TORELLI THEOREM FOR THE DELIGNE–HITCHIN MODULI SPACE, II

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Abstract. Let X and X' be compact Riemann surfaces of genus at least three. Let G and G' be nontrivial connected semisimple linear algebraic groups over $\mathbb{C}$. If some components $\mathcal{M}^d_{\rm DH}(X,G)$ and $\mathcal{M}^{d'}_{\rm DH}(X',G')$ of the associated Deligne–Hitchin moduli spaces are biholomorphic, then X' is isomorphic to X or to the conjugate Riemann surface $\overline{X}$.

1. Introduction

Let X be a compact connected Riemann surface of genus $g \geq 3$. Let $\overline{X}$ denote the conjugate Riemann surface; by definition, it consists of the real manifold underlying X and the almost complex structure $J_{\overline{X}} := -J_X$. Let G be a nontrivial connected semisimple linear algebraic group over $\mathbb{C}$. The topological types of holomorphic principal G–bundles E over X correspond to elements of $\pi_1(G)$. Let $\mathcal{M}^d_{\rm Higgs}(X,G)$ denote the moduli space of semistable Higgs G–bundles (E,θ) over X with E of topological type $d \in \pi_1(G)$.

The *Deligne–Hitchin moduli space* [Si3] is a complex analytic space $\mathcal{M}^d_{\rm DH}(X,G)$ associated to X, G and d. It is the twistor space for the hyper-Kähler structure on $\mathcal{M}^d_{\rm Higgs}(X,G)$; see [Hi2, §9]. Deligne [De] has constructed it together with a surjective holomorphic map

$$\mathcal{M}^d_{\rm DH}(X,G) \twoheadrightarrow \mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}.$$

The inverse image of $\mathbb{C} \subseteq \mathbb{CP}^1$ is the moduli space $\mathcal{M}^d_{\rm Hod}(X,G)$ of holomorphic principal G–bundles over X endowed with a λ–connection. In particular, every fiber over $\mathbb{C}^* \subset \mathbb{CP}^1$ is isomorphic to the moduli space of holomorphic G–connections over X. The fiber over $0 \in \mathbb{CP}^1$ is $\mathcal{M}^d_{\rm Higgs}(X,G)$, and the fiber over $\infty \in \mathbb{CP}^1$ is $\mathcal{M}^{-d}_{\rm Higgs}(\overline{X},G)$.

In this paper, we study the dependence of these moduli spaces on X. Our main result, Theorem 5.3, states that the complex analytic space $\mathcal{M}^d_{\rm DH}(X,G)$ determines the unordered pair $\{X,\overline{X}\}$ up to isomorphism. We also prove that $\mathcal{M}^d_{\rm Higgs}(X,G)$ and $\mathcal{M}^d_{\rm Hod}(X,G)$ each determine X up to isomorphism; see Theorem 5.1 and Theorem 5.2.

The key technical result is Proposition 3.1, which says the following: Let Z be an irreducible component of the fixed point locus for the natural $\mathbb{C}^*$–action on a moduli space $\mathcal{M}^d_{\rm Higgs}(X,G)$ of Higgs G–bundles. Then,

$$\dim Z \leq (g-1)\cdot \dim_{\mathbb{C}} G,$$

with equality holding only for $Z = \mathcal{M}^d(X,G)$.

In [BGHL], the case of $G = {\rm SL}(r,\mathbb{C})$ was considered.

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2. Some moduli spaces associated to a compact Riemann surface

Let X be a compact connected Riemann surface of genus $g \geq 3$. Let G be a nontrivial connected semisimple linear algebraic group defined over $\mathbb{C}$, with Lie algebra $\mathfrak{g}$.

2.1. **Principal G–bundles.** We consider holomorphic principal G–bundles E over X. Recall that the topological type of E is given by an element $d \in \pi_1(G)$ [Ra]; this is a finite abelian group. The *adjoint vector bundle* of E is the holomorphic vector bundle

$$\mathrm{ad}(E) := E \times^G \mathfrak{g}$$

over X, using the adjoint action of G on $\mathfrak{g}$. E is called *stable* (respectively, *semistable*) if

$$\text{degree}(\mathrm{ad}(E_P)) < 0 \quad (\text{respectively, } \leq 0) \tag{1}$$

for every maximal parabolic subgroup $P \subsetneq G$ and every holomorphic reduction of structure group E_P of E to P; here $\mathrm{ad}(E_P) \subset \mathrm{ad}(E)$ is the adjoint vector bundle of E_P.

Let $\mathcal{M}^d(X, G)$ denote the moduli space of semistable holomorphic principal G–bundles E over X of topological type $d \in \pi_1(G)$. It is known that $\mathcal{M}^d(X, G)$ is an irreducible normal projective variety of dimension $(g-1) \cdot \dim_{\mathbb{C}} G$ over $\mathbb{C}$.

2.2. **Higgs G–bundles.** The holomorphic cotangent bundle of X will be denoted by K_X.

A *Higgs G–bundle* over X is a pair (E, θ) consisting of a holomorphic principal G–bundle E over X and a holomorphic section

$$\theta \in \mathrm{H}^0(X, \mathrm{ad}(E) \otimes K_X),$$

the so-called *Higgs field* [Hi1, Si1]. The pair (E, θ) is called *stable* (respectively, *semistable*) if the inequality (1) holds for every holomorphic reduction of structure group E_P of E to a maximal parabolic subgroup $P \subsetneq G$ such that $\theta \in \mathrm{H}^0(X, \mathrm{ad}(E_P) \otimes K_X)$.

Let $\mathcal{M}^d_{\mathrm{Higgs}}(X, G)$ denote the moduli space of semistable Higgs G–bundles (E, θ) over X such that E is of topological type $d \in \pi_1(G)$. It is known that $\mathcal{M}^d_{\mathrm{Higgs}}(X, G)$ is an irreducible normal quasiprojective variety of dimension $2(g-1) \cdot \dim_{\mathbb{C}} G$ over $\mathbb{C}$ [Si2]. We regard $\mathcal{M}^d(X, G)$ as a closed subvariety of $\mathcal{M}^d_{\mathrm{Higgs}}(X, G)$ by means of the embedding

$$\mathcal{M}^d(X, G) \hookrightarrow \mathcal{M}^d_{\mathrm{Higgs}}(X, G), \qquad E \longmapsto (E, 0).$$

There is a natural algebraic symplectic structure on $\mathcal{M}^d_{\mathrm{Higgs}}(X, G)$; see [Hi1, BR].

2.3. **Representations of the surface group in G.** Fix a base point $x_0 \in X$. The fundamental group of X admits a standard presentation

$$\pi_1(X, x_0) \cong \langle a_1, \dots, a_g, b_1, \dots, b_g | \prod_{i=1}^{g} a_i b_i a_i^{-1} b_i^{-1} = 1 \rangle$$

which we choose in such a way that it is compatible with the orientation of X. We identify the fundamental group of G with the kernel of the universal covering $\tilde{G} \twoheadrightarrow G$. The *type* $d \in \pi_1(G)$ of a homomorphism $\rho : \pi_1(X, x_0) \longrightarrow G$ is defined by

$$d := \prod_{i=1}^{g} \alpha_i \beta_i \alpha_i^{-1} \beta_i^{-1} \in \pi_1(G) \subset \tilde{G}$$

for any choice of lifts $\alpha_i, \beta_i \in \tilde{G}$ of $\rho(a_i), \rho(b_i) \in G$. This is also the topological type of the principal G–bundle E_ρ over X given by ρ. The space $\mathrm{Hom}^d(\pi_1(X, x_0), G)$ of all homomorphisms $\rho : \pi_1(X, x_0) \longrightarrow G$ of type $d \in \pi_1(G)$ is an irreducible affine variety over $\mathbb{C}$, and G acts on it by conjugation. The GIT quotient

$$\mathcal{M}^d_{\mathrm{Rep}}(X, G) := \mathrm{Hom}^d(\pi_1(X, x_0), G)/\!\!/G$$

doesn't depend on x_0. It is an affine variety of dimension $2(g-1) \cdot \dim_{\mathbb{C}} G$ over $\mathbb{C}$, which carries a natural symplectic form [AB, Go]. Its points represent equivalence classes of completely reducible homomorphisms ρ. There is a natural bijective map

$$\mathcal{M}^d_{\mathrm{Rep}}(X, G) \longrightarrow \mathcal{M}^d_{\mathrm{Higgs}}(X, G)$$

given by a variant of the Kobayashi–Hitchin correspondence [Si1]. This bijective map is not holomorphic.

2.4. **Holomorphic G–connections.** Let $p : E \longrightarrow X$ be a holomorphic principal G–bundle. Because the vertical tangent space at every point of the total space E is canonically isomorphic to $\mathfrak{g}$, there is a natural exact sequence

$$0 \longrightarrow E \times \mathfrak{g} \longrightarrow TE \xrightarrow{dp} p^*TX \longrightarrow 0$$

of G-equivariant holomorphic vector bundles over E. Taking the G-invariant direct image under p, it follows that the *Atiyah bundle* for E

$$\mathrm{At}(E) := p_*(TE)^G \subset p_*(TE)$$

sits in a natural exact sequence of holomorphic vector bundles

$$0 \longrightarrow \mathrm{ad}(E) \longrightarrow \mathrm{At}(E) \xrightarrow{dp} TX \longrightarrow 0 \qquad (2)$$

over X. This exact sequence is called the *Atiyah sequence*. A *holomorphic connection* on E is a splitting of the Atiyah sequence, or in other words a holomorphic homomorphism

$$D : TX \longrightarrow \mathrm{At}(E)$$

such that $dp \circ D = \mathrm{id}_{TX}$. It always exists if E is semistable [At, AzBi]. The curvature of D is a holomorphic 2–form with values in $\mathrm{ad}(E)$, so D is automatically flat.

A *holomorphic G–connection* is a pair (E, D) where E is a holomorphic principal G–bundle over X, and D is a holomorphic connection on E. Such a pair is automatically semistable, because the degree of a flat vector bundle is zero.

Let $\mathcal{M}^d_{\mathrm{conn}}(X, G)$ denote the moduli space of holomorphic G–connections (E, D) over X such that E is of topological type $d \in \pi_1(G)$. It is known that $\mathcal{M}^d_{\mathrm{conn}}(X, G)$ is an irreducible quasiprojective variety of dimension $2(g-1) \cdot \dim_{\mathbb{C}} G$ over $\mathbb{C}$.

Sending each holomorphic G–connection to its monodromy defines a map

$$\mathcal{M}^d_{\mathrm{conn}}(X, G) \longrightarrow \mathcal{M}^d_{\mathrm{Rep}}(X, G) \qquad (3)$$

which is biholomorphic, but not algebraic; it is called Riemann–Hilbert correspondence. The inverse map sends a homomorphism $\rho : \pi_1(X, x_0) \longrightarrow G$ to the associated principal G–bundle E_ρ, endowed with the induced holomorphic connection D_ρ.

2.5. **λ–connections.** Let $p : E \longrightarrow X$ be a holomorphic principal G–bundle. For any $\lambda \in \mathbb{C}$, a λ*–connection* on E is a holomorphic homomorphism of vector bundles

$$D : TX \longrightarrow \mathrm{At}(E)$$

such that $dp \circ D = \lambda \cdot \mathrm{id}_{TX}$ for the epimorphism dp in the Atiyah sequence (2). Therefore, a 0–connection is a Higgs field, and a 1–connection is a holomorphic connection.

If D is a λ–connection on E with $\lambda \neq 0$, then $\lambda^{-1}D$ is a holomorphic connection on E. In particular, the pair (E, D) is automatically semistable in this case.

Let $\mathcal{M}^d_{\mathrm{Hod}}(X, G)$ denote the moduli space of triples (λ, E, D), where $\lambda \in \mathbb{C}$, E is a holomorphic principal G–bundle over X of topological type $d \in \pi_1(G)$, and D is a semistable λ–connection on E; see [Si2]. There is a canonical algebraic map

$$\mathrm{pr} = \mathrm{pr}_X : \mathcal{M}^d_{\mathrm{Hod}}(X, G) \longrightarrow\!\!\!\!\!\rightarrow \mathbb{C}, \qquad (\lambda, E, D) \longmapsto \lambda. \tag{4}$$

Its fibers over $\lambda = 0$ and $\lambda = 1$ are $\mathcal{M}^d_{\mathrm{Higgs}}(X, G)$ and $\mathcal{M}^d_{\mathrm{conn}}(X, G)$, respectively. The Riemann–Hilbert correspondence (3) allows to define a holomorphic open embedding

$$j = j_X : \mathbb{C}^* \times \mathcal{M}^d_{\mathrm{Rep}}(X, G) \hookrightarrow \mathcal{M}^d_{\mathrm{Hod}}(X, G), \qquad (\lambda, \rho) \longmapsto (\lambda, E_\rho, \lambda D_\rho)$$

with image $\mathrm{pr}^{-1}(\mathbb{C}^*)$. This map commutes with the projections onto $\mathbb{C}^*$.

2.6. **The Deligne–Hitchin moduli space.** The compact Riemann surface X provides an underlying real C^∞ manifold $X_\mathbb{R}$, and an almost complex structure $J_X : TX_\mathbb{R} \longrightarrow TX_\mathbb{R}$. Since any almost complex structure in real dimension two is integrable,

$$\overline{X} := (X_\mathbb{R}, -J_X)$$

is a compact Riemann surface as well. It has the opposite orientation, so

$$\mathcal{M}^d_{\mathrm{Rep}}(X, G) = \mathcal{M}^{-d}_{\mathrm{Rep}}(\overline{X}, G). \tag{5}$$

The *Deligne–Hitchin moduli space* $\mathcal{M}^d_{\mathrm{DH}}(X, G)$ is the complex analytic space obtained by gluing $\mathcal{M}^d_{\mathrm{Hod}}(X, G)$ and $\mathcal{M}^{-d}_{\mathrm{Hod}}(\overline{X}, G)$ along their common open subspace

$$\mathcal{M}^d_{\mathrm{Hod}}(X, G) \xleftarrow{j_X} \mathbb{C}^* \times \mathcal{M}^d_{\mathrm{Rep}}(X, G) \cong \mathbb{C}^* \times \mathcal{M}^{-d}_{\mathrm{Rep}}(\overline{X}, G) \xrightarrow{j_{\overline{X}}} \mathcal{M}^{-d}_{\mathrm{Hod}}(\overline{X}, G)$$

where the isomorphism in the middle sends (λ, ρ) to $(1/\lambda, \rho)$; see [Si3, De]. The projections pr_X on $\mathcal{M}^d_{\mathrm{Hod}}(X, G)$ and $1/\mathrm{pr}_{\overline{X}}$ on $\mathcal{M}^{-d}_{\mathrm{Hod}}(\overline{X}, G)$ patch together to a holomorphic map

$$\mathcal{M}^d_{\mathrm{DH}}(X, G) \longrightarrow\!\!\!\!\!\rightarrow \mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}.$$

Its fiber over any $\lambda \in \mathbb{C}^*$ is biholomorphic to the representation space (5), whereas its fibers over $\lambda = 0$ and $\lambda = \infty$ are $\mathcal{M}^d_{\mathrm{Higgs}}(X, G)$ and $\mathcal{M}^{-d}_{\mathrm{Higgs}}(\overline{X}, G)$, respectively.

3. Fixed points of the natural $\mathbb{C}^*$–action

The group $\mathbb{C}^*$ acts algebraically on the moduli space $\mathcal{M}^d_{\mathrm{Higgs}}(X, G)$, via the formula

$$t \cdot (E, \theta) := (E, t\theta). \tag{6}$$

The fixed point locus $\mathcal{M}^d_{\mathrm{Higgs}}(X, G)^{\mathbb{C}^*}$ contains the closed subvariety $\mathcal{M}^d(X, G)$.

Proposition 3.1. *Let Z be an irreducible component of $\mathcal{M}^d_{\mathrm{Higgs}}(X, G)^{\mathbb{C}^*}$. Then one has*

$$\dim Z \leq (g - 1) \cdot \dim_\mathbb{C} G,$$

with equality holding only for $Z = \mathcal{M}^d(X, G)$.

Proof. Let (E, θ) be a stable Higgs G–bundle over X. Its infinitesimal deformations are, according to [BR, Theorem 2.3], governed by the complex of vector bundles

$$C^0 := \mathrm{ad}(E) \xrightarrow{\mathrm{ad}(\theta)} \mathrm{ad}(E) \otimes K_X =: C^1 \tag{7}$$

over X. Since (E, θ) is stable, it has no infinitesimal automorphisms, so

$$\mathbb{H}^0(X, C^\bullet) = 0 .$$

The Killing form on $\mathfrak{g}$ induces isomorphisms $\mathfrak{g}^* \cong \mathfrak{g}$ and $\mathrm{ad}(E)^* \cong \mathrm{ad}(E)$. Hence the vector bundle $\mathrm{ad}(E)$ has degree 0. Serre duality allows us to conclude

$$\mathbb{H}^2(X, C^\bullet) = 0 .$$

Using all this, the Riemann–Roch formula yields

$$\dim \mathbb{H}^1(X, C^\bullet) = 2(g-1) \cdot \dim_{\mathbb{C}} G . \tag{8}$$

From now on, we assume that the point (E, θ) is fixed by $\mathbb{C}^*$, and we also assume $\theta \neq 0$. Then $(E, \theta) \cong (E, t\theta)$ for all $t \in \mathbb{C}^*$, so the sequence of complex algebraic groups

$$1 \longrightarrow \mathrm{Aut}(E, \theta) \longrightarrow \mathrm{Aut}(E, \mathbb{C}\theta) \longrightarrow \mathrm{Aut}(\mathbb{C}\theta) = \mathbb{C}^* \longrightarrow 1$$

is exact. Because (E, θ) is stable, $\mathrm{Aut}(E, \theta)$ is finite. Consequently, the identity component of $\mathrm{Aut}(E, \mathbb{C}\theta)$ is isomorphic to $\mathbb{C}^*$. This provides an embedding

$$\mathbb{C}^* \hookrightarrow \mathrm{Aut}(E) , \qquad t \longmapsto \varphi_t ,$$

and an integer $w \neq 0$ with $\varphi_t(\theta) = t^w \cdot \theta$ for all $t \in \mathbb{C}^*$. We may assume that $w \geq 1$.

Choose a point $e_0 \in E$. Then there is a unique group homomorphism

$$\iota : \mathbb{C}^* \longrightarrow G$$

such that $\varphi_t(e_0) = e_0 \cdot \iota(t)$ for all $t \in \mathbb{C}^*$. The conjugacy class of ι doesn't depend on e_0, since the space of conjugacy classes $\mathrm{Hom}(\mathbb{C}^*, G)/G$ is discrete. The subset

$$E_H := \{e \in E : \varphi_t(e) = e \cdot \iota(t) \text{ for all } t \in \mathbb{C}^*\}$$

of E is a holomorphic reduction of structure group to the centralizer H of $\iota(\mathbb{C}^*)$ in G. Let

$$\mathfrak{g} = \bigoplus_{n \in \mathbb{Z}} \mathfrak{g}_n \tag{9}$$

denote the eigenspace decomposition given by the adjoint action of $\mathbb{C}^*$ on $\mathfrak{g}$ via ι.

Let $N \in \mathbb{Z}$ be maximal with $\mathfrak{g}_N \neq 0$. Let $P \subset G$ be the parabolic subgroup with

$$\mathrm{Lie}(P) = \bigoplus_{n \geq 0} \mathfrak{g}_n \subset \mathfrak{g} .$$

Since $H \subset G$ has Lie algebra $\mathfrak{g}_0$, it is a Levi subgroup in P. Choose subgroups

$$\iota(\mathbb{C}^*) \subseteq T \subseteq B \subseteq P \subset G$$

such that T is a maximal torus in $H \subset G$ and B is a Borel subgroup in G. Let

$$\alpha_j : T \longrightarrow \mathbb{C}^* \qquad \text{and} \qquad \alpha_j^\vee : \mathbb{C}^* \longrightarrow T$$

be the resulting simple roots and coroots of G. We denote by $\langle \_, \_ \rangle$ the natural pairing between characters and cocharacters of T. Let α_j be a simple root of G with $\langle \alpha_j, \iota \rangle > 0$, and let β be a root of G with $\langle \beta, \iota \rangle = N$. Then the elementary reflection

$$s_j(\beta) = \beta - \langle \beta, \alpha_j^\vee \rangle \alpha_j$$

is a root of G, so $\langle s_j(\beta), \iota\rangle \leq N$; this implies that $\langle \beta, \alpha_j^\vee\rangle \geq 0$. The sum of all such roots β with $\langle \beta, \iota\rangle = N$ is the restriction $\chi|_T$ of the determinant

$$\chi : P \longrightarrow \mathrm{Aut}(\mathfrak{g}_N) \xrightarrow{\det} \mathbb{C}^* \tag{10}$$

of the adjoint action of P on $\mathfrak{g}_N$. Hence we conclude $\langle \chi|_T, \alpha_j^\vee\rangle \geq 0$ for all simple roots α_j with $\langle \alpha_j, \iota\rangle > 0$. This means that the character χ of P is dominant.

The decomposition (9) of $\mathfrak{g}$ induces a vector bundle decomposition

$$\mathrm{ad}(E) \;=\; \bigoplus_{n\in\mathbb{Z}} E_H \times^H \mathfrak{g}_n\,.$$

Since $\mathbb{C}^*$ acts with weight w on the Higgs field θ by construction, we have

$$\theta \;\in\; \mathrm{H}^0\big(X, (E_H \times^H \mathfrak{g}_w) \otimes K_X\big)\,. \tag{11}$$

In particular, $\theta \in \mathrm{H}^0(X, \mathrm{ad}(E_P) \otimes K_X)$ for the reduction $E_P := E_H \times^H P \subseteq E$ of the structure group to P. The Higgs version of the stability criterion [Ra, Lemma 2.1] yields

$$\mathrm{degree}(E_H \times^H \mathfrak{g}_N) \;\leq\; 0$$

since P acts on $\det(\mathfrak{g}_N)$ via the dominant character χ in (10). Now Riemann–Roch implies the following:

$$\dim \mathrm{H}^1(X, E_H \times^H \mathfrak{g}_N) \;\geq\; (g-1)\cdot \dim_{\mathbb{C}} \mathfrak{g}_N \;>\; 0\,. \tag{12}$$

The complex $C^\bullet$ in (7) is, due to (11), the direct sum of its subcomplexes $C_n^\bullet$ given by

$$C_n^0 \;:=\; E_H \times^H \mathfrak{g}_n \;\xrightarrow{\mathrm{ad}(\theta)}\; (E_H \times^H \mathfrak{g}_{n+w}) \otimes K_X \;=:\; C_n^1\,.$$

Thus the hypercohomology of $C^\bullet$ decomposes as well; in particular, we have

$$\mathbb{H}^1(X, C^\bullet) \;=\; \bigoplus_{n\in\mathbb{Z}} \mathbb{H}^1(X, C_n^\bullet)\,.$$

In the last nonzero summand $C_N^\bullet$, we have $C_N^1 = 0$ and hence

$$\dim \mathbb{H}^1(X, C_N^\bullet) \;=\; \dim \mathrm{H}^1(X, E_H \times^H \mathfrak{g}_N) \;>\; 0$$

due to (12). Since $\mathfrak{g}_n^* \cong \mathfrak{g}_{-n}$ via the Killing form on $\mathfrak{g}$, Serre duality yields in particular

$$\dim \mathbb{H}^1(X, C_0^\bullet) = \dim \mathbb{H}^1(X, C_{-w}^\bullet).$$

Taken together, the last three formulas and the equation (8) imply that

$$\dim \mathbb{H}^1(X, C_0^\bullet) \;<\; \frac{1}{2}\dim \mathbb{H}^1(X, C^\bullet) \;=\; (g-1)\cdot \dim_{\mathbb{C}} G\,.$$

But $\mathbb{H}^1(X, C_0^\bullet)$ parameterizes infinitesimal deformations of pairs (E_H, θ) consisting of a principal H–bundle E_H and a section θ as in (11); see [BR, Theorem 2.3]. This proves that

$$\dim Z < (g-1)\cdot \dim_{\mathbb{C}} G$$

for every irreducible component Z of the fixed point locus $\mathcal{M}^d_{\mathrm{Higgs}}(X, G)^{\mathbb{C}^*}$ such that Z contains stable Higgs G–bundles (E, θ) with $\theta \neq 0$.

The non-stable points in $\mathcal{M}^d_{\mathrm{Higgs}}(X, G)$ correspond to polystable Higgs G–bundles (E, θ). Polystability means that E admits a reduction of structure group E_L to a Levi subgroup $L \subsetneq G$ of a parabolic subgroup in G such that θ is a section of the subbundle

$$\mathrm{ad}(E_L) \otimes K_X \;\subset\; \mathrm{ad}(E) \otimes K_X$$

and the pair (E_L, θ) is stable. Let $C \subseteq L$ be the identity component of the center, and let $\mathfrak{c} \subseteq \mathfrak{l}$ be their Lie algebras. Then $E_{L/C} := E_L/C$ is a principal (L/C)–bundle over X, and

$$\mathrm{ad}(E_L) \cong (\mathfrak{c} \otimes \mathcal{O}_X) \oplus \mathrm{ad}(E_{L/C})$$

since $\mathfrak{l} = \mathfrak{c} \oplus [\mathfrak{l}, \mathfrak{l}]$, where the subalgebra $[\mathfrak{l}, \mathfrak{l}] \subseteq \mathfrak{l}$ is also the Lie algebra of L/C. We have

$$\dim_{\mathbb{C}} G - \dim_{\mathbb{C}} L \; \geq \; 2 \dim_{\mathbb{C}} C$$

because maximal Levi subgroups in G have 1-dimensional center and at least one pair of opposite roots less than G; the other Levi subgroups can be reached by iterating this.

Now suppose that $\mathbb{C}^*$ fixes the point (E, θ). Then $(E_L, \theta) \cong (E_L, t\theta)$ for all $t \in \mathbb{C}^*$. But the action of $\mathrm{Aut}(E_L)$ on the direct summand $\mathfrak{c} \otimes \mathcal{O}_X$ of $\mathrm{ad}(E_L)$ is trivial, since the adjoint action of L on $\mathfrak{c}$ is trivial. So θ lives in the other summand of $\mathrm{ad}(E_L)$, meaning

$$\theta \,\in\, \mathrm{H}^0\big(X, \mathrm{ad}(E_{L/C}) \otimes K_X\big)\,.$$

The Higgs (L/C)–bundle $(E_{L/C}, \theta)$ is still stable and fixed by $\mathbb{C}^*$; we have already proved that the locus of such has dimension $\leq (g-1) \cdot \dim_{\mathbb{C}}(L/C)$. The abelian variety $\mathcal{M}^0(X, C)$ acts simply transitively on lifts of $E_{L/C}$ to a principal L–bundle E_L, so these lifts form a family of dimension $g \cdot \dim_{\mathbb{C}} C$. Hence the pairs (E_L, θ) in question have at most

$$(g-1) \cdot \dim_{\mathbb{C}}(L/C) + g \cdot \dim_{\mathbb{C}} C < (g-1) \cdot \dim_{\mathbb{C}} G$$

moduli. This implies that $\dim Z \; < \; (g-1) \cdot \dim_{\mathbb{C}} G$ for each non-stable component Z of the fixed point locus, since there are only finitely many possibilities for L up to conjugation. □

The algebraic $\mathbb{C}^*$–action (6) on $\mathcal{M}^d_{\mathrm{Higgs}}(X, G)$ extends naturally to an algebraic $\mathbb{C}^*$–action on $\mathcal{M}^d_{\mathrm{Hod}}(X, G)$, which is given by the formula

$$t \cdot (\lambda, E, D) := (t\lambda, E, tD). \tag{13}$$

A point (λ, E, D) can only be fixed by this action if $\lambda = 0$, so Proposition 3.1 yields the following corollary:

Corollary 3.2. *Let Z be an irreducible component of $\mathcal{M}^d_{\mathrm{Hod}}(X, G)^{\mathbb{C}^*}$. Then one has*

$$\dim Z \leq (g-1) \cdot \dim_{\mathbb{C}} G,$$

with equality only for $Z = \mathcal{M}^d(X, G)$.

The algebraic $\mathbb{C}^*$–action (13) on $\mathcal{M}^d_{\mathrm{Hod}}(X, G)$ extends naturally to a holomorphic $\mathbb{C}^*$–action on $\mathcal{M}^d_{\mathrm{DH}}(X, G)$, which is on the other open patch $\mathcal{M}^{-d}_{\mathrm{Hod}}(\overline{X}, G)$ given by the formula

$$t \cdot (\lambda, E, D) := (t^{-1}\lambda, E, t^{-1}D).$$

Applying Corollary 3.2 to both $\mathcal{M}^d_{\mathrm{Hod}}(X, G)$ and $\mathcal{M}^{-d}_{\mathrm{Hod}}(\overline{X}, G)$, one immediately gets

Corollary 3.3. *Let Z be an irreducible component of $\mathcal{M}^d_{\mathrm{DH}}(X, G)^{\mathbb{C}^*}$. Then one has*

$$\dim Z \leq (g-1) \cdot \dim_{\mathbb{C}} G,$$

with equality only for $Z = \mathcal{M}^d(X, G)$ and for $Z = \mathcal{M}^{-d}(\overline{X}, G)$.

4. Vector fields on the moduli spaces

A stable principal G–bundle E over X is called *regularly stable* if the automorphism group $\mathrm{Aut}(E)$ is just the center of G. The regularly stable locus

$$\mathcal{M}^{d,\mathrm{rs}}(X,G) \subseteq \mathcal{M}^{d}(X,G)$$

is open, and coincides with the smooth locus of $\mathcal{M}^{d}(X,G)$; see [BH, Corollary 3.4].

Proposition 4.1. *There are no nonzero holomorphic vector fields on $\mathcal{M}^{d,\mathrm{rs}}(X,G)$.*

Proof. This statement is contained in [Fa, Corollary III.3]. □

Proposition 4.2. *There are no nonzero holomorphic 1–forms on $\mathcal{M}^{d,\mathrm{rs}}(X,G)$.*

Proof. The moduli space of Higgs G–bundles is equipped with the *Hitchin map*

$$\mathcal{M}^{d}_{\mathrm{Higgs}}(X,G) \longrightarrow \bigoplus_{i=1}^{\mathrm{rank}(G)} H^0(X, K_X^{\otimes n_i})$$

where the n_i are the degrees of generators for the algebra $\mathrm{Sym}(\mathfrak{g}^*)^G$; see [Hi1, § 4], [La]. Any sufficiently general fiber of this Hitchin map is a complex abelian variety A, and

$$\varphi : A \dashrightarrow \mathcal{M}^{d,\mathrm{rs}}(X,G), \qquad (E,\theta) \longmapsto E,$$

is a dominant rational map. This rational map φ is defined outside a closed subscheme of codimension at least two; see [Fa, Theorem II.6].

Let ω be a holomorphic 1–form on $\mathcal{M}^{d,\mathrm{rs}}(X,G)$. Then $\varphi^*\omega$ extends to a holomorphic 1–form on A by Hartog's theorem. As any holomorphic 1–form on A is closed, it follows that ω is closed. Since $H^1(\mathcal{M}^{d,\mathrm{rs}}(X,G),\mathbb{C}) = 0$ by [AB], we conclude $\omega = df$ for a holomorphic function f on $\mathcal{M}^{d,\mathrm{rs}}(X,G)$. But any such function f is constant, so $\omega = 0$. □

We denote by $\mathcal{M}^{d,\mathrm{rs}}_{\mathrm{Higgs}}(X,G) \subseteq \mathcal{M}^{d}_{\mathrm{Higgs}}(X,G)$ the open locus of Higgs G–bundles (E,θ) for which E is regularly stable. The forgetful map

$$\mathcal{M}^{d,\mathrm{rs}}_{\mathrm{Higgs}}(X,G) \longrightarrow \mathcal{M}^{d,\mathrm{rs}}(X,G), \qquad (E,\theta) \longmapsto E, \tag{14}$$

is an algebraic vector bundle with fibers $\mathrm{H}^0(X, \mathrm{ad}(E) \otimes K_X) \cong \mathrm{H}^1(X, \mathrm{ad}(E))^*$, so it is the cotangent bundle of $\mathcal{M}^{d,\mathrm{rs}}(X,G)$.

Corollary 4.3. *The restriction of the algebraic tangent bundle*

$$T\mathcal{M}^{d,\mathrm{rs}}_{\mathrm{Higgs}}(X,G) \longrightarrow \mathcal{M}^{d,\mathrm{rs}}_{\mathrm{Higgs}}(X,G)$$

to the subvariety $\mathcal{M}^{d,\mathrm{rs}}(X,G) \subseteq \mathcal{M}^{d,\mathrm{rs}}_{\mathrm{Higgs}}(X,G)$ has no nonzero holomorphic sections.

Proof. The subvariety in question is the zero section of the vector bundle (14). Given a vector bundle $V \longrightarrow M$ with zero section $M \subseteq V$, there is a natural isomorphism

$$(TV)|_M \cong TM \oplus V \tag{15}$$

of vector bundles over M. In our situation, both summands have no nonzero holomorphic sections, according to Proposition 4.1 and Proposition 4.2. □

Let $\mathcal{M}^{d,\mathrm{rs}}_{\mathrm{conn}}(X,G) \subseteq \mathcal{M}^{d}_{\mathrm{conn}}(X,G)$ denote the open locus of holomorphic G–connections (E,D) for which E is regularly stable.

Proposition 4.4. *There are no holomorphic sections for the forgetful map*

$$\mathcal{M}^{d,\mathrm{rs}}_{\mathrm{conn}}(X,G) \longrightarrow \mathcal{M}^{d,\mathrm{rs}}(X,G), \qquad (E,D) \longmapsto E. \tag{16}$$

Proof. The map (16) is a holomorphic torsor under the cotangent bundle of $\mathcal{M}^{d,\mathrm{rs}}(X,G)$. As such, it is isomorphic to the torsor of holomorphic connections on the line bundle

$$\mathcal{L} \longrightarrow \mathcal{M}^{d,\mathrm{rs}}(X,G)$$

with fibers $\det \mathrm{H}^1(X, \mathrm{ad}(E))$; see [Fa, Lemma IV.4]. Since $\mathcal{L}$ is ample, its first Chern class is nonzero, so $\mathcal{L}$ admits no global holomorphic connections. □

Let $\mathcal{M}^{d,\mathrm{rs}}_{\mathrm{Hod}}(X,G) \subseteq \mathcal{M}^{d}_{\mathrm{Hod}}(X,G)$ denote the open locus of triples (λ, E, D) for which E is regularly stable. The forgetful maps in (14) and (16) extend to the forgetful map

$$\mathcal{M}^{d,\mathrm{rs}}_{\mathrm{Hod}}(X,G) \longrightarrow \mathcal{M}^{d,\mathrm{rs}}(X,G), \qquad (\lambda, E,D) \longmapsto E, \tag{17}$$

which is an algebraic vector bundle. It contains the cotangent bundle (14) as a subbundle; the quotient is a line bundle, which is trivialized by the projection pr in (4).

Corollary 4.5. *The vector bundle* (17) *has no nonzero holomorphic sections.*

Proof. Let s be a holomorphic section of the vector bundle (17). Then $\mathrm{pr} \circ s$ is a holomorphic function on $\mathcal{M}^{d,\mathrm{rs}}(X,G)$, and hence constant. This constant vanishes because of Proposition 4.4. So $\mathrm{pr} \circ s = 0$, which implies that $s = 0$ using Proposition 4.2. □

Corollary 4.6. *The restriction of the algebraic tangent bundle*

$$T\mathcal{M}^{d,\mathrm{rs}}_{\mathrm{Hod}}(X,G) \longrightarrow \mathcal{M}^{d,\mathrm{rs}}_{\mathrm{Hod}}(X,G)$$

to the subvariety $\mathcal{M}^{d,\mathrm{rs}}(X,G) \subseteq \mathcal{M}^{d,\mathrm{rs}}_{\mathrm{Hod}}(X,G)$ has no nonzero holomorphic sections.

Proof. Use the decomposition (15), Proposition 4.1, and Corollary 4.5. □

5. Torelli theorems

Let X, X' be compact connected Riemann surfaces of genus ≥ 3. Let G, G' be nontrivial connected semisimple linear algebraic groups over $\mathbb{C}$. Fix $d \in \pi_1(G)$ and $d' \in \pi_1(G')$.

Theorem 5.1. *If $\mathcal{M}^{d'}_{\mathrm{Higgs}}(X',G')$ is biholomorphic to $\mathcal{M}^{d}_{\mathrm{Higgs}}(X,G)$, then $X' \cong X$.*

Proof. Corollary 4.3 implies that the subvariety $\mathcal{M}^{d}(X,G)$ is fixed pointwise by every holomorphic $\mathbb{C}^*$–action on $\mathcal{M}^{d}_{\mathrm{Higgs}}(X,G)$. All other complex analytic subvarieties with that property have smaller dimension, due to Proposition 3.1. Thus we get a biholomorphic map from $\mathcal{M}^{d'}(X',G')$ to $\mathcal{M}^{d}(X,G)$ by restriction. Using [BH], this implies that $X' \cong X$. □

Theorem 5.2. *If $\mathcal{M}^{d'}_{\mathrm{Hod}}(X',G')$ is biholomorphic to $\mathcal{M}^{d}_{\mathrm{Hod}}(X,G)$, then $X' \cong X$.*

Proof. The argument is exactly the same as in the previous proof. It suffices to replace Corollary 4.3 by Corollary 4.6, and Proposition 3.1 by Corollary 3.2. □

Theorem 5.3. *If $\mathcal{M}^{d'}_{\mathrm{DH}}(X',G')$ is biholomorphic to $\mathcal{M}^{d}_{\mathrm{DH}}(X,G)$, then $X' \cong X$ or $X' \cong \overline{X}$.*

Proof. The argument is similar. Corollary 4.6 implies that the two subvarieties $\mathcal{M}^{d}(X,G)$ and $\mathcal{M}^{-d}(\overline{X},G)$ are fixed pointwise by every holomorphic $\mathbb{C}^*$–action on $\mathcal{M}^{d}_{\mathrm{DH}}(X,G)$. All other complex analytic subvarieties with that property have smaller dimension, due to Corollary 3.3. Thus we get a biholomorphic map from $\mathcal{M}^{d'}(X',G')$ to either $\mathcal{M}^{d}(X,G)$ or $\mathcal{M}^{-d}(\overline{X},G)$ by restriction. Using [BH], this implies that either $X' \cong X$ or $X' \cong \overline{X}$. □

References

[At] M. F. Atiyah, Complex analytic connections in fibre bundles, *Trans. Amer. Math. Soc.* **85** (1957), 181–207.

[AB] M. F. Atiyah and R. Bott, The Yang–Mills equations over Riemann surfaces, *Phil. Trans. Roy. Soc. Lond.* **308** (1982), 523–615.

[AzBi] H. Azad and I. Biswas, On holomorphic principal bundles over a compact Riemann surface admitting a flat connection, *Math. Ann.* **322** (2002), 333–346.

[BGHL] I. Biswas, T. L. Gómez, N. Hoffmann and M. Logares, Torelli theorem for the Deligne–Hitchin moduli space, *Commun. Math. Phy.* **290** (2009), 357–369.

[BH] I. Biswas and N. Hoffmann, A Torelli theorem for moduli spaces of principal bundles over a curve, *Ann. Inst. Fourier* (to appear), arXiv:1003.4061.

[BR] I. Biswas and S. Ramanan, An infinitesimal study of the moduli of Hitchin pairs, *Jour. London Math. Soc.* **49** (1994), 219–231.

[De] P. Deligne, Letter to C. T. Simpson (March 20, 1989).

[Fa] G. Faltings, Stable G–bundles and projective connections, *Jour. Alg. Geom.* **2** (1993), 507–568.

[Go] W. M. Goldman, The symplectic nature of fundamental groups of surfaces, *Adv. Math.* **54** (1984), 200–225.

[Hi1] N. J. Hitchin, Stable bundles and integrable systems, *Duke Math. Jour.* **54** (1987), 91–114.

[Hi2] N. J. Hitchin, The self-duality equations on a Riemann surface, *Proc. Lond. Math. Soc.* **55** (1987), 59–126.

[La] G. Laumon, Un analogue global du cône nilpotent, *Duke Math. Jour.* **57** (1988), 647–671.

[Ra] A. Ramanathan, Stable principal bundles on a compact Riemann surface, *Math. Ann.* **213** (1975), 129–152.

[Si1] C. T. Simpson, Higgs bundles and local systems, *Inst. Hautes Études Sci. Publ. Math.* **75** (1992), 5–95.

[Si2] C. T. Simpson, Moduli of representations of the fundamental group of a smooth projective variety. II, *Inst. Hautes Études Sci. Publ. Math.* **80** (1994), 5–79.

[Si3] C. T. Simpson, A weight two phenomenon for the moduli of rank one local systems on open varieties, *From Hodge theory to integrability and TQFT tt*-geometry,* 175–214, Proc. Sympos. Pure Math., 78, Amer. Math. Soc., Providence, RI, 2008.

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