4 + (n-1)a_7} = D_1 \text{ (say)}. \tag{4.4}$$

In particular, M becomes an Einstein manifold provided

$$ka_0 - (n-1)k(a_2 + a_3 + a_5 + a_6) + a_7 r = 0.$$

2. *If $a_4 = 0$ and $a_7 \neq 0$, then*

$$r = -\frac{k\,(a_0 + na_1 + a_2 + a_3 + a_5 + a_6)}{a_7}. \tag{4.5}$$

3. *If $a_4 = 0$ and $a_7 = 0$, then either $k = 0$ or*

$$a_0 + na_1 + a_2 + a_3 + a_5 + a_6 = 0. \tag{4.6}$$

Proof. By (2.2) and (3.1), we get

$$\begin{aligned}
\mathcal{T}_a(X,Y,\xi,W) \;=\;& a_0\, R(X,Y,\xi,W) \\
&+ a_1\, S(Y,\xi)\, g(X,W) + a_2 S(X,\xi)\, g(Y,W) \\
&+ a_3 \varepsilon S(X,Y)\eta(W) + a_4 \varepsilon \eta(Y) S(X,W) \\
&+ a_5 \varepsilon \eta(X) S(Y,W) + a_6\, g(X,Y) S(\xi,W) \\
&+ a_7 \varepsilon\, r\, (\eta(Y) g(X,W) - \eta(X) g(Y,W)).
\end{aligned} \tag{4.7}$$

Using $Y = \xi$ in (4.7), we get

$$\begin{aligned}
\mathcal{T}_a(X,\xi,\xi,W) \;=\;& a_0\, R(X,\xi,\xi,W) \\
&+ a_1\, S(\xi,\xi)\, g(X,W) + a_2\, S(X,\xi)\, g(\xi,W) \\
&+ a_3 \varepsilon S(X,\xi)\eta(W) + a_4 \varepsilon \eta(\xi) S(X,W) \\
&+ a_5 \varepsilon \eta(X) S(\xi,W) + a_6\, g(X,\xi) S(\xi,W) \\
&+ a_7 \varepsilon r\, (\eta(\xi) g(X,W) - \eta(X) g(\xi,W)).
\end{aligned} \tag{4.8}$$

Case 1. If $a_4 \neq 0$ and $a_4 + (n-1)a_7 \neq 0$, then using (3.1), (3.20) (3.23) and (3.25) and the fact that M is ξ-$\mathcal{T}_a$-flat in (4.8), we get (4.1) and (4.4).
Case 2. If $a_4 = 0$ and $a_7 \neq 0$, then by using (3.1), (3.20) (3.23) and (3.25) and the fact that M is ξ-$\mathcal{T}_a$-flat in (4.8), we get

$$\begin{aligned}\varepsilon\left(a_0k + (n-1)ka_1 + a_7 r\right) g(Y,Z) &= \left(-(n-1)k(a_2+a_3+a_5+a_6)\right. \\ &\quad \left. +\ (a_0k + a_7\, r)\right) \eta(Y)\eta(Z).\end{aligned}$$

Contracting the above equation, we get

$$a_7 r = -k\left(a_0 + na_1 + a_2 + a_3 + a_5 + a_6\right). \tag{4.9}$$

Since $a_7 \neq 0$, we get (4.5).
Case 3. If $a_4 = 0$ and $a_7 = 0$, then from (4.9) either $k = 0$ or (4.6) is satisfied. This proves the result. ■

Theorem 4.3 *Let M be an n-dimensional $(N(k),\xi)$-semi-Riemannian manifold such that $a_4 \neq 0$ and $a_4 + (n-1)a_7 \neq 0$. If M satisfies* (4.1)*, then*

$$\begin{aligned}\mathcal{T}_a(X,Y)\xi &= (\varepsilon ka_0 + \varepsilon k(n-1)a_1 + \varepsilon a_4 D_2 + \varepsilon a_7 D_1)\eta(Y)X \\ &\quad + (-\varepsilon ka_0 + \varepsilon k(n-1)a_2 + \varepsilon a_5 D_2 - \varepsilon a_7 D_1)\eta(X)Y \\ &\quad + (k(n-1)a_6 + a_3 D_2)g(X,Y)\xi \\ &\quad + (a_3 + a_4 + a_5)D_3\eta(X)\eta(Y)\xi.\end{aligned} \tag{4.10}$$

Remark 4.4 If M is ξ-conformally flat $(N(k),\xi)$-semi-Riemannian manifold, then from (4.4), the scalar curvature r is in indeterminate form.

Suppose that a $(N(k),\xi)$-semi-Riemannian manifold is η-Einstein. Then there are functions α and β such that

$$S(X,Y) = \alpha g(X,Y) + \beta\eta(X)\eta(Y). \tag{4.11}$$

On contracting (4.11), we get

$$r = \alpha n + \beta\varepsilon. \tag{4.12}$$

Taking $X = \xi = Y$ in (4.11), we get

$$k(n-1) = \alpha + \beta\varepsilon. \tag{4.13}$$

Using (4.12) in (4.13) yields

$$r = (k+\alpha)(n-1). \tag{4.14}$$

Theorem 4.5 *Let M be an η-Einstein $(N(k),\xi)$-semi-Riemannian manifold. Then*

$$\begin{aligned}\mathcal{T}_a(X,Y,\xi,V) &= \varepsilon\left(ka_0 + (\alpha+\beta)a_1 + \alpha a_4 + (k+\alpha)(n-1)a_7\right)\eta(Y)g(X,V) \\ &\quad + \varepsilon\left(-ka_0 + (\alpha+\beta)a_2 + \alpha a_5 - (k+\alpha)(n-1)a_7\right)\eta(X)g(Y,V) \\ &\quad + \varepsilon\left(\alpha a_3 + (\alpha+\beta)a_6\right)\eta(V)g(X,Y) \\ &\quad + \varepsilon\beta(a_3+a_4+a_5)\eta(X)\eta(Y)\eta(V).\end{aligned} \tag{4.15}$$

Proof. Let M be an η-Einstein $(N(k),\xi)$-semi-Riemannian manifold. Taking $Z = \xi$ in (2.2) and the using (3.18), (3.23), (3.24) and (4.14), we get (4.15). ■

In view of Theorem 4.2, we have the following Corollaries:

Corollary 4.6 *Let M be an n-dimensional ξ-quasi-conformally flat $(N(k),\xi)$-semi-Riemannian manifold such that $a_1 \neq 0$ and $a_0 + (n-2)a_1 \neq 0$. Then we have the following table*:

M	$S =$
$N(k)$-contact metric	$k(n-1)g$
Sasakian	$(n-1)g$
Kenmotsu	$-(n-1)g$
(ε)-Sasakian	$\varepsilon(n-1)g$
para-Sasakian	$-(n-1)g$
(ε)-para-Sasakian	$-\varepsilon(n-1)g$

Corollary 4.7 *Let M be an n-dimensional ξ-conformally flat $(N(k),\xi)$-semi-Riemannian manifold. Then we have the following table*:

M	$S=$
$N(k)$-*contact metric* [7]	$\left(\frac{r}{n-1}-k\right)g+\left(nk-\frac{r}{n-1}\right)\eta\otimes\eta$
Sasakian	$\left(\frac{r}{n-1}-1\right)g+\left(n-\frac{r}{n-1}\right)\eta\otimes\eta$
Kenmotsu	$\left(\frac{r}{n-1}+1\right)g-\left(n+\frac{r}{n-1}\right)\eta\otimes\eta$
(ε)-*Sasakian*	$\left(\frac{r}{n-1}-\varepsilon\right)g+\varepsilon\left(\varepsilon n-\frac{r}{n-1}\right)\eta\otimes\eta$
para-Sasakian	$\left(\frac{r}{n-1}+1\right)g-\left(n+\frac{r}{n-1}\right)\eta\otimes\eta$
(ε)-*para-Sasakian*	$\left(\frac{r}{n-1}+\varepsilon\right)g-\varepsilon\left(\varepsilon n+\frac{r}{n-1}\right)\eta\otimes\eta$

Corollary 4.8 *Let M be an n-dimensional ξ-conharmonically flat $(N(k),\xi)$-semi-Riemannian manifold. Then we have the following table*:

M	$S=$
$N(k)$-*contact metric*	$-kg+kn\eta\otimes\eta$
Sasakian [12]	$-g+n\eta\otimes\eta$
Kenmotsu	$g-n\eta\otimes\eta$
(ε)-*Sasakian*	$-\varepsilon g+n\eta\otimes\eta$
para-Sasakian	$g-n\eta\otimes\eta$
(ε)-*para-Sasakian*	$\varepsilon g-n\eta\otimes\eta$

Corollary 4.9 *Let M be an n-dimensional ξ-concircularly flat $(N(k),\xi)$-semi-Riemannian manifold. Then we have the following table*:

M	$r=$
$N(k)$-*contact metric*	$kn(n-1)$
Sasakian	$n(n-1)$
Kenmotsu	$-n(n-1)$
(ε)-*Sasakian*	$\varepsilon n(n-1)$
para-Sasakian	$-n(n-1)$
(ε)-*para-Sasakian*	$-\varepsilon n(n-1)$

Corollary 4.10 *Let M be an n-dimensional ξ-pseudo-projectively flat $(N(k),\xi)$-semi-Riemannian manifold such that $a_0+(n-1)a_1\neq 0$. Then we have the following table*:

M	$r=$
$N(k)$-*contact metric*	$kn(n-1)$
Sasakian	$n(n-1)$
Kenmotsu	$-n(n-1)$
(ε)-*Sasakian*	$\varepsilon n(n-1)$
para-Sasakian	$-n(n-1)$
(ε)-*para-Sasakian*	$-\varepsilon n(n-1)$

Corollary 4.11 *Let M be an n-dimensional ξ-$\mathcal{M}$-flat $(N(k),\xi)$-semi-Riemannian manifold. Then we have the following table*:

M	$S=$
$N(k)$-*contact metric*	$k(n-1)g$
Sasakian	$(n-1)g$
Kenmotsu	$-(n-1)g$
(ε)-*Sasakian*	$\varepsilon(n-1)g$
para-Sasakian	$-(n-1)g$
(ε)-*para-Sasakian*	$-\varepsilon(n-1)g$

Corollary 4.12 *Let M be an n-dimensional ξ-$\mathcal{W}_2$-flat $(N(k),\xi)$-semi-Riemannian manifold. Then we have the following table*:

M	$S =$
$N(k)$-*contact metric*	$k(n-1)g$
Sasakian	$(n-1)g$
Kenmotsu	$-(n-1)g$
(ε)-*Sasakian*	$\varepsilon(n-1)g$
para-Sasakian	$-(n-1)g$
(ε)-*para-Sasakian*	$-\varepsilon(n-1)g$

Corollary 4.13 *Let M be an n-dimensional ξ-$\mathcal{W}_3$-flat $(N(k),\xi)$-semi-Riemannian manifold. Then we have the following table*:

M	$S =$
$N(k)$-*contact metric*	$-k(n-1)g + 2k(n-1)\eta\otimes\eta$
Sasakian	$-(n-1)g + 2(n-1)\eta\otimes\eta$
Kenmotsu	$(n-1)g - 2(n-1)\eta\otimes\eta$
(ε)-*Sasakian*	$-\varepsilon(n-1)g + 2(n-1)\eta\otimes\eta$
para-Sasakian	$(n-1)g - 2(n-1)\eta\otimes\eta$
(ε)-*para-Sasakian*	$\varepsilon(n-1)g - 2(n-1)\eta\otimes\eta$

Corollary 4.14 *Let M be an n-dimensional ξ-$\mathcal{W}_7$-flat $(N(k),\xi)$-semi-Riemannian manifold. Then we have the following table*:

M	$S =$
$N(k)$-*contact metric*	$k(n-1)\eta\otimes\eta$
Sasakian	$(n-1)\eta\otimes\eta$
Kenmotsu	$-(n-1)\eta\otimes\eta$
(ε)-*Sasakian*	$(n-1)\eta\otimes\eta$
para-Sasakian	$-(n-1)\eta\otimes\eta$
(ε)-*para-Sasakian*	$-(n-1)\eta\otimes\eta$

Corollary 4.15 *Let M be an n-dimensional ξ-$\mathcal{W}_9$-flat $(N(k),\xi)$-semi-Riemannian manifold. Then we have the following table*:

M	$S =$
$N(k)$-*contact metric*	$k(n-1)g$
Sasakian	$(n-1)g$
Kenmotsu	$-(n-1)g$
(ε)-*Sasakian*	$\varepsilon(n-1)g$
para-Sasakian	$-(n-1)g$
(ε)-*para-Sasakian*	$-\varepsilon(n-1)g$

Remark 4.16 For projective curvature tensor, $\mathcal{W}_0$-curvature tensor, $\mathcal{W}_1^*$-curvature tensor, $\mathcal{W}_6$-curvature tensor and $\mathcal{W}_8$-curvature tensor, the equation (4.6) is true. For $\mathcal{W}_0^*$-curvature tensor, $\mathcal{W}_1$-curvature tensor, $\mathcal{W}_4$-curvature tensor and $\mathcal{W}_5$-curvature tensor (4.6) do not hold.

In view of Theorem 4.2 and Theorem 4.3 , we have the following

Corollary 4.17 *Let M be an n-dimensional $(N(K),\xi)$-semi-Riemannian manifold. Then the following statements are true:*

(a) *For $\mathcal{T}_a \in \{\mathcal{C},\mathcal{L}\}$, M is ξ-$\mathcal{T}_a$-flat if and only if it is η-Einstein.*

(b) *For $\mathcal{T}_a \in \{\mathcal{C}_*,\mathcal{M},\mathcal{W}_2\}$, M is ξ-$\mathcal{T}_a$-flat if and only if it is Einstein.*

5 T-recurrent manifolds

Definition 5.1 Let T be a $(1,3)$-type tensor. A semi-Riemannian manifold (M,g) is said to be T*-recurrent* if it satisfies

$$(\nabla_U T)(X,Y)Z = \alpha(U)T(X,Y)Z, \tag{5.1}$$

for some nonzero 1-form α. In particular, if T is equal to $\mathcal{T}_a$, R, $\mathcal{C}_*$, $\mathcal{C}$, $\mathcal{L}$, $\mathcal{V}$, $\mathcal{P}_*$, $\mathcal{P}$, $\mathcal{M}$, $\mathcal{W}_0$, $\mathcal{W}_0^*$, $\mathcal{W}_1$, $\mathcal{W}_1^*$, $\mathcal{W}_2$, $\mathcal{W}_3$, $\mathcal{W}_4$, $\mathcal{W}_5$, $\mathcal{W}_6$, $\mathcal{W}_7$, $\mathcal{W}_8$, $\mathcal{W}_9$, then it becomes $\mathcal{T}_a$-recurrent, recurrent, quasi-conformal recurrent, Weyl recurrent, conharmonic recurrent, concircular recurrent, pseudo-projective recurrent, projective recurrent, $\mathcal{M}$-recurrent, $\mathcal{W}_0$-recurrent, $\mathcal{W}_0^*$-recurrent, $\mathcal{W}_1$-recurrent, $\mathcal{W}_1^*$-recurrent, $\mathcal{W}_2$-recurrent, $\mathcal{W}_3$-recurrent, $\mathcal{W}_4$-recurrent, $\mathcal{W}_5$-recurrent, $\mathcal{W}_6$-recurrent, $\mathcal{W}_7$-recurrent, $\mathcal{W}_8$-recurrent, $\mathcal{W}_9$-recurrent, respectively.

Definition 5.2 Let T be a $(1,3)$-type tensor. A semi-Riemannian manifold (M,g) is said to be T*-symmetric* if it satisfies if

$$\nabla T = 0.$$

In particular, if T is equal to $\mathcal{T}_a$, R, $\mathcal{C}_*$, $\mathcal{C}$, $\mathcal{L}$, $\mathcal{V}$, $\mathcal{P}_*$, $\mathcal{P}$, $\mathcal{M}$, $\mathcal{W}_0$, $\mathcal{W}_0^*$, $\mathcal{W}_1$, $\mathcal{W}_1^*$, $\mathcal{W}_2$, $\mathcal{W}_3$, $\mathcal{W}_4$, $\mathcal{W}_5$, $\mathcal{W}_6$, $\mathcal{W}_7$, $\mathcal{W}_8$, $\mathcal{W}_9$, then it becomes $\mathcal{T}_a$-symmetric, symmetric, quasi-conformal symmetric, Weyl symmetric, conharmonic symmetric, concircular symmetric, pseudo-projective symmetric, projective symmetric, $\mathcal{M}$-symmetric, $\mathcal{W}_0$-symmetric, $\mathcal{W}_0^*$-symmetric, $\mathcal{W}_1$-symmetric, $\mathcal{W}_1^*$-symmetric, $\mathcal{W}_2$-symmetric, $\mathcal{W}_3$-symmetric, $\mathcal{W}_4$-symmetric, $\mathcal{W}_5$-symmetric, $\mathcal{W}_6$-symmetric, $\mathcal{W}_7$-symmetric, $\mathcal{W}_8$-symmetric, $\mathcal{W}_9$-symmetric, respectively.

Definition 5.3 Let T be a $(1,3)$-type tensor. A semi-Riemannian manifold (M,g) is said to be T-*semisymmetric* if it satisfies if

$$R(V,U)\cdot T = 0,$$

where $R\,(V,U)$ acts as a derivation on T. In particular, if T is equal to $\mathcal{T}_a$, R, $\mathcal{C}_*$, $\mathcal{C}$, $\mathcal{L}$, $\mathcal{V}$, $\mathcal{P}_*$, $\mathcal{P}$, $\mathcal{M}$, $\mathcal{W}_0$, $\mathcal{W}_0^*$, $\mathcal{W}_1$, $\mathcal{W}_1^*$, $\mathcal{W}_2$, $\mathcal{W}_3$, $\mathcal{W}_4$, $\mathcal{W}_5$, $\mathcal{W}_6$, $\mathcal{W}_7$, $\mathcal{W}_8$, $\mathcal{W}_9$, then it becomes $\mathcal{T}_a$-semisymmetric, semisymmetric, quasi-conformal semisymmetric, Weyl semisymmetric, conharmonic semisymmetric, concircular semisymmetric, pseudo-projective semisymmetric, projective semisymmetric, $\mathcal{M}$-semisymmetric, $\mathcal{W}_0$-semisymmetric, $\mathcal{W}_0^*$-semisymmetric, $\mathcal{W}_1$-semisymmetric, $\mathcal{W}_1^*$-semisymmetric, $\mathcal{W}_2$-semisymmetric, $\mathcal{W}_3$-semisymmetric, $\mathcal{W}_4$-semisymmetric, $\mathcal{W}_5$ -semisymmetric, $\mathcal{W}_6$-semisymmetric, $\mathcal{W}_7$-semisymmetric, $\mathcal{W}_8$-semisymmetric, $\mathcal{W}_9$-semi symmetric, respectively.

Theorem 5.4 *Let M be a semi-Riemannian manifold. If M is T-recurrent or T-symmetric then it is T-semisymmetric.*

Proof. Let us suppose that $T \neq 0$ and M be a T-recurrent semi-Riemannian manifold. Then using (5.1), we get

$$\nabla_Y(g(T,T)) = 2\alpha(Y)g(T,T)$$

and

$$\nabla_X\nabla_Y(g(T,T)) = 2(X\alpha(Y))g(T,T) + 4\alpha(X)\alpha(Y)g(T,T),$$

where the metric g is extended to the inner product between the tensor fields in the standard fashion [27]. Therefore

$$\begin{aligned} 0 &= (\nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]})g(T,T) \\ &= 4d\alpha(X,Y)g(T,T). \end{aligned}$$

Since $g(T,T) \neq 0$, therefore $d\alpha(X,Y) = 0$. Therefore the 1-form α is closed.

Now, from (5.1) we have

$$(\nabla_V\nabla_U T)(X,Y)Z = (V\alpha(U) + \alpha(V)\alpha(U))T(X,Y)Z.$$

Hence

$$(\nabla_V\nabla_U T - \nabla_U\nabla_V T - \nabla_{[V,U]}T)(X,Y)Z = 0.$$

Therefore, we have $R(V,U)\cdot T = 0$. Similarly, we can prove that if M is T-symmetric semi-Riemannian manifold, then it is T-semisymmetric. This proves the result. ■

6 $(\mathcal{T}_a, \mathcal{T}_b)$-semisymmetry

It is well known that every $(1,1)$ tensor field $\mathcal{A}$ on a differentiable manifold determines a derivation $\mathcal{A}\cdot$ of the tensor algebra on the manifold, commuting with contractions. For example, the $(1,1)$ tensor fields $\mathcal{B}(V,U)$ induces the derivation $\mathcal{B}(V,U)\cdot$, thus associating with a $(0,s)$ tensor field $\mathcal{K}$, give the $(0,s+2)$ tensor $\mathcal{B}\cdot\mathcal{K}$. The condition $\mathcal{B}\cdot\mathcal{K}$ is defined by

$$\begin{aligned}(\mathcal{B}\cdot\mathcal{K})(X_1,\ldots,X_s,X,Y) &= (\mathcal{B}(X,Y)\cdot\mathcal{K})(X_1,\ldots,X_s)\\ &= -\mathcal{K}(\mathcal{B}(X,Y)X_1,\ldots,X_s) - \cdots\\ &\quad -\mathcal{K}(X_1,\ldots,\mathcal{B}(X,Y)X_s).\end{aligned} \tag{6.1}$$

Definition 6.1 A semi-Riemannian manifold (M,g) is said to be $(\mathcal{T}_a, \mathcal{T}_b)$-semisymmetric if

$$\mathcal{T}_a(X,Y)\cdot\mathcal{T}_b = 0, \tag{6.2}$$

where $\mathcal{T}_a(X,Y)$ acts as a derivation on $\mathcal{T}_b$. In particular, it is said to be $(R, \mathcal{T}_a)$-semisymmetric if

$$R(X,Y)\cdot\mathcal{T}_a = 0, \tag{6.3}$$

which, in brief, is said to be $\mathcal{T}_a$-semisymmetric.

Theorem 6.2 *Let M be an n-dimensional $(\mathcal{T}_a, \mathcal{T}_b)$-semisymmetric $(N(k),\xi)$-semi-Riemannian manifold. Then*

$$
\begin{aligned}
&- \varepsilon b_0(ka_0 + \varepsilon k(n-1)a_4 + a_7 r)R(U,V,W,X) - \varepsilon a_1 b_0 S(X, R(U,V)W) \\
= \quad & -2k(n-1)a_3(kb_0 + k(n-1)b_4 + b_7 r)\eta(X)\eta(U)g(V,W) \\
&-2k(n-1)a_3(-kb_0 + k(n-1)b_5 - b_7 r)\eta(X)\eta(V)g(U,W) \\
&+ \varepsilon a_1 b_4 S^2(X,U)g(V,W) + \varepsilon a_1 b_5 S^2(X,V)g(U,W) \\
&+ \varepsilon a_1 b_6 S^2(X,W)g(U,V) - a_5(b_1+b_3)S^2(X,V)\eta(U)\eta(W) \\
&- a_5(b_1+b_2)S^2(X,W)\eta(U)\eta(V) - a_5(b_2+b_3)S^2(X,U)\eta(V)\eta(W) \\
&- 2a_6 b_1 S^2(V,W)\eta(X)\eta(U) - 2a_6 b_2 S^2(U,W)\eta(X)\eta(V) \\
&- 2a_6 b_3 S^2(U,V)\eta(X)\eta(W) - 2k^2(n-1)a_3 b_6 g(U,V)\eta(X)\eta(W) \\
&- 2\left(k(n-1)a_3 b_1 + a_6(kb_0 + k(n-1)b_4 + b_7 r)\right)\eta(X)\eta(U)S(V,W) \\
&- 2\left(k(n-1)a_3 b_2 + a_6(-kb_0 + k(n-1)b_5 - b_7 r)\right)\eta(X)\eta(V)S(U,W) \\
&- 2k(n-1)(a_3 b_3 + a_6 b_6)S(U,V)\eta(X)\eta(W) \\
&+ \varepsilon(b_4(ka_0 + k(n-1)a_4 + a_7 r) - a_1(kb_0 + k(n-1)b_4))S(X,U)g(V,W) \\
&+ \varepsilon(b_5(ka_0 + k(n-1)a_4 + a_7 r) - a_1(-kb_0 + k(n-1)b_5))S(X,V)g(U,W) \\
&+ \varepsilon b_6(ka_0 + k(n-1)(a_4 - a_1) + a_7 r)S(X,W)g(U,V) \\
&- \varepsilon(kb_0 + k(n-1)b_4)(ka_0 + k(n-1)a_4 + a_7 r)g(X,U)g(V,W) \\
&- \varepsilon(-kb_0 + k(n-1)b_5)(ka_0 + k(n-1)a_4 + a_7 r)g(U,W)g(X,V) \\
&- \varepsilon k(n-1)b_6(ka_0 + k(n-1)a_4 + a_7 r)g(X,W)g(U,V) \\
&- k(n-1)\left((b_2+b_3)(ka_0 + k(n-1)a_4 + a_7 r)\right. \\
&\left. + (a_2+a_4)(-kb_0 + k(n-1)(b_5+b_6) - b_7 r)\right)g(X,U)\eta(V)\eta(W) \\
&- k(n-1)\left((b_1+b_3)(ka_0 + k(n-1)a_4 + a_7 r)\right. \\
&\left. + (a_2+a_4)(kb_0 + k(n-1)(b_4+b_6) + b_7 r)\right)g(X,V)\eta(U)\eta(W) \\
&- \left((b_1+b_3)(-ka_0 + k(n-1)(a_1+a_2) - a_7 r)\right. \\
&\left. + (a_1+a_5)(kb_0 + k(n-1)(b_4+b_6) + b_7 r)\right)S(X,V)\eta(U)\eta(W) \\
&- \left((b_2+b_3)(-ka_0 + k(n-1)(a_1+a_2) - a_7 r)\right. \\
&\left. + (a_1+a_5)(kb_0 + k(n-1)(b_5+b_6) + b_7 r)\right)S(X,U)\eta(V)\eta(W) \\
&- \left((b_1+b_2)(-ka_0 + k(n-1)(a_1+a_2) - a_7 r)\right. \\
&\left. + k(n-1)(b_4+b_5)(a_1+a_5)\right)S(X,W)\eta(U)\eta(V) \\
&- k(n-1)\left(k(n-1)(b_4+b_5)(a_2+a_4)\right. \\
&\left. + (b_1+b_2)(ka_0 + k(n-1)a_4 + a_7 r)\right)g(X,W)\eta(U)\eta(V). \qquad (6.4)
\end{aligned}
$$

In particular, if M is an n-dimensional $(\mathcal{T}_a, \mathcal{T}_a)$-semisymmetric $(N(k), \xi)$-semi-Riemannian manifold, then

$$
\begin{aligned}
&- \varepsilon a_0(ka_0 + \varepsilon k(n-1)a_4 + a_7 r)R(U,V,W,X) - \varepsilon a_0 a_1 S(X, R(U,V)W)\\
= \quad &- 2k(n-1)a_3(ka_0 + k(n-1)a_4 + a_7 r)\eta(X)\eta(U)g(V,W)\\
&- 2k(n-1)a_3(-ka_0 + k(n-1)a_5 - a_7 r)\eta(X)\eta(V)g(U,W)\\
&+ \varepsilon a_1 a_4 S^2(X,U)g(V,W) + \varepsilon a_1 a_5 S^2(X,V)g(U,W)\\
&+ \varepsilon a_1 a_6 S^2(X,W)g(U,V) - a_5(a_1 + a_3)S^2(X,V)\eta(U)\eta(W)\\
&- a_5(a_1 + a_2)S^2(X,W)\eta(U)\eta(V) - a_5(a_2 + a_3)S^2(X,U)\eta(V)\eta(W)\\
&- 2a_1 a_6 S^2(V,W)\eta(X)\eta(U) - 2a_2 a_6 S^2(U,W)\eta(X)\eta(V)\\
&- 2a_3 a_6 S^2(U,V)\eta(X)\eta(W) - 2k^2(n-1)a_3 a_6 g(U,V)\eta(X)\eta(W)\\
&- 2\left(k(n-1)a_1 a_3 + a_6(ka_0 + k(n-1)a_4 + a_7 r)\right)\eta(X)\eta(U)S(V,W)\\
&- 2\left(k(n-1)a_2 a_3 + a_6(-ka_0 + k(n-1)a_5 - a_7 r)\right)\eta(X)\eta(V)S(U,W)\\
&- 2k(n-1)(a_3 a_3 + a_6 a_6)S(U,V)\eta(X)\eta(W)\\
&+ \varepsilon(a_4(ka_0 + k(n-1)a_4 + a_7 r) - a_1(ka_0 + k(n-1)a_4))S(X,U)g(V,W)\\
&+ \varepsilon(a_5(ka_0 + k(n-1)a_4 + a_7 r) - a_1(-ka_0 + k(n-1)a_5))S(X,V)g(U,W)\\
&+ \varepsilon a_6(ka_0 + k(n-1)(a_4 - a_1) + a_7 r)S(X,W)g(U,V)\\
&- \varepsilon(ka_0 + k(n-1)a_4)(ka_0 + k(n-1)a_4 + a_7 r)g(X,U)g(V,W)\\
&- \varepsilon(-ka_0 + k(n-1)a_5)(ka_0 + k(n-1)a_4 + a_7 r)g(U,W)g(X,V)\\
&- \varepsilon k(n-1)a_6(ka_0 + k(n-1)a_4 + a_7 r)g(X,W)g(U,V)\\
&- k(n-1)\left((a_2 + a_3)(ka_0 + k(n-1)a_4 + a_7 r)\right.\\
&\left.+ (a_2 + a_4)(-ka_0 + k(n-1)(a_5 + a_6) - a_7 r)\right)g(X,U)\eta(V)\eta(W)\\
&- k(n-1)\left((a_1 + a_3)(ka_0 + k(n-1)a_4 + a_7 r)\right.\\
&\left.+ (a_2 + a_4)(ka_0 + k(n-1)(a_4 + a_6) + a_7 r)\right)g(X,V)\eta(U)\eta(W)\\
&- \left((a_1 + a_3)(-ka_0 + k(n-1)(a_1 + a_2) - a_7 r)\right.\\
&\left.+ (a_1 + a_5)(ka_0 + k(n-1)(a_4 + a_6) + a_7 r)\right)S(X,V)\eta(U)\eta(W)\\
&- \left((a_2 + a_3)(-ka_0 + k(n-1)(a_1 + a_2) - a_7 r)\right.\\
&\left.+ (a_1 + a_5)(ka_0 + k(n-1)(a_5 + a_6) + a_7 r)\right)S(X,U)\eta(V)\eta(W)\\
&- \left((a_1 + a_2)(-ka_0 + k(n-1)(a_1 + a_2) - a_7 r)\right.\\
&\left.+ k(n-1)(a_4 + a_5)(a_1 + a_5)\right)S(X,W)\eta(U)\eta(V)\\
&- k(n-1)\left(k(n-1)(a_4 + a_5)(a_2 + a_4)\right.\\
&\left.+ (a_1 + a_2)(ka_0 + k(n-1)a_4 + a_7 r)\right)g(X,W)\eta(U)\eta(V).
\end{aligned}
$$

Proof. Let M be an n-dimensional $(\mathcal{T}_a, \mathcal{T}_b)$-semisymmetric $(N(k), \xi)$-semi-Riemannian manifold. Then

$$(\mathcal{T}_a(Z,X) \cdot \mathcal{T}_b)(U,V)W = 0. \tag{6.5}$$

Taking $Z = \xi$ in (6.5), we get

$$(\mathcal{T}_a(\xi,X) \cdot \mathcal{T}_b)(U,V)W = 0$$

which gives

$$[\mathcal{T}_a(\xi,X), \mathcal{T}_b(U,V)]\, W - \mathcal{T}_b(\mathcal{T}_a(\xi,X)U,V)W - \mathcal{T}_b(U,\mathcal{T}_a(\xi,X)V)W = 0,$$

that is,

$$
\begin{aligned}
0 = {}& \mathcal{T}_a(\xi,X)\mathcal{T}_b(U,V)W - \mathcal{T}_b(\mathcal{T}_a(\xi,X)U,V)W\\
& - \mathcal{T}_b(U,\mathcal{T}_a(\xi,X)V)W - \mathcal{T}_b(U,V)\mathcal{T}_a(\xi,X)W.
\end{aligned} \tag{6.6}
$$

Taking the inner product of (6.6) with ξ, we get

$$
\begin{aligned}
0 = {}& \mathcal{T}_a(\xi,X,\mathcal{T}_b(U,V)W,\xi) - \mathcal{T}_b(\mathcal{T}_a(\xi,X)U,V,W,\xi)\\
& - \mathcal{T}_b(U,\mathcal{T}_a(\xi,X)V,W,\xi) - \mathcal{T}_b(U,V,\mathcal{T}_a(\xi,X)W,\xi).
\end{aligned} \tag{6.7}
$$

By using (3.29),...,(3.34) in (6.7), we get (6.4). ■

Theorem 6.3 *Let M be an n-dimensional $(\mathcal{T}_a, \mathcal{T}_b)$-semisymmetric $(N(k), \xi)$-semi-Riemannian manifold. Then*

$$\begin{aligned}
& \varepsilon \{a_5 b_0 + n a_5 b_1 + a_5 b_2 + a_5 b_6 + a_5 b_3 + a_5 b_5\} S^2(V,W) \\
= \quad & (\{(nb_1 + b_2 + b_3 + b_5 + b_6 + b_0)(\varepsilon k a_0 + \varepsilon b_7 r) \\
& \quad - \varepsilon k(n-1)(2a_5 b_6 + a_2 b_3 + a_1 b_6 + a_1 b_3 + a_1 b_5 \\
& \quad + a_1 b_1 + a_1 b_2 + a_2 b_2 + a_2 b_6 + n a_2 b_1 + a_1 b_0 + a_2 b_0)\} \\
& - \varepsilon(n-1) a_1 b_7 r - \varepsilon n a_5 b_7 r - \varepsilon b_4 a_5 r - \varepsilon a_1 b_4 r) S(V,W) \\
& + \{-\varepsilon k(n-1)(nb_1 + b_2 + b_3 + b_5 + b_6 + b_0)(a_7 r + k a_0 + k(n-1) a_4) \\
& \quad - \varepsilon k(n-1) r((n-1) b_7 a_2 + (n-1) b_7 a_4 + a_2 b_4 + a_4 b_4)\} g(V,W) \\
& + (a_1 + a_2 + 2a_3 + a_4 + a_5 + 2a_6) \left\{-k^2(n-1)^2(nb_1 + b_2 + b_3 + b_5 + b_6 + b_0) \right. \\
& \quad \left. - k(n-1)^2 b_7 r - k(n-1) b_4 r\right\} \eta(V)\eta(W). \qquad (6.8)
\end{aligned}$$

In particular, if M is an n-dimensional $(\mathcal{T}_a, \mathcal{T}_a)$-semisymmetric $(N(k), \xi)$-semi-Riemannian manifold, then

$$\begin{aligned}
& \varepsilon \left\{a_5 a_0 + n a_5 a_1 + a_5 a_2 + a_5 a_6 + a_5 a_3 + a_5^2\right\} S^2(V,W) \\
= \quad & (\{(na_1 + a_2 + a_3 + a_5 + a_6 + a_0)(\varepsilon k a_0 + \varepsilon a_7 r) \\
& \quad - \varepsilon k(n-1)(2a_5 a_6 + a_2 a_3 + a_1 a_6 + a_1 a_3 + a_1 a_5 \\
& \quad + a_1^2 + a_1 a_2 + a_2^2 + a_2 a_6 + n a_1 a_2 + a_0 a_1 + a_0 a_2)\} \\
& - \varepsilon(n-1) a_1 a_7 r - \varepsilon n a_5 a_7 r - \varepsilon a_4 a_5 r - \varepsilon a_1 a_4 r) S(V,W) \\
& + \{-\varepsilon k(n-1)(na_1 + a_2 + a_3 + a_5 + a_6 + a_0)(a_7 r + k a_0 + k(n-1) a_4) \\
& \quad - \varepsilon k(n-1) r((n-1) a_2 a_7 + (n-1) a_4 a_7 + a_2 a_4 + a_4^2)\} g(V,W) \\
& + (a_1 + a_2 + 2a_3 + a_4 + a_5 + 2a_6) \left\{-k^2(n-1)^2(na_1 + a_2 + a_3 + a_5 + a_6 + a_0) \right. \\
& \quad \left. - k(n-1)^2 a_7 r - k(n-1) a_4 r\right\} \eta(V)\eta(W).
\end{aligned}$$

Proof. By contracting (6.4) we get (6.8). ■

From Theorem 6.2, we get

Theorem 6.4 *Let M be an n-dimensional $\mathcal{T}_a$-semisymmetric $(N(k), \xi)$-semi-Riemannian manifold. Then*

$$\begin{aligned}
-\varepsilon a_0 k R(U,V,W,X) = \; & \varepsilon k a_4 S(X,U) g(V,W) + \varepsilon k a_5 S(X,V) g(U,W) \\
& + \varepsilon k a_6 S(X,W) g(U,V) - \varepsilon k^2(n-1) a_6 g(X,W) g(U,V) \\
& - \varepsilon k(k a_0 + k(n-1) a_4) g(V,W) g(X,U) \\
& - \varepsilon k(-k a_0 + k(n-1) a_5) g(U,W) g(X,V) \\
& - k^2(n-1)(a_2 + a_3) g(X,U) \eta(V)\eta(W) \\
& - k^2(n-1)(a_1 + a_3) g(X,V) \eta(U)\eta(W) \\
& - k^2(n-1)(a_1 + a_2) g(X,W) \eta(U)\eta(V) \\
& + k(a_2 + a_3) S(X,U) \eta(V)\eta(W) \\
& + k(a_1 + a_3) S(X,V) \eta(U)\eta(W) \\
& + k(a_1 + a_2) S(X,W) \eta(U)\eta(V). \qquad (6.9)
\end{aligned}$$

Theorem 6.5 *Let M be an n-dimensional $\mathcal{T}_a$-semisymmetric $(N(k), \xi)$-semi-Riemannian manifold such that $a_0 + a_5 + a_6 \neq 0$. Then*

(a)

$$S(V,W) = B_1 g(V,W) + B_2 \eta(V)\eta(W) \qquad (6.10)$$

and

$$\begin{aligned}
-a_0 R(U,V,W,X) \;=\;& (a_4 B_1 - k a_0 - k(n-1)a_4)\, g(X,U)g(V,W) \\
&+ (a_5 B_1 + k a_0 - k(n-1)a_5)\, g(X,V)g(U,W) \\
&+ (a_6 B_1 - k(n-1)a_6)\, g(X,W)g(U,V) \\
&+ \varepsilon(a_2+a_3)\,(B_1 - k(n-1))\, g(X,U)\eta(V)\eta(W) \\
&+ \varepsilon(a_1+a_3)\,(B_1 - k(n-1))\, g(X,V)\eta(U)\eta(W) \\
&+ \varepsilon(a_1+a_2)\,(B_1 - k(n-1))\, g(X,W)\eta(U)\eta(V) \\
&+ 2\varepsilon B_2(a_1+a_2+a_3)\eta(X)\eta(U)\eta(V)\eta(W) \\
&+ a_4 B_2 g(V,W)\eta(X)\eta(U) \\
&+ a_5 B_2 g(U,W)\eta(X)\eta(V) \\
&+ a_6 B_2 g(U,V)\eta(X)\eta(W), \qquad (6.11)
\end{aligned}$$

where

$$B_1 = -\frac{a_4 r - n(k a_0 + k(n-1)a_4) - (-k a_0 + k(n-1)a_5) - k(n-1)a_6}{a_0 + a_5 + a_6},$$

$$B_2 = -\frac{\varepsilon k(a_2+a_3)(r - n(n-1)k)}{a_0 + a_5 + a_6}.$$

(b) *If $a_0 + a_2 + a_3 + n a_4 + a_5 + a_6 \neq 0$, then it is an Einstein manifold and a manifold of constant curvature k.*

Proof. Let M be an n-dimensional $\mathcal{T}_a$-semisymmetric $(N(k),\xi)$-semi-Riemannian manifold such that $a_0 + a_5 + a_6 \neq 0$.
Case (a). Contracting (6.9), we get (6.10).
Case (b). Let $a_0 + a_2 + a_3 + n a_4 + a_5 + a_6 \neq 0$, then contracting (6.10), we get

$$r = kn(n-1). \qquad (6.12)$$

Since $a_0 + a_5 + a_6 \neq 0$, then by (6.9) and (6.12) we get

$$S(V,W) = k(n-1)g(V,W). \qquad (6.13)$$

Using (6.13) and (6.12) in (6.9), we get

$$R(U,V,W,X) = k\,(g(V,W)g(X,U) - g(U,W)g(X,V)), \qquad (6.14)$$

which proves the result. ■

In view of Theorem 6.5, we have the following

Corollary 6.6 *Let M be an n-dimensional $\mathcal{T}_a$-semisymmetric $(N(k),\xi)$-semi-Riemannian manifold such that*

$$\mathcal{T}_a \in \{R, \mathcal{V}, \mathcal{P}, \mathcal{M}, \mathcal{W}_0, \mathcal{W}_0^*, \mathcal{W}_1, \mathcal{W}_1^*, \mathcal{W}_3, \ldots, \mathcal{W}_8\}$$

Then we have the following two tables:

M	$S =$
$N(k)$-contact metric	$k(n-1)g$
Sasakian	$(n-1)g$
Kenmotsu	$-(n-1)g$
(ε)-Sasakian	$\varepsilon(n-1)g$
para-Sasakian	$-(n-1)g$
(ε)-para-Sasakian	$-\varepsilon(n-1)g$

M	$R(X,Y)Z =$
$N(k)$-*contact metric*	$k\,(g(Y,Z)X - g(X,Z)Y)$
Sasakian	$g(Y,Z)X - g(X,Z)Y$
Kenmotsu	$-\,(g(Y,Z)X - g(X,Z)Y)$
(ε)-*Sasakian*	$\varepsilon\,(g(Y,Z)X - g(X,Z)Y)$
para-Sasakian	$-\,(g(Y,Z)X - g(X,Z)Y)$
(ε)-*para-Sasakian*	$-\,\varepsilon\,(g(Y,Z)X - g(X,Z)Y)$

Corollary 6.7 *Let M be an n-dimensional quasi-conformal semisymmetric $(N(k),\xi)$-semi-Riemannian manifold such that $a_0 - a_1 \neq 0$ and $a_0 + (n-2)a_1 \neq 0$. Then we have the following two tables*:

M	$S =$
$N(k)$-*contact metric*	$k(n-1)g$
Sasakian [9]	$(n-1)g$
Kenmotsu	$-\,(n-1)g$
(ε)-*Sasakian*	$\varepsilon(n-1)g$
para-Sasakian	$-\,(n-1)g$
(ε)-*para-Sasakian*	$-\,\varepsilon(n-1)g$

M	$R(X,Y)Z =$
$N(k)$-*contact metric*	$k\,(g(Y,Z)X - g(X,Z)Y)$
Sasakian	$g(Y,Z)X - g(X,Z)Y$
Kenmotsu	$-\,(g(Y,Z)X - g(X,Z)Y)$
(ε)-*Sasakian*	$\varepsilon\,(g(Y,Z)X - g(X,Z)Y)$
para-Sasakian	$-\,(g(Y,Z)X - g(X,Z)Y)$
(ε)-*para-Sasakian*	$-\,\varepsilon\,(g(Y,Z)X - g(X,Z)Y)$

Corollary 6.8 *Let M be an n-dimensional pseudo-projective semisymmetric $(N(k),\xi)$-semi-Riemannian manifold such that $a_0 \neq 0$ and $a_0 - a_1 \neq 0$. Then we have the following two tables*:

M	$S =$
$N(k)$-*contact metric*	$k(n-1)g$
Sasakian	$(n-1)g$
Kenmotsu	$-\,(n-1)g$
(ε)-*Sasakian*	$\varepsilon(n-1)g$
para-Sasakian	$-\,(n-1)g$
(ε)-*para-Sasakian*	$-\,\varepsilon(n-1)g$

M	$R(X,Y)Z =$
$N(k)$-*contact metric*	$k\,(g(Y,Z)X - g(X,Z)Y)$
Sasakian	$g(Y,Z)X - g(X,Z)Y$
Kenmotsu	$-\,(g(Y,Z)X - g(X,Z)Y)$
(ε)-*Sasakian*	$\varepsilon\,(g(Y,Z)X - g(X,Z)Y)$
para-Sasakian	$-\,(g(Y,Z)X - g(X,Z)Y)$
(ε)-*para-Sasakian*	$-\,\varepsilon\,(g(Y,Z)X - g(X,Z)Y)$

Here, we give the well known results of Okumura [28] and Koufogiorgos [19].

Theorem 6.9 [28, Lemma 2.2] *If an n-dimensional Sasakian manifold is conformally flat, then the scalar curvature has a positive constant value $n(n-1)$.*

Theorem 6.10 [19, Corollary 3.3] *Let M be an η-Einstein contact metric manifold of dimension $2m+1 > 5$. If ξ belongs to the k-nullity distribution, then $k = 1$ and the structure is Sasakian.*

Corollary 6.11 *Let M be an n-dimensional Weyl-semisymmetric $(N(k),\xi)$-semi-Riemannian manifold. Then we have the following two tables*:

M	$S =$
$N(k)$-contact metric	$k(n-1)g$
Sasakian	$(n-1)g$
Kenmotsu	$-(n-1)g$
(ε)-Sasakian	$\varepsilon(n-1)g$
para-Sasakian [49]	$-(n-1)g$
(ε)-para-Sasakian	$-\varepsilon(n-1)g$

M	$R(X,Y)Z =$
$N(k)$-contact metric	$k\,(g(Y,Z)X - g(X,Z)Y)$
Sasakian [6]	$g(Y,Z)X - g(X,Z)Y$
Kenmotsu	$-\,(g(Y,Z)X - g(X,Z)Y)$
(ε)-Sasakian	$\varepsilon\,(g(Y,Z)X - g(X,Z)Y)$
para-Sasakian	$-\,(g(Y,Z)X - g(X,Z)Y)$
(ε)-para-Sasakian	$-\varepsilon\,(g(Y,Z)X - g(X,Z)Y)$

Proof. By putting the value for conformal curvature tensor in (6.10), we get

$$S = \left(\frac{r}{n-1} - k\right) g + \varepsilon k \left(nk - \frac{r}{n-1}\right) \eta \otimes \eta. \tag{6.15}$$

Case 1. Let $k \neq 1$. Contracting (6.15), we get

$$r = kn(n-1), \quad k \neq 1.$$

Using the value of r in (6.15), we get

$$S = k(n-1)g.$$

Using this in (6.11), we get

$$R(X,Y)Z = k\,(g(Y,Z)X - g(X,Z)Y)\,.$$

Case 2. Let $k = 1$. By putting the value for conformal curvature tensor in (6.11), we get $\mathcal{C} = 0$. Then using the result of [28] and [19], we get $r = n(n-1)$. ■

Corollary 6.12 *Let M be an n-dimensional conharmonic semisymmetric $(N(k),\xi)$-semi-Riemannian manifold. Then we have the following two tables*:

M	$S =$
$N(k)$-contact metric	$k(n-1)g$
Sasakian	$\left(\frac{r}{n-1} - 1\right) g + \left(n - \frac{r}{n-1}\right) \eta \otimes \eta$
Kenmotsu	$-(n-1)g$
(ε)-Sasakian	$\varepsilon(n-1)g$
para-Sasakian	$-(n-1)g$
(ε)-para-Sasakian	$-\varepsilon(n-1)g$

M	$R(X,Y)Z =$
$N(k)$-contact metric	$k\,(g(Y,Z)X - g(X,Z)Y)$
Sasakian	$\frac{1}{n-2}\left(\frac{r}{n-1} - 2\right)(g(Y,Z)X - g(X,Z)Y)$ $+\frac{1}{n-2}\left(n - \frac{r}{n-1}\right)(\eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y$ $+\,g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi)$
Kenmotsu	$-\,(g(Y,Z)X - g(X,Z)Y)$
(ε)-Sasakian	$\varepsilon\,(g(Y,Z)X - g(X,Z)Y)$
para-Sasakian	$-\,(g(Y,Z)X - g(X,Z)Y)$
(ε)-para-Sasakian	$-\varepsilon\,(g(Y,Z)X - g(X,Z)Y)$

Proof. By putting the value for conharmonic curvature tensor in (6.10), we get

$$S = \left(\frac{r}{n-1} - k\right) g + \varepsilon k \left(nk - \frac{r}{n-1}\right) \eta \otimes \eta. \tag{6.16}$$

Let $k \neq 1$. Contracting (6.16), we get

$$r = kn(n-1), \quad k \neq 1.$$

Using the value of r in (6.16), we get

$$S = k(n-1)g.$$

Using this in (6.11), we get

$$R(X,Y)Z = k\left(g(Y,Z)X - g(X,Z)Y\right).$$

Corollary 6.13 *Let M be an n-dimensional $\mathcal{W}_2$-semisymmetric $(N(k),\xi)$-semi-Riemannian manifold. If M is one of $N(k)$-contact metric manifold, Sasakian manifold, Kenmotsu manifold, (ε)-Sasakian manifold, para-Sasakian manifold or (ε)-para-Sasakian manifold, then*

$$S = \frac{r}{n} g$$

and

$$R(X,Y)Z = \frac{r}{n(n-1)}\left(g(Y,Z)X - g(X,Z)Y\right).$$

Proof. By putting the values for $\mathcal{W}_2$-curvature tensor in (6.10) and (6.11), we get the result. ■

Corollary 6.14 *Let M be an n-dimensional $\mathcal{W}_9$-semisymmetric $(N(k),\xi)$-semi-Riemannian manifold. Then we have the following two tables*:

M	$S =$
$N(k)$-contact metric	$k(n-1)g$
Sasakian	$\left(\frac{r}{n-1} - 1\right) g + \left(n - \frac{r}{n-1}\right) \eta \otimes \eta$
Kenmotsu	$-(n-1)g$
(ε)-Sasakian	$\varepsilon(n-1)g$
para-Sasakian	$-(n-1)g$
(ε)-para-Sasakian	$-\varepsilon(n-1)g$

M	$R(X,Y)Z =$
$N(k)$-contact metric	$k\left(g(Y,Z)X - g(X,Z)Y\right)$
Sasakian	$\frac{1}{n-1}\left(\frac{r}{n-1} - 1\right)\left(g(Y,Z)X - g(X,Z)Y\right)$ $+ \frac{1}{n-1}\left(n - \frac{r}{n-1}\right)\left(\eta(Y)\eta(Z)X + \eta(X)\eta(Z)Y\right)$ $- \frac{2}{n-1}\left(n - \frac{r}{n-1}\right)\eta(X)\eta(Y)\eta(Z)\xi$ $+ \frac{1}{n-1}\left(n - \frac{r}{n-1}\right) g(Y,Z)\eta(X)\xi$
Kenmotsu	$-\left(g(Y,Z)X - g(X,Z)Y\right)$
(ε)-Sasakian	$\varepsilon\left(g(Y,Z)X - g(X,Z)Y\right)$
para-Sasakian	$-\left(g(Y,Z)X - g(X,Z)Y\right)$
(ε)-para-Sasakian	$-\varepsilon\left(g(Y,Z)X - g(X,Z)Y\right)$

Proof. By putting the value for $\mathcal{W}_9$-curvature tensor in (6.10), we get

$$S = \left(\frac{r}{n-1} - k\right) g + \varepsilon k \left(nk - \frac{r}{n-1}\right) \eta \otimes \eta. \tag{6.17}$$

Let $k \neq 1$. Contracting (6.17), we get

$$r = kn(n-1), \quad k \neq 1.$$

Using the value of r in (6.17), we get

$$S = k(n-1)g.$$

Using this in (6.11), we get

$$R(X,Y)Z = k\left(g(Y,Z)X - g(X,Z)Y\right).$$

Remark 6.15 By Theorem 5.4, we conclude that the same results hold if the condition of $\mathcal{T}_a$-semisymmetric is replaced by $\mathcal{T}_a$-recurrent or $\mathcal{T}_a$-symmetric.

Remark 6.16 Some of the results related to the above Corollaries have been proved by authors Miyazawa and Yamaguchi ([23],[26]), Mishra [24], Adati and Matsumoto [1], Sato and Matsumoto [43], Adati and Miyazawa [2], Maralabhavi [21], Ojha [29], De and Ghosh [8], Rahman [40], Ghosh and Sharma [15], Mishra and Ojha [25], Tarafdar and Sengupta [50], Blair et al. [5], Jun et al. [17], Özgür and De [32], Tripathi et al. [51].

Remark 6.17 There are 400 combinations of derivations for the 20 curvature tensors mentioned as particular cases in Definition 2.1. Here, we discussed the results for 6 different structures for each derivation condition. So there are total 2400 results for different structures and curvature tensors. Out of these 2400 cases, we have discussed only 120 cases in this paper. The remaining 2280 cases can be obtained by putting the appropriate value for the curvature tensors, ε and k in (6.4) and (6.8). Out of the remaining 2280 cases, some are mentioned below.

Corollary 6.18 [16] *A $(2m+1)$-dimensional Kenmotsu manifold M satisfies $\mathcal{V}(\xi, X)\cdot R = 0$ if and only if M is either of constant scalar curvature or of constant curvature -1.*

Corollary 6.19 [16] *A $(2m+1)$-dimensional Kenmotsu manifold M satisfies $\mathcal{V}(\xi, X)\cdot \mathcal{V} = 0$ if and only if M is either of constant scalar curvature or of constant curvature -1.*

Corollary 6.20 [33] *An n-dimensional para-Sasakian manifold M satisfies $\mathcal{V}(\xi, X)\cdot \mathcal{V} = 0$ if and only if either the scalar curvature r of M is $r = n(1-n)$ or M is locally isometric to the Hyperbolic space $H^n(-1)$.*

Corollary 6.21 [33] *An n-dimensional para-Sasakian manifold M satisfies $\mathcal{V}(\xi, X)\cdot R = 0$ if and only if either M is locally isometric to the Hyperbolic space $H^n(-1)$ or M has constant scalar curvature $r = n(1-n)$.*

Corollary 6.22 [33] *An n-dimensional para-Sasakian manifold M satisfies $\mathcal{V}(\xi, X)\cdot \mathcal{C} = 0$ if and only if either M has scalar curvature $r = n(1-n)$ or M is conformally flat, in which case M is a special para-Sasakian manifold.*

Corollary 6.23 [5] *A $(2m+1)$-dimensional $N(k)$-contact metric manifold M satisfies $\mathcal{V}(\xi, X)\cdot \mathcal{V} = 0$ if and only if M is locally isometric to the sphere $S^{2m+1}(1)$ or M is 3-dimensional and flat.*

Corollary 6.24 [5] *A $(2m+1)$-dimensional $N(k)$-contact metric manifold M satisfies $\mathcal{V}(\xi, X)\cdot R = 0$ if and only if M is locally isometric to the sphere $S^{2m+1}(1)$ or M is 3-dimensional and flat.*

Theorem 6.25 *Let M be an n-dimensional $\mathcal{T}_a$-semisymmetric $(N(k), \xi)$-semi-Riemannian manifold such that $a_0 + a_5 + a_6 \neq 0$ and*

$$a_0 + a_2 + a_3 + na_4 + a_5 + a_6 \neq 0,$$

then M is $\mathcal{T}_a$-flat if either $k = 0$ or $k \neq 0$ and

$$a_0 + a_1(n-1) + a_4(n-1) + a_7 n(n-1) = 0,$$

$$-a_0 + a_2(n-1) + a_5(n-1) - a_7 n(n-1) = 0,$$

$$a_3 + a_6 = 0.$$

Theorem 6.26 *Let M be an n-dimensional $(N(k), \xi)$-semi-Riemannian manifold of constant curvature k and*

$$\begin{gathered} a_0 + a_1(n-1) + a_4(n-1) + a_7 n(n-1) = 0, \\ -a_0 + a_2(n-1) + a_5(n-1) - a_7 n(n-1) = 0, \\ a_3 + a_6 = 0, \end{gathered}$$

then it is $\mathcal{T}_a$-semisymmetric.

Corollary 6.27 *Let M be an $(N(k), \xi)$-semi-Riemannian manifold such that*

$$\mathcal{T}_a \in \{\mathcal{C}_*, \mathcal{C}, \mathcal{V}, \mathcal{P}_*, \mathcal{P}, \mathcal{M}, \mathcal{W}_0, \mathcal{W}_1^*, \mathcal{W}_2\}.$$

Then it is $\mathcal{T}_a$-semisymmetric if and only if it is a manifold of constant curvature k.

Theorem 6.28 *Let M be $\mathcal{T}_a$-semisymmetric $(N(k), \xi)$-semi-Riemannian manifold such that $a_0+a_5+a_6 \neq 0$ and*

$$a_0 + a_2 + a_3 + na_4 + a_5 + a_6 \neq 0,$$

then M is $\mathcal{T}_a$-conservative.

Proof. Using (6.13) and (6.12) in (2.3), we get div $\mathcal{T}_a = 0$. ■

Example 6.29 [20] *Let M be an n-dimensional $N(k)$-contact metric manifold. If the φ-sectional curvature of any point of M is independent of the choice of φ-section at the point, then it is constant on M and the curvature tensor is given by*

$$\begin{aligned} 4R(X,Y)Z &= (c+3)(g(Y,Z)X - g(X,Z)Y) \\ &\quad +(c+3-4k)(\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X \\ &\quad +g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi) \\ &\quad +(c-1)(2g(X,\varphi Y)\varphi Z + g(X,\varphi Z)\varphi Y \\ &\quad -g(Y,\varphi Z)\varphi X), \end{aligned} \tag{6.18}$$

where c is the constant φ-sectional curvature.

Contracting (6.18)*, we get*

$$4S(Y,Z) = E_1 g(Y,Z) - E_2 \eta(Y)\eta(Z), \tag{6.19}$$

where

$$E_1 = (n-1)(c+3) - (c+3-4k) + 3(c-1)$$

and

$$E_2 = (n-2)(c+3-4k) + 3(c-1).$$

Consider a $\mathcal{T}_a$-semisymmetric $N(k)$-contact metric manifold M with constant φ-sectional curvature c such that

$$\mathcal{T}_a \in \{R, \mathcal{C}_*, \mathcal{C}, \mathcal{L}, \mathcal{V}, \mathcal{P}_*, \mathcal{P}, \mathcal{M}, \mathcal{W}_0, \mathcal{W}_0^*, \mathcal{W}_1, \mathcal{W}_1^*, \mathcal{W}_3, \ldots, \mathcal{W}_9\},$$

then by Corollaries 6.6, 6.7, 6.8, 6.11, 6.12 *and* 6.14*, we have*

$$S = k(n-1)g. \tag{6.20}$$

By (6.19) *and* (6.20)*, we get*

$$E_1 g(Y,Z) - E_2 \eta(Y)\eta(Z) = 4k(n-1)g(Y,Z)$$

Contracting above equation, we get

$$E_1 n - E_2 = 4kn(n-1).$$

By using the value of E_1 and E_2, we get

$$c = \frac{4kn^2 - 12nk - 3n^2 + 8k + 12n - 9}{n^2 - 1}.$$

If M is a $\mathcal{W}_2$-semisymmetric $N(k)$-contact metric manifold of constant φ-sectional curvature c, then by using Corollary 6.13*, we have*

$$c = \frac{4r - 8nk - 3n^2 + 8k + 12n - 9}{n^2 - 1}.$$

7 $(\mathcal{T}_a, S_{\mathcal{T}_b})$-semisymmetry

Definition 7.1 A semi-Riemannian manifold is said to be $(\mathcal{T}_a, S_{\mathcal{T}_b})$-semisymmetric if

$$\mathcal{T}_a(V,U)\cdot S_{\mathcal{T}_b} = 0. \tag{7.1}$$

In particular, it is said to be $(\mathcal{T}_a, S)$-semisymmetric or, in short, $\mathcal{T}_a$-*Ricci-semisymmetric* if it satisfies

$$\mathcal{T}_a(V,U)\cdot S = 0, \tag{7.2}$$

where $\mathcal{T}_a(V,U)$ acts as a derivation on S. In particular, if in (7.2), $\mathcal{T}_a$ is equal to R, $\mathcal{C}_*$, $\mathcal{C}$, $\mathcal{L}$, $\mathcal{V}$, $\mathcal{P}_*$, $\mathcal{P}$, $\mathcal{M}$, $\mathcal{W}_0$, $\mathcal{W}_0^*$, $\mathcal{W}_1$, $\mathcal{W}_1^*$, $\mathcal{W}_2$, $\mathcal{W}_3$, $\mathcal{W}_4$, $\mathcal{W}_5$, $\mathcal{W}_6$, $\mathcal{W}_7$, $\mathcal{W}_8$, $\mathcal{W}_9$, then it becomes Ricci-semisymmetric [2], $\mathcal{C}_*$-Ricci-semisymmetric, $\mathcal{C}$-Ricci-semisymmetric (or, Weyl Ricci-semisymmetric [31]), $\mathcal{L}$-Ricci-semisymmetric, $\mathcal{V}$-Ricci-semisymmetric (concircular Ricci-semisymmetric [16]), $\mathcal{P}_*$-Ricci-semisymmetric, $\mathcal{P}$-Ricci-semisymmetric, $\mathcal{M}$-Ricci-semisymmetric, $\mathcal{W}_0$-Ricci-semisymmetric, $\mathcal{W}_0^*$-Ricci-semisymmetric, $\mathcal{W}_1$-Ricci-semisymmetric, $\mathcal{W}_1^*$-Ricci-semisymmetric, $\mathcal{W}_2$-Ricci-semisymmetric, $\mathcal{W}_3$-Ricci-semisymmetric, $\mathcal{W}_4$-Ricci-semisymmetric, $\mathcal{W}_5$-Ricci-semisymmetric, $\mathcal{W}_6$-Ricci-semisymmetric, $\mathcal{W}_7$-Ricci-semisymmetric, $\mathcal{W}_8$-Ricci-semisymmetric, $\mathcal{W}_9$-Ricci-semisymmetric, respectively.

Lemma 7.2 *Let M be an n-dimensional $(\mathcal{T}_a, S_{\mathcal{T}_b})$-semisymmetric $(N(k),\xi)$-semi-Riemannian manifold. Then*

$$\begin{aligned} 0 \;=\;& \varepsilon a_5(b_0+nb_1+b_2+b_3+b_5+b_6)S^2(Y,U) \\ &+ \{\varepsilon(b_0+nb_1+b_2+b_3+b_5+b_6)\times \\ &(-ka_0+k(n-1)a_1+k(n-1)a_2-a_7r) \\ &\quad + \varepsilon(a_1+a_5)(b_4r+(n-1)b_7r)\}\, S(Y,U) \\ &+ \{\varepsilon k(n-1)(a_2+a_4)(b_4r+(n-1)b_7r) \\ &\quad + \varepsilon k(n-1)(b_0+nb_1+b_2+b_3+b_5+b_6)\times \\ &(ka_0+k(n-1)a_4+a_7r)\}\, g(Y,U) \\ &+ k(n-1)(a_1+a_2+2a_3+a_4+a_5+2a_6)\times \\ &\{(b_4r+(n-1)b_7r) \\ &+ k(n-1)(b_0+nb_1+b_2+b_3+b_5+b_6)\}\, \eta(Y)\eta(U). \end{aligned} \tag{7.3}$$

In particular, if M is $(\mathcal{T}_a, S_{\mathcal{T}_a})$-semisymmetric $(N(k),\xi)$-semi-Riemannian manifold, then

$$\begin{aligned} 0 \;=\;& \varepsilon a_5(a_0+na_1+a_2+a_3+a_5+a_6)S^2(Y,U) \\ &+ \{\varepsilon(a_0+na_1+a_2+a_3+a_5+a_6)\times \\ &(-ka_0+k(n-1)a_1+k(n-1)a_2-a_7r) \\ &\quad + \varepsilon(a_1+a_5)(a_4r+(n-1)a_7r)\}\, S(Y,U) \\ &+ \{\varepsilon k(n-1)(a_2+a_4)(a_4r+(n-1)a_7r) \\ &\quad + \varepsilon k(n-1)(a_0+na_1+a_2+a_3+a_5+a_6)\times \\ &(ka_0+k(n-1)a_4+a_7r)\}\, g(Y,U) \\ &+ k(n-1)(a_1+a_2+2a_3+a_4+a_5+2a_6)\times \\ &\{(a_4r+(n-1)a_7r) \\ &+ k(n-1)(a_0+na_1+a_2+a_3+a_5+a_6)\}\, \eta(Y)\eta(U). \end{aligned} \tag{7.4}$$

Proof. Let M be an n-dimensional $(\mathcal{T}_a, S_{\mathcal{T}_b})$-semisymmetric $(N(k),\xi)$-semi-Riemannian manifold. Then

$$\left(\mathcal{T}_a(X,Y)\cdot S_{\mathcal{T}_b}\right)(U,V) = 0. \tag{7.5}$$

Taking $X = \xi = V$ in (7.5), we get

$$(\mathcal{T}_a(\xi,Y)\cdot S_{\mathcal{T}_b})(U,\xi) = 0,$$

which gives

$$S_{\mathcal{T}_b}(\mathcal{T}_a(\xi,Y)U,\xi) + S_{\mathcal{T}_b}(U,\mathcal{T}_a(\xi,Y)\xi) = 0. \tag{7.6}$$

Using (3.1), (3.23), (3.30), (3.31), (3.35) and (3.36) in (7.6), we get (7.3). ■

By Lemma 7.2, we have the following

Theorem 7.3 *Let M be a $(R, S_{\mathcal{T}_a})$-semisymmetric $(N(k), \xi)$-semi-Riemannian manifold such that*

$$a_0 + na_1 + a_2 + a_3 + a_5 + a_6 \neq 0.$$

Then

$$S(X, Y) = k(n-1)g(X, Y).$$

Theorem 7.4 *An Einstein manifold is $(R, S_{\mathcal{T}_a})$-semisymmetric.*

Proof. Let M be an Einstein manifold. Then we have

$$S(X, Y) = d_1 g(X, Y),$$

where d_1 is smooth function on the manifold. By (2.4), we have

$$\begin{aligned} S_{\mathcal{T}_a}(X, Y) &= d_1(a_0 + na_1 + a_2 + a_3 + na_4 + a_5 + a_6 + n(n-1)a_7)g(X, Y) \\ &= d_2 g(X, Y), \end{aligned} \tag{7.7}$$

where

$$d_2 = d_1(a_0 + na_1 + a_2 + a_3 + na_4 + a_5 + a_6 + n(n-1)a_7).$$

Then by condition $(R(X, Y) \cdot S_{\mathcal{T}_a})(U, V)$ and (7.7), we get

$$-S_{\mathcal{T}_a}(R(X, Y)U, V) - S_{\mathcal{T}_a}(U, R(X, Y)V) = 0.$$

Hence this proves the result. ■

Theorem 7.5 *Let M be an Einstein manifold such that*

$$\mathcal{T}_a \in \{R, \mathcal{C}_*, \mathcal{C}, \mathcal{L}, \mathcal{V}, \mathcal{M}, \mathcal{W}_0, \mathcal{W}_0^*, \mathcal{W}_3\}.$$

Then it is $\mathcal{T}_a$-Ricci-semisymmetric.

Proof. Let M be an Einstein manifold such that

$$\mathcal{T}_a \in \{R, \mathcal{C}_*, \mathcal{C}, \mathcal{L}, \mathcal{V}, \mathcal{M}, \mathcal{W}_0, \mathcal{W}_0^*, \mathcal{W}_3\}.$$

Then we have

$$S = \alpha g, \tag{7.8}$$

where α is smooth function on the manifold M. Using condition $(\mathcal{T}_a(X, Y) \cdot S)(U, V)$ and (7.8), we get

$$-S(\mathcal{T}_a(X, Y)U, V) - S(U, \mathcal{T}_a(X, Y)V) = 0.$$

Hence this proves the result. ■

By Lemma 7.2, we have the following

Theorem 7.6 *Let M be a $\mathcal{T}_a$-Ricci-semisymmetric $(N(k), \xi)$-semi-Riemannian manifold. Then*

$$\varepsilon a_5\, S^2(X, Y) = E\, S(X, Y) + F g(X, Y) + G\eta(X)\eta(Y),$$

where

$$E = (\varepsilon k a_0 + \varepsilon a_7 r - \varepsilon k(n-1)a_1 - \varepsilon k(n-1)a_2),$$

$$F = -\varepsilon k(n-1)(k a_0 + k(n-1)a_4 + a_7 r),$$

$$G = -k^2(n-1)^2(a_1 + a_2 + 2a_3 + a_4 + a_5 + 2a_6).$$

In particular, if $a_5 = 0 \neq E$, then M is η-Einstein manifold.

In view of Theorem 7.6, we have the following

Corollary 7.7 *Let M be an n-dimensional Ricci-semisymmetric $(N(k),\xi)$-semi-Riemannian manifold. Then we have the following table*:

M	$S=$
$N(k)$-contact metric [35]	$k(n-1)g$
Sasakian [35]	$(n-1)g$
Kenmotsu [16]	$-(n-1)g$
(ε)-Sasakian	$\varepsilon(n-1)g$
para-Sasakian ([2], [34], [44])	$-(n-1)g$
(ε)-para-Sasakian [51]	$-\varepsilon(n-1)g$

Corollary 7.8 *Let M be an n-dimensional $\mathcal{C}_*$-Ricci-semisymmetric $(N(k),\xi)$-semi-Riemannian manifold. Then we have the following table*:

M	$S^2=$
$N(k)$-contact metric	$-\left(\left(k-\frac{r}{n(n-1)}\right)\frac{a_0}{a_1}-\frac{2r}{n}\right)S$ $+k(n-1)\left(\left(k-\frac{r}{n(n-1)}\right)\frac{a_0}{a_1}+k(n-1)-\frac{2r}{n}\right)g$
Sasakian	$-\left(\left(1-\frac{r}{n(n-1)}\right)\frac{a_0}{a_1}-\frac{2r}{n}\right)S$ $+(n-1)\left(\left(1-\frac{r}{n(n-1)}\right)\frac{a_0}{a_1}+(n-1)-\frac{2r}{n}\right)g$
Kenmotsu	$\left(\left(1+\frac{r}{n(n-1)}\right)\frac{a_0}{a_1}+\frac{2r}{n}\right)S$ $+(n-1)\left(\left(1+\frac{r}{n(n-1)}\right)\frac{a_0}{a_1}+(n-1)+\frac{2r}{n}\right)g$
(ε)-Sasakian	$-\varepsilon\left(\left(1-\frac{\varepsilon r}{n(n-1)}\right)\frac{a_0}{a_1}-\frac{2\varepsilon r}{n}\right)S$ $+\varepsilon(n-1)\left(\left(\varepsilon-\frac{r}{n(n-1)}\right)\frac{a_0}{a_1}+\varepsilon(n-1)-\frac{2r}{n}\right)g$
para-Sasakian	$\left(\left(1+\frac{r}{n(n-1)}\right)\frac{a_0}{a_1}+\frac{2r}{n}\right)S$ $+(n-1)\left(\left(1+\frac{r}{n(n-1)}\right)\frac{a_0}{a_1}+(n-1)+\frac{2r}{n}\right)g$
(ε)-para-Sasakian	$\varepsilon\left(\left(1+\frac{\varepsilon r}{n(n-1)}\right)\frac{a_0}{a_1}+\frac{2\varepsilon r}{n}\right)S$ $+\varepsilon(n-1)\left(\left(\varepsilon+\frac{r}{n(n-1)}\right)\frac{a_0}{a_1}+\varepsilon(n-1)+\frac{2r}{n}\right)g$

Corollary 7.9 *Let M be an n-dimensional $\mathcal{C}$-Ricci-semisymmetric $(N(k),\xi)$-semi-Riemannian manifold. Then we have the following table*:

M	$S^2=$
$N(k)$-contact metric	$\left(\frac{r}{n-1}+k(n-2)\right)S-k(r-(n-1)k)g$
Sasakian	$\left(\frac{r}{n-1}+(n-2)\right)S-(r-(n-1))g$
Kenmotsu	$\left(\frac{r}{n-1}-(n-2)\right)S+(r+(n-1))g$
(ε)-Sasakian	$\left(\frac{r}{n-1}+\varepsilon(n-2)\right)S-\varepsilon(r-(n-1)\varepsilon)g$
para-Sasakian [31]	$\left(\frac{r}{n-1}-(n-2)\right)S+(r+(n-1))g$
(ε)-para-Sasakian	$\left(\frac{r}{n-1}-\varepsilon(n-2)\right)S+\varepsilon(r+(n-1)\varepsilon)g$

Corollary 7.10 *Let M be an n-dimensional $\mathcal{L}$-Ricci-semisymmetric $(N(k),\xi)$-semi-Riemannian manifold. Then we have the following table*:

M	$S^2 =$
$N(k)$*-contact metric*	$k(n-2)S+k^2(n-1)g$
Sasakian	$(n-2)S+(n-1)g$
Kenmotsu	$-(n-2)S+(n-1)g$
(ε)*-Sasakian*	$\varepsilon(n-2)S+(n-1)g$
para-Sasakian	$-(n-2)S+(n-1)g$
(ε)*-para-Sasakian*	$-\varepsilon(n-2)S+(n-1)g$

Corollary 7.11 *Let M be an n-dimensional $\mathcal{V}$-Ricci-semisymmetric $(N(k),\xi)$-semi-Riemannian manifold. Then we have the following table*:

M	Result
$N(k)$*-contact metric*	$S=k(n-1)g$ or $r=kn(n-1)$
Sasakian	$S=(n-1)g$ or $r=n(n-1)$
Kenmotsu [16]	$S=-(n-1)g$ or $r=-n(n-1)$
(ε)*-Sasakian*	$S=\varepsilon(n-1)g$ or $r=\varepsilon n(n-1)$
para-Sasakian [33]	$S=-(n-1)g$ or $r=-n(n-1)$
(ε)*-para-Sasakian*	$S=-\varepsilon(n-1)g$ or $r=-\varepsilon n(n-1)$

Corollary 7.12 *Let M be an n-dimensional $\mathcal{P}_*$-Ricci-semisymmetric $(N(k),\xi)$-semi-Riemannian manifold such that $a_0+(n-1)a_1\neq 0$. Then we have the following table*:

M	Result
$N(k)$*-contact metric*	$S=k(n-1)g$ or $r=\dfrac{n(n-1)ka_0}{a_0+(n-1)a_1}$
Sasakian	$S=(n-1)g$ or $r=\dfrac{n(n-1)a_0}{a_0+(n-1)a_1}$
Kenmotsu	$S=-(n-1)g$ or $r=\dfrac{-n(n-1)a_0}{a_0+(n-1)a_1}$
(ε)*-Sasakian*	$S=\varepsilon(n-1)g$ or $r=\dfrac{n(n-1)\varepsilon a_0}{a_0+(n-1)a_1}$
para-Sasakian	$S=-(n-1)g$ or $r=\dfrac{-n(n-1)a_0}{a_0+(n-1)a_1}$
(ε)*-para-Sasakian*	$S=-\varepsilon(n-1)g$ or $r=\dfrac{-n(n-1)\varepsilon a_0}{a_0+(n-1)a_1}$

Corollary 7.13 *Let M be an n-dimensional $\mathcal{P}$-Ricci-semisymmetric $(N(k),\xi)$-semi-Riemannian manifold. Then we have the following table*:

M	$S=$
$N(k)$*-contact metric*	$k(n-1)g$
Sasakian	$(n-1)g$
Kenmotsu	$-(n-1)g$
(ε)*-Sasakian*	$\varepsilon(n-1)g$
para-Sasakian	$-(n-1)g$
(ε)*-para-Sasakian*	$-\varepsilon(n-1)g$
Lorentzian para-Sasakian	$(n-1)g$

Corollary 7.14 *Let M be an n-dimensional $\mathcal{M}$-Ricci-semisymmetric $(N(k),\xi)$-semi-Riemannian mani-*

fold. Then we have the following table:

M	$S^2 =$
$N(k)$-*contact metric*	$2k(n-1)S + k^2(n-1)^2 g$
Sasakian	$2(n-1)S + (n-1)^2 g$
Kenmotsu	$-2(n-1)S + (n-1)^2 g$
(ε)-*Sasakian*	$2\varepsilon(n-1)S + (n-1)^2 g$
para-Sasakian	$-2(n-1)S + (n-1)^2 g$
(ε)-*para-Sasakian*	$-2\varepsilon(n-1)S + (n-1)^2 g$

Corollary 7.15 *Let M be an n-dimensional $\mathcal{W}_0$-Ricci-semisymmetric $(N(k),\xi)$-semi-Riemannian manifold. Then we have the following table*:

M	$S^2 =$
$N(k)$-*contact metric*	$2k(n-1)S - k^2(n-1)^2 g$
Sasakian	$2(n-1)S - (n-1)^2 g$
Kenmotsu	$-2(n-1)S - (n-1)^2 g$
(ε)-*Sasakian*	$2\varepsilon(n-1)S - (n-1)^2 g$
para-Sasakian	$-2(n-1)S - (n-1)^2 g$
(ε)-*para-Sasakian*	$-2\varepsilon(n-1)S - (n-1)^2 g$

Corollary 7.16 *Let M be an n-dimensional $\mathcal{W}_0^*$-Ricci-semisymmetric $(N(k),\xi)$-semi-Riemannian manifold. Then we have the following table*:

M	$S^2 =$
$N(k)$-*contact metric*	$k^2(n-1)^2 g$
Sasakian	$(n-1)^2 g$
Kenmotsu	$(n-1)^2 g$
(ε)-*Sasakian*	$(n-1)^2 g$
para-Sasakian	$(n-1)^2 g$
(ε)-*para-Sasakian*	$(n-1)^2 g$

Corollary 7.17 *Let M be an n-dimensional $\mathcal{W}_1$-Ricci-semisymmetric $(N(k),\xi)$-semi-Riemannian manifold. Then we have the following table*:

M	$S =$
$N(k)$-*contact metric*	$k(n-1)g$
Sasakian	$(n-1)g$
Kenmotsu	$-(n-1)g$
(ε)-*Sasakian*	$\varepsilon(n-1)g$
para-Sasakian	$-(n-1)g$
(ε)-*para-Sasakian*	$-\varepsilon(n-1)g$

Corollary 7.18 *Let M be an n-dimensional $\mathcal{W}_1^*$-Ricci-semisymmetric $(N(k),\xi)$-semi-Riemannian manifold. Then we have the following table*:

M	$S =$
$N(k)$-*contact metric*	$k(n-1)g$
Sasakian	$(n-1)g$
Kenmotsu	$-(n-1)g$
(ε)-*Sasakian*	$\varepsilon(n-1)g$
para-Sasakian	$-(n-1)g$
(ε)-*para-Sasakian*	$-\varepsilon(n-1)g$

Corollary 7.19 *Let M be an n-dimensional $\mathcal{W}_2$-Ricci-semisymmetric $(N(k),\xi)$-semi-Riemannian mani-*

fold. Then we have the following table:

M	$S^2 =$
$N(k)$-*contact metric*	$k(n-1)S$
Sasakian	$(n-1)S$
Kenmotsu	$-(n-1)S$
(ε)-*Sasakian*	$\varepsilon(n-1)S$
para-Sasakian	$-(n-1)S$
(ε)-*para-Sasakian*	$-\varepsilon(n-1)S$

Corollary 7.20 *Let M be an n-dimensional $\mathcal{W}_3$-Ricci-semisymmetric $(N(k), \xi)$-semi-Riemannian manifold. Then we have the following table*:

M	$S =$
$N(k)$-*contact metric*	$k(n-1)g$
Sasakian	$(n-1)g$
Kenmotsu	$-(n-1)g$
(ε)-*Sasakian*	$\varepsilon(n-1)g$
para-Sasakian	$-(n-1)g$
(ε)-*para-Sasakian*	$-\varepsilon(n-1)g$

Corollary 7.21 *Let M be an n-dimensional $\mathcal{W}_4$-Ricci-semisymmetric $(N(k), \xi)$-semi-Riemannian manifold. Then we have the following table*:

M	$S^2 =$
$N(k)$-*contact metric*	$k(n-1)S - k^2(n-1)^2 g + \varepsilon k^2(n-1)^2 \eta \otimes \eta$
Sasakian	$(n-1)S - (n-1)^2 g + (n-1)^2 \eta \otimes \eta$
Kenmotsu	$-(n-1)S - (n-1)^2 g + (n-1)^2 \eta \otimes \eta$
(ε)-*Sasakian*	$\varepsilon(n-1)S - (n-1)^2 g + \varepsilon(n-1)^2 \eta \otimes \eta$
para-Sasakian	$-(n-1)S - (n-1)^2 g + (n-1)^2 \eta \otimes \eta$
(ε)-*para-Sasakian*	$-\varepsilon(n-1)S - (n-1)^2 g + \varepsilon(n-1)^2 \eta \otimes \eta$

Corollary 7.22 *Let M be an n-dimensional $\mathcal{W}_5$-Ricci-semisymmetric $(N(k), \xi)$-semi-Riemannian manifold. Then we have the following table*:

M	$S^2 =$
$N(k)$-*contact metric*	$2k(n-1)S - k^2(n-1)^2 g$
Sasakian	$2(n-1)S - (n-1)^2 g$
Kenmotsu	$-2(n-1)S - (n-1)^2 g$
(ε)-*Sasakian*	$2\varepsilon(n-1)S - (n-1)^2 g$
para-Sasakian	$-2(n-1)S - (n-1)^2 g$
(ε)-*para-Sasakian*	$-2\varepsilon(n-1)S - (n-1)^2 g$

Corollary 7.23 *Let M be an n-dimensional $\mathcal{W}_6$-Ricci-semisymmetric $(N(k), \xi)$-semi-Riemannian manifold. Then we have the following table*:

M	$2S =$
$N(k)$-*contact metric*	$k(n-1)g + k(n-1)\eta \otimes \eta$
Sasakian	$(n-1)g + (n-1)\eta \otimes \eta$
Kenmotsu	$-(n-1)g - (n-1)\eta \otimes \eta$
(ε)-*Sasakian*	$\varepsilon(n-1)g + (n-1)\eta \otimes \eta$
para-Sasakian	$-(n-1)g - (n-1)\eta \otimes \eta$
(ε)-*para-Sasakian*	$-\varepsilon(n-1)g - (n-1)\eta \otimes \eta$

Corollary 7.24 *Let M be an n-dimensional $\mathcal{W}_7$-Ricci-semisymmetric $(N(k), \xi)$-semi-Riemannian mani-*

fold. Then we have the following table:

M	$S =$
$N(k)$-*contact metric*	$k(n-1)g$
Sasakian	$(n-1)g$
Kenmotsu	$-(n-1)g$
(ε)-*Sasakian*	$\varepsilon(n-1)g$
para-Sasakian	$-(n-1)g$
(ε)-*para-Sasakian*	$-\varepsilon(n-1)g$

Corollary 7.25 *Let M be an n-dimensional $\mathcal{W}_8$-Ricci-semisymmetric $(N(k),\xi)$-semi-Riemannian manifold. Then we have the following table*:

M	$2S =$
$N(k)$-*contact metric*	$k(n-1)g + k(n-1)\eta\otimes\eta$
Sasakian	$(n-1)g + (n-1)\eta\otimes\eta$
Kenmotsu	$-(n-1)g - (n-1)\eta\otimes\eta$
(ε)-*Sasakian*	$\varepsilon(n-1)g + (n-1)\eta\otimes\eta$
para-Sasakian	$-(n-1)g - (n-1)\eta\otimes\eta$
(ε)-*para-Sasakian*	$-\varepsilon(n-1)g - (n-1)\eta\otimes\eta$

Corollary 7.26 *Let M be an n-dimensional $\mathcal{W}_9$-Ricci-semisymmetric $(N(k),\xi)$-semi-Riemannian manifold. Then we have the following table*:

M	$S =$
$N(k)$-*contact metric*	$k(n-1)\eta\otimes\eta$
Sasakian	$(n-1)\eta\otimes\eta$
Kenmotsu	$-(n-1)\eta\otimes\eta$
(ε)-*Sasakian*	$(n-1)\eta\otimes\eta$
para-Sasakian	$-(n-1)\eta\otimes\eta$
(ε)-*para-Sasakian*	$-(n-1)\eta\otimes\eta$

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