# Stability of tautological bundles on the degree two Hilbert scheme of surfaces

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March 14, 2012

**Abstract**

Let $(X, H)$ be a polarized smooth projective surface satisfying $H^1(X, \mathcal{O}_X) = 0$ and let $\mathcal{F}$ be either a rank one torsion-free sheaf or a rank two $\mu_H$-stable vector bundle on $X$. Assume that $c_1(\mathcal{F}) \neq 0$. In this article it is shown that the rank two, respectively rank four tautological sheaf $\mathcal{F}^{[2]}$ associated with $\mathcal{F}$ on the Hilbert square $X^{[2]}$ is $\mu$-stable with respect to a certain polarization.

Keywords: moduli spaces, irreducible holomorphic symplectic manifolds, K3 surfaces
MCS: 14D20, 14J28, 14J60, 14F05

# Contents

# 0 Introduction

Let $X$ be an algebraic K3 surface with polarization $H \in \mathrm{Pic}(X)$ and let $v = (r, c, s) \in \mathbb{N} \oplus \mathrm{NS}(X) \oplus \mathbb{Z}$. Mukai has shown that in many cases - if $v$ is carefully chosen - the moduli space of $H$-semistable sheaves of rank $r$, first Chern class $c$ and second Chern class $s$ is again a smooth compact complex manifold carrying a holomorphic symplectic structure. In fact, all these moduli spaces are deformation equivalent to $\mathrm{Hilb}^n(X)$ for some $n \geq 0$. Now the natural question arises what happens if we start with another hyperkähler manifold and study the geometry of moduli spaces of sheaves on this manifold. Not much is known about this topic and one of the fundamental questions is the following: Does there exist a symplectic structure on these moduli spaces? Of course answering this question in general will be very complicated. But one could hope for at least finding an example of a such a moduli space that does carry such a symplectic structure. Therefore we need examples of vector bundles on higher dimensional hyperkähler manifolds and then have to ask for the stability of these bundles. One big class of examples are the so-called 'tautological bundles' on the Hilbert schemes of K3 surfaces. They arise as the images of vector bundles on a K3 under a Fourier−Mukai transform. We will concentrate on the case of $\mathrm{Hilb}^2(X)$, where $X$ is a projective K3-surface. Schlickewei has shown in [Schl] that in many cases these tautological bundles associated with line bundles on $X$ are stable with respect to a carefully chosen polarization on the Hilbert scheme. We will extend this result by showing that, in fact, every rank two tautological sheaf associated with any rank one torsion-free sheaf having non-vanishing first Chern class is stable with respect to some polarization. Furthermore we will prove that the rank four tautological vector bundle associated with any stable rank two bundle is stable. Again we assume that the first Chern class is nontrivial. This provides us with quite a big variety of stable vector bundles on $\mathrm{Hilb}^2(X)$. In a forthcoming paper it will be shown that in some cases the component of the moduli space of sheaves on $\mathrm{Hilb}^2(X)$ containing the tautological sheaves is smooth and isomorphic to the moduli space of sheaves on the K3 surface. It is therefore an irreducible holomorphic symplectic manifold.
In fact, all results concerning the stability of the tautological sheaves are valid for any smooth projective surface $X$ satisfying $h^1(X, \mathcal{O}_X) = 0$, so they will be presented in this generality.

*Acknowledgements*: I want to thank Klaus Hulek, David Ploog, Marc Nieper-Wißkirchen and Andreas Krug for many useful comments and suggestions.

# 1 The geometric set-up

Let $X$ be a projective surface satisfying $h^1(X, \mathcal{O}_X) = 0$ and choose a polarization $H$. Throughout this text we will consider the following basic blow-up and projections diagram:

$$\begin{tikzcd}
D \arrow[rr, "i"] \arrow[d, "\sigma_D"] & & \widetilde{X \times X} \arrow[d, "\sigma"] \arrow[ddl, bend right, "r_1"'] \arrow[ddr, bend left, "r_2"] \arrow[drr, "\pi"] & & & X^{[2]} \times X \arrow[dl, "p"'] \arrow[dr, "q"] & \\
X \arrow[rr, "\Delta"] & & X \times X \arrow[dl, "\pi_1"] \arrow[dr, "\pi_2"] & & X^{[2]} & & X \\
& X & & X & & &
\end{tikzcd} .$$

Here $\Delta$ is the diagonal embedding, $\sigma$ is the blowing-up morphism (we are blowing up the diagonal), $D \simeq \mathbb{P}(\mathcal{N}_{X|X\times X}) \simeq \mathbb{P}(\mathcal{T}_X)$ denotes the exceptional divisor together with the projection $\sigma_D$, the inclusion $i$ and $\mathcal{O}_D(1)$, the dual of the tautological line bundle. It is well know that $\mathcal{N}_{D|\widetilde{X\times X}} \cong \mathcal{O}_D(-1)$ (see for example Theorem II 8.24 in [Har]). Furthermore $\pi_1$, $\pi_2$, $p$ and $q$ denote the natural projections onto the particular factors and $r_1$ and $r_2$ are the compositions of $\pi_1$ and $\pi_2$ with $\sigma$. Last but not least we have the flat two-to-one covering $\pi$.

We want to summarize the most important facts about the Chow rings and cohomology of the upper diagram. Therefore, for any variety $Y$ denote by $A^*(Y)$ its Chow ring and by $H^*(Y)$ its rational cohomology. The following results are formulated in terms of Chow rings in [Ful] but we will sometimes tacitly use the cycle map to transfer the results to the rational cohomology. By Künneth formula we have $H^*(X \times X) \cong H^*(X)^{\boxtimes 2}$. Now let $\xi$ denote the first Chern class of $\mathcal{O}_D(1)$. By Remark 3.2.4 and Theorem 3.3 in [Ful] we have

$$A^*(D) \cong A^*(X)[\xi]/(\xi^2 + c_1(\mathcal{T}_X)\xi + c_2(\mathcal{T}_X)).$$

Finally, from Proposition 6.7e) in [Ful] it follows that

$$H^*(\widetilde{X \times X}) \cong \sigma^\star H^*(X \times X) \oplus i_\star \sigma_D^\star H^*(X).$$

The ring structure on $A^*(\widetilde{X \times X})$ is explained in c) and d) of the following lemma together with the most fundamental identities in this ring.

**Lemma 1.1** *Let $\alpha, \beta, \gamma \in A^*(X)$. In $A^*(\widetilde{X \times X})$ we have the following identities:*

a)
$$i_\star(\xi \cdot \sigma_D^\star(\alpha)) = \sigma^\star \Delta_\star(\alpha),$$

b)
$$i^\star i_\star \lambda = -\xi \cdot \lambda, \text{ for all } \lambda \in A^*(D),$$

c)
$$i_\star \sigma_D^\star \alpha \cdot \sigma^\star(\beta \otimes \gamma) = i_\star \sigma_D^\star(\alpha \cdot \beta \cdot \gamma),$$

d)
$$i_\star \sigma_D^\star(\alpha) \cdot i_\star \sigma_D^\star(\beta) = -\sigma^\star \Delta_\star(\alpha \cdot \beta).$$

*Proof*: a) Follows from the general formula in Prop. 6.7. a) in [Ful]. The proof in this special case is quite elementary but lenghty.
b) This is the self-intersection formula Cor 6.3 in [Ful]:

$$i^\star i_\star \lambda = c_1(\mathcal{N}_{D|\widetilde{X \times X}}) \cdot \lambda = c_1(\mathcal{O}_D(-1)) \cdot \lambda = -\xi \cdot \lambda.$$

c) We have $\alpha \cdot \beta \cdot \gamma = \alpha \cdot \Delta^\star(\beta \otimes \gamma)$. Applying $\sigma_D^\star$ we get

$$\begin{array}{rcl} \sigma_D^\star(\alpha \cdot \beta \cdot \gamma) & = & \sigma_D^\star(\alpha \cdot \Delta^\star(\beta \otimes \gamma)) \\ = \quad \sigma_D^\star \alpha \cdot \sigma_D^\star \Delta^\star(\beta \otimes \gamma) & = & \sigma_D^\star \alpha \cdot i^\star \sigma^\star(\beta \otimes \gamma). \end{array}$$

Now we apply $i_\star$ and use the projection formula.
d) We use b) to find

$$i_\star \sigma_D^\star(\alpha) \cdot i_\star \sigma_D^\star(\beta) = i_\star(i^\star i_\star \sigma_D^\star(\alpha) \cdot i_\star \sigma_D^\star(\beta)) = -i_\star(\xi \cdot \sigma_D^\star(\alpha) \cdot i_\star \sigma_D^\star(\beta)).$$

Now we apply c) and we are done. □

**Corollary 1.2** *We have*

a)
$$i^\star D = -\xi,$$

*where we denote $i_\star[D] \in A^3(\widetilde{X \times X})$ simply by $D$,*

b)
$$D^2 = -i_\star \xi = -\sigma^\star \Delta,$$

*where $\Delta$ also denotes the cohomology class of the diagonal in $X \times X$, and finally*

c)
$$(\sigma^\star \Delta)^2 = \sigma^\star \Delta_\star(c_2(\mathcal{T}_X)).$$

*Proof*: a) Apply b) of the lemma to $\lambda = [D]$.
b) We use a) and for the second equality we apply a) of the Lemma to $\alpha = [X]$ to get

$$D^2 = i_\star i^\star D = i_\star(-\xi) = -\sigma^\star \Delta.$$

c) Very similarly to the proof of b) in the lemma we have:

$$(\sigma^\star \Delta)^2 = \sigma^\star \Delta_\star \Delta^\star \Delta = \sigma^\star \Delta_\star \Delta^\star \Delta_\star[X] = \sigma^\star \Delta_\star(c_2(\mathcal{N}_{X|X \times X})) = \sigma^\star \Delta_\star(c_2(\mathcal{T}_X)).$$

□

We will continue with some considerations concerning the Picard groups of the varieties we are looking at. We have $\mathrm{Pic}(X \times X) \cong (\mathrm{Pic}X)^{\boxplus 2}$. Here we apply exercise III 12.6b) in [Har] since $h^1(X, \mathcal{O}_X) = 0$. And accordingly we have $\mathrm{Pic}(\widetilde{X \times X}) \cong (\mathrm{Pic}X)^{\boxplus 2} \oplus \mathbb{Z}D$.

So we will write an element of $\mathrm{Pic}(\widetilde{X \times X})$ as $g \otimes 1 + 1 \otimes h + aD$ for some $g, h \in \mathrm{Pic}X$ and $a \in \mathbb{Z}$ and denote the corresponding line bundle by $\mathcal{L}_{(g,h,a)}$.
Furthermore it is well known that $\mathrm{Pic}(X^{[2]}) \cong \mathrm{Pic}(X) \oplus \mathbb{Z}\delta$, where $\delta$ is a class such that $2\delta$ is the exceptional divisor in $X^{[2]}$ coming from the blow-up of the diagonal in the quotient $(X \times X)/S_2$. We will denote the line bundle corresponding to $\delta$ by $\mathcal{L}_\delta$. We have the relations $\pi^\star\delta = D$ and $\pi_\star D = 2\delta$.
Let us finish this chapter by determinig the canonical line bundles of $D$ and $\widetilde{X \times X}$. Therefore we will follow very closely [Ful] Section 15.4, especially Lemma 15.4. On $D = \mathbb{P}(\mathcal{T}_X)$ we have the short exact sequence: $0 \to \mathcal{O}_D(-1) \to \sigma_D^\star \mathcal{T}_X \to \mathcal{Q} \to 0$, where $\mathcal{Q}$ is the universal quotient line bundle. We find $\mathcal{Q} = \mathcal{O}_D(1) \otimes \sigma_D^\star \omega_X^\vee$:

$$0 \to \mathcal{O}_D(-1) \to \sigma_D^\star \mathcal{T}_X \to \mathcal{O}_D(1) \otimes \sigma_D^\star \omega_X^\vee \to 0. \tag{1}$$

On $\widetilde{X \times X}$ we have a short exact sequence:

$$0 \to \mathcal{T}_{\widetilde{X\times X}} \to \sigma^\star \mathcal{T}_{X\times X} \to i_\star(\mathcal{O}_D(1) \otimes \sigma_D^\star \omega_X^\vee) \to 0.$$

So we immediately see $c_1(\mathcal{T}_{\widetilde{X\times X}}) = r_1^\star c_1(\mathcal{T}_X) + r_2^\star c_1(\mathcal{T}_X) - D$ and therefore $\omega_{\widetilde{X\times X}} = \mathcal{L}_{(K_X, K_X, 1)}$.
Next, on $D$ we have the exact sequence:

$$0 \to \mathcal{T}_D \to i^\star \mathcal{T}_{\widetilde{X\times X}} \to \mathcal{O}_D(-1) \to 0.$$

So again, we derive $c_1(\mathcal{T}_D) = 2\sigma_D^\star c_1(\mathcal{T}_X) + 2\xi$, so $\omega_D \simeq \sigma_D^\star \omega_X^{\vee 2} \otimes \mathcal{O}_D(-2)$.

# 2 Tautological bundles

Now let $\mathcal{F}$ be a vector bundle on $X$ of rank $r$ with first Chern class $f$. Recall that in $X^{[2]} \times X$ there is the *universal cycle* $\Xi$ consisting of pairs $(\xi, x)$ such that $x \in \xi$. We define the *tautological bundle associated with* $\mathcal{F}$ to be the image of $\mathcal{F}$ under the Fourier−Mukai transform with the structure sheaf of the universal cycle as kernel:

$$\mathcal{F}^{[2]} := Rp_\star(q^\star \mathcal{F} \otimes \mathcal{O}_\Xi).$$

Since we are only considering the case of the second Hilbert scheme we can simplify this definition. Indeed, the universal cycle $\Xi$ is naturally isomorphic to the blow-up $\widetilde{X \times X}$ of $X \times X$ at the diagonal. And $p$ restricted to $\Xi$ is nothing but the two-to-one cover $\pi$. Thus we end up with the much simpler formula:

$$\mathcal{F}^{[2]} = \pi_\star r_1^\star \mathcal{F}.$$

*Remark:* We see immediately that this process is, in fact, an exact functor and we do not need to derive the pushforward along the finite morphism $\pi$.

Of course, we could also have chosen the second projection to form this vector bundle, but the image under $\pi$ is the same.

Now $\mathcal{F}^{[2]}$ is, of course, a vector bundle on $X^{[2]}$ of rank $2r$ and we have the following formula for its dual:

**Lemma 2.1** *Let $\mathcal{F}$ be a vector bundle on $X$. Then*

$$\mathcal{F}^{[2]\vee} \simeq \mathcal{F}^{\vee[2]} \otimes \mathcal{L}_\delta. \tag{2}$$

*Proof:* Using Grothendieck–Verdier duality we have

$$\begin{array}{rcl} \mathcal{F}^{[2]\vee} & = & \mathcal{H}om_{\mathcal{O}_{X^{[2]}}}(\pi_\star r_1^\star \mathcal{F}, \mathcal{O}_{X^{[2]}}) \quad \simeq \quad \pi_\star \mathcal{H}om_{\mathcal{O}_{\widetilde{X\times X}}}(r_1^\star\mathcal{F}, \mathcal{L}_{(K_X,K_X,1)} \otimes \pi^\star \omega_{X^{[2]}}^\vee) \\ & \simeq & \pi_\star \mathcal{H}om_{\mathcal{O}_{\widetilde{X\times X}}}(r_1^\star \mathcal{F}, \mathcal{L}_{(0,0,1)}) \quad \simeq \quad \pi_\star(r_1^\star \mathcal{F}^\vee \otimes \pi^\star \mathcal{L}_\delta) \\ & \simeq & \mathcal{F}^{\vee[2]} \otimes \mathcal{L}_\delta. \end{array}$$

Note that we have used here the identity $\omega_{X^{[2]}} \simeq \omega_X$. □

Schlickewei showed that the pullback $\pi^\star \mathcal{F}^{[2]}$ fits into a basic exact sequence as follows:

$$0 \to \pi^\star \mathcal{F}^{[2]} \to r_1^\star \mathcal{F} \oplus r_2^\star \mathcal{F} \to i_\star \sigma_D^\star \mathcal{F} \to 0. \tag{3}$$

Note that the exactness of this sequence is a special case of a more general result due to Scala (cf. [Sca]). From sequence (3) we find a simple formula for the first Chern class of $\mathcal{F}^{[2]}$ :

**Lemma 2.2** *We have*

$$c_1(\mathcal{F}^{[2]}) = r_1^\star f + r_2^\star f - rD. \tag{4}$$

*Proof:* It is certainly enough to show that for every sheaf $\mathcal{E}$ on $D$ of rank $r$ we have $c_1(i_\star \mathcal{E}) = rD$. But this follows easily from the Grothendieck–Riemann–Roch theorem:

$$\mathrm{ch}(i_\star \mathcal{E}) = i_\star(\mathrm{ch}(\mathcal{E})\mathrm{td}_i) = i_\star((r + \dots)(1 + \dots)) = i_\star(r + \dots) = rD + \dots.$$

□

In the sequel we will analyze conditions such that $\mathcal{F}^{[2]}$ is stable. As a main ingredient for this to be possible, we will from now on assume that we are given a polarization $H$ of $X$ and that $\mathcal{F}$ is $\mu_H$-stable. More precisely for every subsheaf $\mathcal{E} \subseteq \mathcal{F}$ of rank $0 < r_\mathcal{E} < r$ we have

$$\frac{c_1(\mathcal{E}) \cdot H}{r_\mathcal{E}} < \frac{c_1(\mathcal{F}) \cdot H}{r}.$$

Next we have to fix a polarization on $X^{[2]}$. This is done as follows: For $N \in \mathbb{N}$ we define $H_N := NH - \delta$. This divisor is ample for all sufficiently large $N$, say $N \geq N_0$.
Now let us assume that there is a destabilizing subsheaf $\mathcal{E}' \subseteq \mathcal{F}^{[2]}$. Pulling back both sheaves via $\pi$ we get an inclusion of sheaves on $\widetilde{X \times X}$:

$$\pi^\star \mathcal{E}' =: \mathcal{E} \subseteq \pi^\star \mathcal{F}^{[2]}.$$

Since the slope of a vector bundle is just multiplied by two under the finite pullback $\pi^\star$, $\mathcal{E}$ is also a destabilizing subsheaf of $\pi^\star\mathcal{F}^{[2]}$ with respect to the polarization $\widetilde{H_N} = \pi^\star H_N$ of $\widetilde{X \times X}$. Therefore we will, in fact, consider destabilizing subbundles of $\pi^\star\mathcal{F}^{[2]}$ which come from $X^{[2]}$.
As a first step towards any considerations about the stability of a vector bundle $\pi^\star\mathcal{F}^{[2]}$, we first have to calculate the slope of a sheaf $\mathcal{E}$ with respect to the given polarization. It is defined as

$$\mu_{\widetilde{H_N}}(\mathcal{E}) := \frac{c_1(\mathcal{E})\widetilde{H_N}^3}{r_{\mathcal{E}}},$$

considered as a number by integrating against the fundamental class of $\widetilde{X \times X}$. Thus we first calculate $\widetilde{H_N}^3$ :

$$\begin{aligned}
\widetilde{H_N}^3 &= (NH \otimes 1 + 1 \otimes NH - D)^3 \\
&= (NH \otimes 1 + 1 \otimes NH)^3 - 3(NH \otimes 1 + 1 \otimes NH)^2 D \\
&\quad +3(NH \otimes 1 + 1 \otimes NH)D^2 - D^3 \\
&= 3N^3(H^2 \otimes H + H \otimes H^2) - 3N^2(H^2 \otimes 1 + 2H \otimes H + 1 \otimes H^2)D \\
&\quad -3N\sigma^\star\Delta(H \otimes 1 + 1 \otimes H) + D\sigma^\star\Delta.
\end{aligned}$$

Now let $\mathcal{E}$ be a sheaf on $\widetilde{X \times X}$. We write its first chern class as $c_1(\mathcal{E}) = g\otimes 1 + 1\otimes h + aD$, with $g, h \in \mathrm{Pic}X$ and $a \in \mathbb{Z}$.

**Lemma 2.3** *We have the following formulas for the slopes of $\mathcal{E}$ and $\mathcal{F}^{[2]}$:*

$$\mu_{\widetilde{H_N}}(\mathcal{E}) = \frac{1}{r_{\mathcal{E}}}\left\{3H^2(H.(g+h))N^3 + 12aH^2N^2 - 6(H.(g+h))N - c_2(\mathcal{T}_X)a\right\}, \quad (5)$$

$$\mu_{\widetilde{H_N}}(\pi^\star\mathcal{F}^{[2]}) = \frac{3H^2(H.f)}{r}N^3 - 6H^2N^2 - \frac{6(H.f)}{r}N + \frac{c_2(\mathcal{T}_X)}{2}. \quad (6)$$

*Proof* : At first note, that formula (6) is just the special case of setting $g = h = f$, $a = -r$ and $r_{\mathcal{H}'} = 2r$ in formula (5). Next from Lemma 1.1 c) we deduce that $\sigma^\star(H^i(X\times X))\cdot i_\star\sigma_D^\star(H^j(X)) = 0$ for $i+j > 4$. Thus half of the terms in our computation vanish and we are left with

$$\begin{aligned}
\widetilde{H_N}^3 c_1(\mathcal{E}) &= 3N^3(H^2 \otimes H + H \otimes H^2)(g \otimes 1 + 1 \otimes h) \\
&\quad -3N^2(H^2 \otimes 1 + 2H \otimes H + 1 \otimes H^2)D \cdot aD \\
&\quad -3N(\sigma^\star\Delta)(H \otimes 1 + 1 \otimes H)(g \otimes 1 + 1 \otimes h) + D(\sigma^\star\Delta) \cdot aD,
\end{aligned}$$

from which we easily get (5). □

# 3 Destabilizing line subbundles of tautological bundles

In this section we will show that for $N \geq N_0$ there exist no $H_N$ destabilizing line subbundles $\mathcal{L}' \subseteq \mathcal{F}^{[2]}$ in the case $\mathcal{F} \not\cong \mathcal{O}_X$. So assume that $\mathcal{L}'$ was such a destabilizing line subbundle. The pullback $\mathcal{L} = \pi^\star \mathcal{L}'$ of such a line bundle is a destabilizing line subbundle of the pullback $\pi^\star \mathcal{F}^{[2]}$ with respect to $\widetilde{H_N}$. Composing this inclusion with the one from the basic exact sequence (3) we find $\mathcal{L} \subseteq r_1^\star\mathcal{F} \oplus r_2^\star\mathcal{F}$. We will proceed by showing that $\mathrm{Hom}_{\widetilde{X\times X}}(\mathcal{L}, r_{1/2}^\star\mathcal{F}) = 0$. The situation is completely symmetric, thus we will focus on $\mathrm{Hom}_{\widetilde{X\times X}}(\mathcal{L}, r_1^\star\mathcal{F})$. We write the first Chern class of $\mathcal{L}$ as $c_1(\mathcal{L}) = g\otimes 1 + 1\otimes h + aD$ with $g = c_1(\mathcal{G})$ and $h = c_1(\mathcal{H})$ for some line bundles $\mathcal{G}$ and $\mathcal{H}$ on $X$. In fact, since $\mathcal{L}$ is coming from $X^{[2]}$ this class is invariant under the $S_2$-action, that is, $g = h$. But for later use we will proceed in this generality and denote the line bundle class with first Chern class equal to $g\otimes 1 + 1\otimes h + aD$ simply by $\mathcal{L}_{(g,h,a)}$. We have the following central result:

**Proposition 3.1** *For all $g, h \in \mathrm{Pic}X$ and $a \in \mathbb{Z}$ we have*

$$\mathrm{Hom}_{\widetilde{X\times X}}(\mathcal{L}_{(g,h,a)}, r_1^\star\mathcal{F}) \subseteq \mathrm{Hom}_X(\mathcal{G},\mathcal{F})^{h^2(X,\mathcal{H}\otimes\omega_X)}. \tag{7}$$

*Proof:* Consider the defining exact sequence of the structure sheaf of the exceptional divisor $D$:

$$0 \to \mathcal{L}_{(0,0,-1)} \to \mathcal{O}_{\widetilde{X\times X}} \to \mathcal{O}_D \to 0.$$

Tensoring this sequence with $\mathcal{L}_{(0,0,a)}$ we have

$$0 \to \mathcal{L}_{(0,0,a-1)} \to \mathcal{L}_{(0,0,a)} \to \mathcal{O}_D(-a) \to 0. \tag{8}$$

So we see immediately that $\sigma_\star\mathcal{L}_{(0,0,a)}$ is contained in $\mathcal{O}_{X\times X}$ for all $a \in \mathbb{Z}$. Thus we find that $r_{1\star}(r_2^\star\mathcal{H}^\vee \otimes \mathcal{L}_{(0,0,-a)}) \simeq \pi_{1\star}(\pi_2^\star\mathcal{H}^\vee \otimes \sigma_\star\mathcal{L}_{(0,0,-a)})$ is a subsheaf of $\pi_{1\star}\pi_2^\star\mathcal{H}^\vee \simeq H^0(\mathcal{H}^\vee)\otimes\mathcal{O}_X \simeq \mathcal{O}_X^{h^2(X,\mathcal{H}\otimes\omega_X)}$. Now using the projection formula and adjunction we get:

$$\begin{aligned}
\mathrm{Hom}_{\widetilde{X\times X}}(\mathcal{L}_{(g,h,a)}, r_1^\star\mathcal{F}) &\cong \mathrm{Hom}_{\widetilde{X\times X}}(r_1^\star\mathcal{G}, r_1^\star\mathcal{F}\otimes\mathcal{L}_{(0,-h,-a)}) \cong \\
\mathrm{Hom}_X(\mathcal{G}, r_{1\star}(r_1^\star\mathcal{F}\otimes\mathcal{L}_{(0,-h,-a)})) &\cong \mathrm{Hom}_X(\mathcal{G}, \mathcal{F}\otimes r_{1\star}(r_2^\star\mathcal{H}^\vee\otimes\mathcal{L}_{(0,0,-a)})).
\end{aligned}$$

Together with the inclusion of above we are done. □

**Corollary 3.2** *Let $\mathcal{F}$ be a $\mu_H$-stable vector bundle on $X$ of rank $r$ and first Chern class $c_1(\mathcal{F}) = f$. Assume $\mathcal{F} \not\simeq \mathcal{O}_X$. Then $r_1^\star\mathcal{F}$ contains no line subbundles $\mathcal{L}_{(g,h,a)}$ satisfying:*

$$H.(g+h) \geq \frac{H.f}{r}, \tag{9}$$

*except the case $r = 1$, $h = 0$ and $g = f$.*

*Proof:* So let $\mathcal{L}_{(g,h,a)}$ be a line subbundle of $r_1^\star\mathcal{F}$. We will show that $\mathrm{Hom}_X(\mathcal{G},\mathcal{F})^{h^2(X,\mathcal{H}\otimes\omega_X)} = 0$ which yields a contradiction to Proposition 3.1.
If $H.h > 0$ we have $0 = h^0(X,\mathcal{H}^\vee) = h^2(X,\mathcal{H}\omega_X)$ and we are done.
If $H.h \leq 0$ we see

$$H.g \geq H.(g+h) \geq \frac{H.f}{r}. \tag{10}$$

So if $\mathcal{G} \not\simeq \mathcal{F}$ by the stability of $\mathcal{F}$ we have $\mathrm{Hom}_X(\mathcal{G}, \mathcal{F}) = 0$.
If $\mathcal{G} \simeq \mathcal{F}$ we must have $r = 1$ and equalities everywhere in equation (10), so $H.h = 0$. But then again $h^2(X, \mathcal{H} \otimes \omega_X) = 0$ for all such $\mathcal{H}$ but the trivial line bundle, i.e. $h = 0$.$\square$

Now that we have an explicit description when homomorphisms from a line bundle to $\pi^\star\mathcal{F}^{[2]}$ may exist, let us have a closer look at the destabilzing condition for line subbundles in $\pi^\star\mathcal{F}^{[2]}$.

**Lemma 3.3** *For sufficiently large $N$ a line subbundle $\mathcal{L}_{(g,h,a)}$ in $\pi^\star\mathcal{F}^{[2]}$ is $H_N$-destabilizing if and only if*

$$rH.(g+h) > H.f \quad \text{or} \quad rH.(g+h) = H.f \text{ and } a \geq 0.$$

*Proof:* Equation (5) computes the slope of $\mathcal{L}_{(g,h,a)}$ as

$$\mu_{\widetilde{H_N}}(\mathcal{L}_{(g,h,a)}) = 3H^2(H.(g+h))N^3 + 12aH^2N^2 - 6(H.(g+h))N - c_2(\mathcal{T}_X)a.$$

And we also derived $\mu_{\widetilde{H_N}}(\pi^\star\mathcal{F}^{[2]})$ in (6):

$$\mu_{\widetilde{H_N}}(\pi^\star\mathcal{F}^{[2]}) = \frac{3H^2(H.f)}{r}N^3 - 6H^2N^2 + \frac{6(H.f)}{r}N + \frac{c_2(\mathcal{T}_X)}{2}.$$

Thus $\mathcal{L}_{aD+e}$ is destabilizing if either $rH.(g+h) > H.f$ or $rH.(g+h) = H.f$ and $2a > -1$. Since $a \in \mathbb{Z}$ we can replace the last inequality by $a \geq 0$. $\square$

**Theorem 3.4** *Let $\mathcal{F}$ be a $\mu_H$-stable vector bundle on $X$ of rank $r$ and first Chern class $c_1(\mathcal{F}) = f$. Assume $\mathcal{F} \not\simeq \mathcal{O}_X$. Then for sufficiently large $N$ the tautological vector bundle $\mathcal{F}^{[2]}$ on $X^{[2]}$ has no $\mu_{H_N}$-destabilizing line subbundles.*

*Proof:* As explained before we consider an $S_2$-invariant destabilizing line subbundle $\mathcal{L}_{(g,g,a)}$ of $\pi^\star\mathcal{F}^{[2]}$. The destabilizing condition yields $H.g \geq \frac{H.f}{2r}$. So by Corollary 3.2 such a line subbundle cannot exist. $\square$

# 4 The cases $r = 1$ and $r = 2$

From Theorem 3.4 we immediately deduce:

**Corollary 4.1** *Let $\mathcal{F}$ be a line bundle on $X$ not isomorphic to $\mathcal{O}_X$. Then for sufficiently large $N$, $\mathcal{F}^{[2]}$ is a $\mu_{H_N}$-stable rank two vector bundle on $X^{[2]}$.*

We can generalize this result to arbitrary torsion free rank one sheaves on $X$ with nonvanishing first Chern class:

**Theorem 4.2** *Let $\mathcal{F}$ be a rank one torsion free sheaf on $X$ satisfying $c_1(\mathcal{F}) \neq 0$. Then for sufficiently large $N$, $\mathcal{F}^{[2]}$ is a $\mu_{H_N}$-stable rank two torsion free sheaf on $X^{[2]}$.*

*Proof:* Every torsion free rank one sheaf $\mathcal{F}$ on a surface can be written as $\mathcal{F} \simeq \mathcal{L} \otimes \mathcal{I}_Z$ for some line bundle $\mathcal{L}$ and an ideal sheaf $\mathcal{I}_Z$ of a zero dimensional subscheme $Z \subset X$. We thus have an injection $\mathcal{F} \subseteq \mathcal{L}$ and, of course, $c_1(\mathcal{F}) = c_1(\mathcal{L})$.
Now since $(-)^{[2]}$ is an exact functor we see (cf. Lemma 2.23 in [Sca]) that $\mathcal{F}^{[2]}$ is also torsion free. Furthermore we have an injection $\mathcal{F}^{[2]} \subseteq \mathcal{L}^{[2]}$ and $c_1(\mathcal{F}^{[2]}) = c_1(\mathcal{L}^{[2]})$. So the stability of $\mathcal{F}^{[2]}$ follows immediately. $\square$

Now we want to consider the case $r = \operatorname{rk} \mathcal{F} = 2$. We have seen before that $\mathcal{F}^{[2]}$ cannot contain destabilizing line subbundles. In this section we will prove that in most cases, in fact, $\mathcal{F}^{[2]}$ does not contain any destabilizing subsheaves.

**Theorem 4.3** *Let $\mathcal{F}$ be a rank two $\mu_H$-stable vector bundle on $X$ and assume $f = c_1(\mathcal{F}) \neq 0$. Then for sufficiently large $N$, $\mathcal{F}^{[2]}$ is a $\mu_{H_N}$-stable rank four vector bundle on $X^{[2]}$.*

*Proof:* Let $\mathcal{E}$ be the maximal destabilizing subsheaf of $\pi^\star \mathcal{F}^{[2]}$. Note that it is semistable, reflexive, saturated and $-$ since it is unique $-$ it has to be $S_2$-linearized. By Theorem 3.4, $\mathcal{E}$ cannot have rank one. So let us first consider the case that $\operatorname{rk} \mathcal{E} = 3$ and let us have a look at the corresponding short exact sequence on $\widetilde{X \times X}$:

$$0 \to \mathcal{E} \to \pi^\star \mathcal{F}^{[2]} \to \mathcal{Q} \to 0,$$

where $\mathcal{E} := \pi^\star \mathcal{E}'$ is also destabilizing and $\mathcal{Q}$ is the corresponding destabilizing quotient. Let us write $c_1(\mathcal{E}) = e \otimes 1 + 1 \otimes e + aD$. Using equation (2) we see that the dual of this sequence looks as follows:

$$0 \to \underbrace{\mathcal{H}om_{\mathcal{O}_{\widetilde{X \times X}}}(\mathcal{Q}, \mathcal{O}_{\widetilde{X \times X}})}_{=:\mathcal{Q}'} \to \pi^\star(\mathcal{F}^{\vee[2]}) \otimes \mathcal{L}_{(0,0,1)} \to \mathcal{E}^\vee \to \mathcal{E}xt^1_{\mathcal{O}_X}(\mathcal{Q}, \mathcal{O}_X) \to 0.$$

Since $\mathcal{E}$ is saturated, we may assume $\mathcal{Q}$ to be torsion free and so the support of $\mathcal{E}xt^1_{\mathcal{O}_X}(\mathcal{Q}, \mathcal{O}_X)$ has codimension at least 2, so vanishing first chern class. We compute

$$\begin{aligned} c_1(\mathcal{Q}') &= c_1(\pi^\star(\mathcal{F}^\vee)^{[2]}) + c_1(\mathcal{L}_{(0,0,1)}) \cdot \operatorname{rk}(\pi^\star(\mathcal{F}^\vee)^{[2]}) - c_1(\mathcal{E}^\vee) \\ &= (e - f) \otimes 1 + 1 \otimes (e - f) + (4 + a)D. \end{aligned}$$

Now we may assume that $\mathcal{Q}'$ is reflexive, i.e. locally free. (If necessary we replace $\mathcal{Q}'$ by its reflexive hull which still gives a subsheaf of $\pi^\star(\mathcal{F}^{\vee[2]}) \otimes \mathcal{L}_{(0,0,1)}$ with the same first chern class.) More precisely, $\mathcal{Q}' \simeq \mathcal{L}_{(e-f,e-f,4+a)}$. We have an inclusion $\mathcal{Q}' \otimes \mathcal{L}_{(0,0,-1)} \hookrightarrow \pi^\star \mathcal{F}^{\vee[2]}$. Now the destabilizing condition on $\mathcal{E}$ implies:

$$4H.e \geq 3H.f.$$

Thus $2H.(e - f) \geq -\frac{H.f}{2}$ and by Corollary 3.2 we get a contradiction.
Finally assume that the maximal destabilizing subsheaf of $\pi^\star \mathcal{F}^{[2]}$ is a rank two sheaf $\mathcal{E}$. Again its first chern class can be written as $c_1(\mathcal{E}) = e \otimes 1 + 1 \otimes e + aD$ with $e \in \operatorname{Pic} X$ and $a \in \mathbb{Z}$ and by the fundamental exact sequence (3) we get an injective homomorphism

$\mathcal{E} \hookrightarrow r_1^\star \mathcal{F} \oplus r_2^\star \mathcal{F}$. We will denote its composition with the projection onto the first factor by $\beta \colon \mathcal{E} \to r_1^\star \mathcal{F}$. Now we will distinguish three cases:
a) $\operatorname{rank} \ker \beta = 0$.
So $\ker \beta$ is a torsion subsheaf of $\mathcal{E}$, so it is trivial since $\mathcal{E}$ is torsion-free. So $\beta$ is an isomorphism away from an effective divisor $j \colon Y \hookrightarrow \widetilde{X \times X}$. Thus $\operatorname{coker} \beta$ can be written as $j_\star \mathcal{K}$ for some sheaf $\mathcal{K}$ on $Y$. Let $Y = \bigcup_i Y_i$ be the decomposition into irreducible components, then by Grothendieck−Riemann−Roch we can write its first chern class as $c_1(\operatorname{coker} \beta) = \sum_i (Y_i \cdot \operatorname{rk} \mathcal{K}_i)$, where $\mathcal{K}_i$ is the restriction of $\mathcal{K}$ to $Y_i$. On the other hand we can compute the first chern class of $\operatorname{coker} \beta$ directly:

$$c_1(\operatorname{coker} \beta) = c_1(r_1^\star \mathcal{F}) - c_1(\mathcal{E}) = f \otimes 1 - e \otimes 1 - 1 \otimes e - aD.$$

Now $Y$ is effective. So if $\operatorname{rk} \mathcal{K}_i \neq 0$ for some $i$ we must have $(f-e) \otimes 1 - 1 \otimes e$ effective, i.e. the zero set of a section $s \in H^0(r_1^\star \mathcal{L}_1 \otimes r_2^\star \mathcal{L}_2)$, where $\mathcal{L}_1$, $\mathcal{L}_2$ are the according line bundles on $X$. By the Künneth formula we have $H^0(r_1^\star \mathcal{L}_1 \otimes r_2^\star \mathcal{L}_2) \cong H^0(X, \mathcal{L}_1) \otimes H^0(X, \mathcal{L}_2)$. So both factors must not vanish and we see that $f - e$ as well as $-e$ have to be effective on $X$. In particular, $H.e < 0$ and $H.e < H.f$. Together with the destabilizing condition on $\mathcal{E}$ − which implies $2.He \geq H.f$ − we get a contradiction. If $\operatorname{rk} \mathcal{K}_i = 0 \ \forall i$, i.e. $c_1(\operatorname{coker} \beta) = 0$ we must have $f = 0$ which we excluded.
b) $\operatorname{rank} \ker \beta = 2$.
This says that on an open subset $\beta$ has to vanish which by symmetry contradicts the fact that $\mathcal{E}$ injects into $r_1^\star \mathcal{F} \oplus r_2^\star \mathcal{F}$.
c) $\operatorname{rank} \ker \beta = 1$.
Now $\operatorname{im}\beta$ is a rank one quotient sheaf of $\mathcal{E}$ and we write its first Chern class $c_1(\operatorname{im}\beta) = g \otimes 1 + 1 \otimes h + bD$. The stability of $\mathcal{E}$ yields

$$H.e \leq H.(g + h).$$

At the same time $\operatorname{im}\beta$ is a rank one subsheaf of $r_1^\star \mathcal{F}$. Denote by $\operatorname{im}\beta^{\vee\vee}$ its reflexive hull. This is a reflexive rank one sheaf, thus a line bundle. And it has the same first chern class as $\operatorname{im}\beta$, so $\operatorname{im}\beta^{\vee\vee} = \mathcal{L}_{(g,h,b)}$. The destabilizing condition on $\mathcal{E}$ implies $2H.e \geq H.f$. Putting things together we find a line subbundle $\mathcal{L}_{(g,h,b)}$ in $r_1^\star \mathcal{F}$ satisfying $2H.g + 2H.h \geq H.f$. This is a contradiction to Corollary 3.2. $\square$

# 5 The case of the trivial line bundle

In the previous section we explicitly excluded the case $\mathcal{F} \simeq \mathcal{O}_X$. We did this not only because the above proof does not work, but because the corresponding result is false. The following example shows, that indeed the tautological vector bundle associated with the structure sheaf is not stable:

**Example 5.1** Consider the defining exact sequence of the structure sheaf of the universal cycle $\Xi$ in $X^{[2]} \times X$:

$$0 \to I_\Xi \to \mathcal{O}_{X^{[2]} \times X} \to \mathcal{O}_\Xi \to 0.$$

We will now analyze the derived pushforward of the upper sequence via the projection $p$, whose fibres are isomorphic to $X$. Since $X$ is a projective surface, we find $R^0 p_\star \mathcal{O}_{X^{[2]}\times X} = \mathcal{O}_{X^{[2]}}$. Furthermore by definition $R^0 p_\star \mathcal{O}_\Xi = \mathcal{O}_X^{[2]}$. Looking a bit closer at $I_\Xi$ we see that the fibre of $R^0 p_\star I_\Xi$ at a point $\xi \in X^{[2]}$ is isomorphic to $H^0(X, I_\xi)$, which is definitely trivial. Alltogether we see that the pushforward of the upper sequence yields an inclusion

$$0 \to \mathcal{O}_{X^{[2]}} \to \mathcal{O}_X^{[2]}.$$

Now we claim that $\mathcal{O}_{X^{[2]}}$ is a destabilizing line subbundle of $\mathcal{O}_X^{[2]}$ with respect to the polarizations of $X^{[2]}$ defined above. To see this, let us calculate the slopes of the bundles of interest. We will again consider the pullbacks under $\pi$. We have:

$$\begin{array}{rcl} \mu_{\widetilde{H_N}}(\pi^\star \mathcal{O}_{X^{[2]}}) & = & 0 \quad \text{and} \\ \mu_{\widetilde{H_N}}(\pi^\star \mathcal{O}_X^{[2]}) & = & -\frac{1}{2} D \widetilde{H_N}^3 \\ & = & -\frac{3}{2} N^2 \sigma^\star \Delta (H^2 \otimes 1 + 2H \otimes H + 1 \otimes H^2) + \frac{1}{2}(\sigma^\star \Delta)^2 \\ & = & -6N^2 H^2 + \frac{1}{2}(\sigma^\star \Delta)^2 \leq 0. \end{array}$$

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