Almost split Kac-Moody groups over ultrametric fields

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Abstract

For a split Kac-Moody group G over an ultrametric field K, S. Gaussent and the author defined an ordered affine hovel on which the group acts; it generalizes the Bruhat-Tits building which corresponds to the case when G is reductive. This construction was generalized by C. Charignon to the almost split case when K is a local field. We explain here these constructions with more details and prove many new properties *e.g.* that the hovel of an almost split Kac-Moody group is an ordered affine hovel, as defined in a previous article.

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Introduction

Split Kac-Moody groups over ultrametric local fields were first studied by H. Garland in the case of loop groups [G95]. In [Ro06] we constructed a "microaffine" building for every split Kac-Moody group over a field K endowed with a non trivial real valuation. It is a (non discrete) building with the good usual properties, but it looks not like a Bruhat-Tits building, rather like the border of this building in its Satake (or polyhedral) compactification.

A more direct generalization of the Bruhat-Tits construction was made by S. Gaussent and the author, in the case where the residue field of K contains $\mathbb{C}$ [GR08]. This enabled us to deduce interesting consequences in representation theory. In [Ro12] the restriction about the residue field was removed. So, for a split Kac-Moody group G over K, one can build an hovel $\mathscr{I}$ on which G acts. As for the Bruhat-Tits building, $\mathscr{I}$ is covered by apartments corresponding to split maximal tori; but it is no longer true that any two points are in a same apartment (this corresponds to the fact that the Cartan decomposition fails in G). This is the reason why the word "building" was changed to "hovel". Nevertheless this hovel has interesting properties: it is an ordered affine hovel as defined in [Ro11]. As a consequence the residues in each point of $\mathscr{I}$ are twin buildings, there exist on $\mathscr{I}$ a preorder invariant by G and, at infinity, we get twin buildings and two microaffine buildings. These are the twin buildings of G introduced by B. Rémy [Re02] and the microaffine buildings of [Ro11].

Cyril Charignon undertook the construction of hovels for almost split Kac-Moody groups [Ch10], [Ch11]. Actually he considered the disjoint union of the hovel and of some hovels at infinity called façades. This union is called a bordered hovel, it looks like the Satake compactification of a Bruhat-Tits building; in addition to the main hovel it contains the microaffine buildings. He elaborates an abstract theory of bordered hovels associated to a generating root datum, a valuation and a family of parahoric subgroups. He proves an abstract descent theorem and succeeds in using it to build a bordered hovel associated to an almost split Kac-Moody group over a field endowed with a discrete valuation and a perfect residue field. As a corollary the microaffine buildings are also defined in this situation.

In this article we give more details about these constructions and improve many results. In particular the fixed point theorem in $\mathbb{R}-$buildings proved recently by K. Struyve [S11] enables us to prove the existence of bordered hovels in new cases (with a non discrete valuation). With these additional details and results it is possible to define spherical Hecke algebras for any almost split Kac-Moody group over a local field [GR12].

In section 1 we explain the general framework of our study: abstract generating root data, their associated twin buildings and split Kac-Moody groups (as defined by J. Tits [T87]).

Section 2 is devoted to B. Rémy's theory of almost split Kac-Moody groups [Re02]. We improve a few results, *e.g.* on geometric realizations of the associated twin buildings and on imaginary relative root groups.

In section 3 we define the affine apartments associated to a valuation of an abstract root datum. We explain the interesting subsets or filters of subsets inside them (facets, sectors, chimneys, enclosures,...) and embed them in their bordered apartments. There are several possible choices for these apartments, their imaginary roots or walls and for the façades at infinity. So this leads to several choices for all these objects and none of them is better in all circumstances.

Section 4 is devoted to Charignon's abstract construction of the bordered hovel associated to a good family of parahoric subgroups in a valuated root datum [Ch10], [Ch11]. We select two other conditions he considered for parahoric families and a new third one to define what is a very good family. Then we are able to generalize abstractly the constructions of [GR08] and prove that the abstract hovel we get is an ordered affine hovel in the sense of [Ro11] (slightly generalized). This involves an enclosure map ($cl_{\mathbb{R}}^{\Delta^{ti}}$) which gives (too) small enclosures.

In section 5 we mix these abstract results and the results of [Ro12] to define the bordered hovel of a split Kac-Moody group over a field endowed with a non trivial real valuation. One of the problems is to extend the results to general apartments, neither essential as in Charignon's or Rémy's works, nor associated directly to the group as in [Ro12]. We prove that these bordered hovels are functorial, uniquely defined (in the sense that the very good family of parahorics is unique) and that their residue twin buildings are associated to a generating root datum.

These results are generalized to almost split Kac-Moody groups in section 6. We explain the abstract descent theorem of Charignon (generalizing the analogous theorem of F. Bruhat and J. Tits [BrT72]). To apply it to an almost split Kac-Moody group G, we need the same condition as for reductive groups: G is assumed to become quasi split over a finite tamely ramified Galois extension L/K [Ro77]. There is no need for another condition, even for a non discrete valuation, as is now clear from Struyve's work [S11]. We explain Charignon's results in this almost split case and generalize them to more general apartments. So we get a bordered hovel and we prove that the hovel inside it (its main façade) is an ordered affine hovel (in the sense of [Ro11]) with an enclosure map ($cl_{L/K}^{K\Delta^r}$) which is better than the one in section 4 (but certainly still not the best one).

1 Root data and split Kac-Moody groups

Most of the following definitions or results may be found in [Re02], [Ro06], [Ro11] or [Ro12].

1.1 Root generating systems

1) We consider a Kac-Moody matrix (or generalized Cartan matrix) $\mathbb{M} = (a_{i,j})_{i,j\in I}$, with rows and columns indexed by a finite set I. Let Q be a free $\mathbb{Z}-$module with basis $(\alpha_i)_{i\in I}$ and $Q^+ = \sum_{i\in I} \mathbb{Z}_{\geq 0}.\alpha_i \subset Q$, $Q^- = -Q^+$. The *(vectorial) Weyl group* W^v associated to $\mathbb{M}$ is a Coxeter group with generating system the set $\Sigma = \{\, s_i \mid i \in I\}$ of automorphisms of Q defined by $s_i(\alpha_j) = \alpha_j - a_{i,j}\alpha_i$. The associated system of *real roots* is $\Phi = \{\, w(\alpha_i) \mid w \in W^v, i \in I\,\}$ [K90]; it is a real root system (with free basis $(\alpha_i)_{i\in I}$) in the sense of [MoP89] or [MoP95], see also [Ba96] and [H91]. If $\alpha \in \Phi$, then $s_\alpha = w.s_i.w^{-1}$ is well defined by α independently of the choice of w and i such that $\alpha = w(\alpha_i)$. We say that we are in the *classical case* when W^v is finite, then $\mathbb{M}$ is a Cartan matrix and Φ a root system in the sense of [B-Lie]. For $J \subset I$, $\mathbb{M}(J) = (a_{i,j})_{i,j\in J}$ is a Kac-Moody matrix; with obvious notations, $Q(J)$ is a submodule of Q and $\Phi^m(J) = \Phi \cap Q(J)$ the root system associated to $\mathbb{M}(J)$, its Weyl group is $W^v(J) = \langle s_i \mid i \in J\rangle$.

2) A *root generating system* (or RGS) [Ba96] will be (for our purpose) a quadruple $\mathcal{S} = (\mathbb{M}, Y, (\overline{\alpha_i})_{i\in I}, (\alpha_i^\vee)_{i\in I})$ where $\mathbb{M}$ is a Kac-Moody matrix, Y a free $\mathbb{Z}-$module of finite rank n, $(\overline{\alpha_i})_{i\in I}$ a family (of *simple roots*) in its dual $X = Y^*$ and $(\alpha_i^\vee)_{i\in I}$ a family (of *simple coroots*) in Y. These data have to satisfy the condition: $a_{i,j} = \overline{\alpha_j}(\alpha_i^\vee)$.

The Weyl group W^v acts on X (and dually on Y) by $s_i(\chi) = \chi - \chi(\alpha_i^\vee)\overline{\alpha_i}$.

We say that $\mathcal{S}$ is *free* (resp. *adjoint*) if $(\overline{\alpha_i})_{i\in I}$ is free in (resp. generates) X. For example the minimal adjoint RGS $\mathcal{S}_{\mathbb{M}m} = (\mathbb{M}, Q^*, (\alpha_i)_{i\in I}, (\alpha_i^\vee)_{i\in I})$ (with an obvious definition of the $\alpha_i^\vee$) is free and adjoint.

There is a group homomorphism $bar : Q \to X$, $\alpha \mapsto \overline{\alpha}$ such that $bar(\alpha_i) = \overline{\alpha_i}$; it is W^v-equivariant and $bar^* : Y \to Q^*$ is a commutative extension of RGS $\mathcal{S} \to \mathcal{S}_{\mathbb{M}m}$ [Ro12, 1.1.7]. When $\mathcal{S}$ is free, Q is identified with $\overline{Q} = bar(Q) \subset X$.

For $J \subset I$, $\mathcal{S}(J) = (\mathbb{M}(J), Y, (\overline{\alpha_i})_{i\in J}, (\alpha_i^\vee)_{i\in J})$ is also a RGS.

3) The *complex Kac-Moody algebra* $\mathfrak{g}_\mathcal{S}$ associated to $\mathcal{S}$ is generated by the *Cartan subalgebra* $\mathfrak{h}_\mathcal{S} = Y \otimes_\mathbb{Z} \mathbb{C}$ and elements $(e_i, f_i)_{i\in I}$ with well known relations [K90]. This Lie algebra has a gradation by Q : $\mathfrak{g}_\mathcal{S} = \mathfrak{h}_\mathcal{S} \oplus (\oplus_{\alpha\in\Delta}\, \mathfrak{g}_\alpha)$ where $\Delta \subset Q \setminus \{0\}$ is the *root system* of $\mathfrak{g}_\mathcal{S}$ or of $\mathbb{M}$.

We have $\mathfrak{h}_\mathcal{S} = (\mathfrak{g}_\mathcal{S})_0$, $\mathfrak{g}_{\alpha_i} = \mathbb{C}e_i$, $\mathfrak{g}_{-\alpha_i} = \mathbb{C}f_i$ and $\Phi \subset \Delta$ (as Δ is W^v-stable).

If $\Delta^+ = \Delta \cap Q^+$ (resp. $\Delta^- = -\Delta^+$) is the set of *positive* (resp. *negative*) roots, then $\Delta = \Delta^- \bigsqcup \Delta^+$. We set $\Phi^\pm = \Phi \cap \Delta^\pm = -\Phi^\mp$. The *imaginary* roots are the roots in $\Delta \setminus \Phi = \Delta_{im}$; we set $\Delta_{re} = \Phi$, ${\Delta^\pm}_{re} = \Phi^\pm$ and ${\Delta^\pm}_{im} = \Delta_{im} \cap \Delta^\pm$.

For $J \subset I$, $\mathfrak{g}_{\mathcal{S}(J)}$ is a Lie subalgebra of $\mathfrak{g}_\mathcal{S}$ and $\Delta^m(J) = \Delta \cap Q(J)$ a subroot system of Δ.

In the classical case, $\mathfrak{g}_{\mathcal{S}}$ is a reductive finite dimensional Lie algebra and $\Delta_{im} = \emptyset$.

1.2 Vectorial apartments

1) We consider a free RGS $\mathcal{S} = (\mathbb{M}, Y, (\alpha_i)_{i\in I}, (\alpha_i^\vee)_{i\in I})$ and the real vector space $V = V^{\mathcal{S}} = Y \otimes \mathbb{R} = \mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{R})$. Each element in X or in $Q \subset X$ induces a linear form on V. A *vectorial wall* in V is the kernel of some $\alpha \in \Phi$. The *positive* (resp. *negative*) *fundamental chamber* in V is $C^v_+ = \{\, v \in V \mid \alpha_i(v) > 0, \forall i \in I \,\}$ (resp. $C^v_- = -C^v_+$). If $J \subset I$, let $F^v_+(J) = \{\, v \in V \mid \alpha_i(v) = 0, \forall i \in J, \alpha_i(v) > 0, \forall i \in I \setminus J \,\}$ and $F^v_-(J) = -F^v_+(J)$, they are cones in V. Then the closed positive fundamental chamber is $\overline{C^v_+} = \bigsqcup_{J\subset I} F^v_+(J)$ and symmetrically for $\overline{C^v_-}$.

The Weyl group W^v acts faithfully on V, we identify W^v with its image in $GL(V)$. For $w \in W^v$ and $J \subset I$, $wF^v_+(J)$ (resp. $wF^v_-(J)$) is called a *positive* (resp. *negative*) *vectorial facet of type* J. The fixator or stabilizer of $F^v_\pm(J)$ is $W^v(J)$; if this group is finite we say that J or $wF^v_\pm(J)$ is *spherical*. These positive facets are disjoint and their union $\mathcal{T}_+$ is a cone: the *Tits cone*. The inclusion in the closure gives an order relation on these facets. The *star* of a facet F^v is the set F^{v*} of all facets F^v_1 such that $F^v \subset \overline{F^v_1}$.

The properties of the action of W^v on the set of positive facets allows one to identify this poset (or to be short $\mathcal{T}_+$) with the Coxeter complex of (W^v, Σ). The interior of $\mathcal{T}_+$ is the union of its spherical facets. The symmetric results for $\mathcal{T}_- = -\mathcal{T}_+$ are also true.

We call $\mathbb{A}^v = \mathcal{T}_+ \cup \mathcal{T}_-$ the *vectorial fundamental twin apartment* associated to $\mathcal{S}$ and set $\overrightarrow{\mathbb{A}^v} = V$ (vector space generated by $\mathbb{A}^v$). A *generic subspace* of $\mathbb{A}^v$ is an intersection of $\mathbb{A}^v$ with a vector subspace of $\overrightarrow{\mathbb{A}^v}$ which meets the interior of $\mathbb{A}^v$; for example a wall is a generic subspace. An *half-apartment* in $\mathbb{A}^v$ is the intersection with $\mathbb{A}^v$ of one of the two closed half-spaces of $\overrightarrow{\mathbb{A}^v}$ limited by a wall.

In V the subspace $V_0 = F_\pm{}^v(I) = \bigcap_{i\in I} \mathrm{Ker}(\alpha_i)$ (trivial facet) is the intersection of all vectorial walls. Acting by translations it stabilizes all facets and the two Tits cones. So the essentialization of V or $\mathbb{A}^v$ is $V^e = V/V_0$ or $\mathbb{A}^{ve} = \mathbb{A}^v/V_0$.

One may generalize these definitions to the case when the chamber C^v_+ defined by $(\overline{\alpha_i})_{i\in I}$ (for a non free RGS) is non empty in $V = Y \otimes \mathbb{R}$ [Ba96, p. 113, 114] but we shall avoid this.

2) The smallest example for V associated to $\mathbb{M}$ and $\Phi \subset Q$ corresponds to $\mathcal{S} = \mathcal{S}_{\mathbb{M}m}$. Then $V = V^q = V^{\mathbb{M}} = Q^* \otimes \mathbb{R} = \mathrm{Hom}_{\mathbb{Z}}(Q, \mathbb{R})$. In the above notations we add an exponent ^{q} to all names. We get thus the *essential vectorial fundamental twin apartment* $\mathbb{A}^{vq}$. Actually V^q and $\mathbb{A}^{vq}$ are canonically the essentializations of any V or $\mathbb{A}^v$ in 1) : $V^e = V^q$ and $\mathbb{A}^{ve} = \mathbb{A}^{vq}$

3) If $\mathcal{S}$ is a given free RGS, we shall write $V^x = V^{\mathcal{S}}$ and add an exponent ^{x} to all names in 1) *e.g.* $V^q = V^{xe} = V^x/V_0^x$ and $\mathbb{A}^q = \mathbb{A}^{xe} = \mathbb{A}^x/V_0^x$. We get thus the *normal vectorial fundamental twin apartment* $\mathbb{A}^{vx}$.

If $\mathcal{S}$ is a given (non necessarily free) RGS, we may consider the free RGS $\mathcal{S}^l$ of [Ro12, 1.3d]: $\mathcal{S}^l = (\mathbb{M}, Y^{xl}, (\alpha_i^{xl})_{i\in I}, (\alpha_i^{xl\vee})_{i\in I})$ with $Y^{xl} = Y \oplus Q^*$, $\alpha_i^{xl} = \overline{\alpha}_i + \alpha_i \in X^{xl} = X \oplus Q$ and $\alpha_i^{xl\vee} = \alpha_i^\vee \in Y \subset Y^{xl} = Y \oplus Q^* = (X \oplus Q)^*$. Then $V^{xl} = \mathrm{Hom}_{\mathbb{Z}}(X \oplus Q, \mathbb{R})$ and we add an exponent ^{xl} to all names in 1). We get thus the *extended vectorial fundamental twin apartment* $\mathbb{A}^{vxl}$.

4) More generally we may consider a quadruple as in [Ro11, 1.1]: $(V, W^v, (\alpha_i)_{i\in I}, (\alpha_i^\vee)_{i\in I})$ with α_i free in V^* and $a_{i,j} = \alpha_j(\alpha_i^\vee)$ hence $\Phi \subset Q \subset V^*$. The same things as in 1) (*e.g.* $F^v_\pm(J)$, $\mathbb{A}^v_\pm = \mathcal{T}_\pm$, ..) may be defined in V and we have $V^q = V/V_0$ for $V_0 = \bigcap_{i\in I} \mathrm{Ker}\alpha_i$. For example we may take for V a quotient of a $V^{\mathcal{S}}$ as in 1) by any subspace V_{00} of V_0.

We get thus many geometric realizations of the Coxeter complex of (W^v, Σ).

1.3 The split Kac-Moody group $\mathfrak{G}_{\mathcal{S}}$

As defined by J. Tits [T87], this group $\mathfrak{G}_{\mathcal{S}}$ is a functor from the category of (commutative) rings to the category of groups.

One considers first the torus $\mathfrak{T}_{\mathcal{S}} = \mathfrak{T}_Y = \mathfrak{Spec}(\mathbb{Z}[X])$ with character group $X(\mathfrak{T}_{\mathcal{S}}) = X$ and cocharacter group $Y(\mathfrak{T}_{\mathcal{S}}) = Y$. For any ring R, $\mathfrak{T}_Y(R) = Y \otimes_{\mathbb{Z}} R^* = \mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{R}^*)$. Then the group $\mathfrak{G}_{\mathcal{S}}(R)$ is generated by $\mathfrak{T}_{\mathcal{S}}(R)$ and elements $\mathfrak{x}_\alpha(r)$ for $\alpha \in \Phi$ and $r \in R$; for the precise relations see [T87], [Re02] or [Ro12].

Actually $\mathfrak{T}_{\mathcal{S}}$ is a sub-group-functor of $\mathfrak{G}_{\mathcal{S}}$, the *standard maximal split subtorus*. For $\alpha \in \Phi$, there is an injective homomorphism $\mathfrak{x}_\alpha : \mathfrak{Add} \to \mathfrak{G}_{\mathcal{S}}$, $r \in R \mapsto \mathfrak{x}_\alpha(r)$; the sub-group-functor of $\mathfrak{G}_{\mathcal{S}}$ image of $\mathfrak{x}_\alpha$ is written $\mathfrak{U}_\alpha$. The *standard positive* (resp. *negative*) *maximal unipotent subgroup* is the sub-group-functor $\mathfrak{U}^{\pm}_{\mathcal{S}}$ such that, for all ring R, $\mathfrak{U}^{+}_{\mathcal{S}}(R)$ (resp. $\mathfrak{U}^{-}_{\mathcal{S}}(R)$) is generated by all $\mathfrak{U}_\alpha(R)$ for $\alpha \in \Phi^+$ (resp. $\alpha \in \Phi^-$); it depends actually only of $\mathbb{M}$, not of $\mathcal{S}$. Then the *standard positive* (resp. *negative*) *Borel subgroup* is the semi-direct product $\mathfrak{B}^+_{\mathcal{S}} = \mathfrak{T}_{\mathcal{S}} \ltimes \mathfrak{U}^+_{\mathcal{S}}$ (resp. $\mathfrak{B}^-_{\mathcal{S}} = \mathfrak{T}_{\mathcal{S}} \ltimes \mathfrak{U}^-_{\mathcal{S}}$).

The construction of $\mathfrak{G}_{\mathcal{S}}$ uses a $Q-$graded $\mathbb{Z}-$form $\mathcal{U}_{\mathcal{S}\mathbb{Z}}$ of the universal enveloping algebra of $\mathfrak{g}_{\mathcal{S}}$, we call it the *Tits enveloping algebra* of $\mathfrak{G}_{\mathcal{S}}$ over $\mathbb{Z}$. It is a filtered $\mathbb{Z}-$bialgebra; the first term of its filtration is $\mathbb{Z} \oplus \mathfrak{g}_{\mathcal{S}\mathbb{Z}}$, where $\mathfrak{g}_{\mathcal{S}\mathbb{Z}}$ is a $\mathbb{Z}-$form of the Lie algebra $\mathfrak{g}_{\mathcal{S}}$. There is a functorial adjoint representation $Ad : \mathfrak{G}_{\mathcal{S}} \to \mathfrak{Aut}(\mathcal{U}_{\mathcal{S}\mathbb{Z}})$, see [Re02] and/or [Ro12] for details. In the classical case $\mathfrak{G}_{\mathcal{S}}$ is a reductive group and $\mathcal{U}_{\mathcal{S}\mathbb{Z}}$ is often called the Kostant's $\mathbb{Z}-$form. By analogy with this case we define the *reductive rank* (resp. *semi-simple rank*) of $\mathfrak{G}_{\mathcal{S}}$ or $\mathcal{S}$ as $rrk(\mathcal{S}) = n = dim(X)$ (resp. $ssrk(\mathcal{S}) = |I|$); there is no a priori inequality between these two ranks.

In the following we shall almost always consider a field K and restrict the above functors to the category $\mathcal{S}ep(K)$ of algebraic separable field extensions of K contained in a given separable closure K_s. The groups associated to K by these functors are then written with roman letters: $G_{\mathcal{S}} = \mathfrak{G}_{\mathcal{S}}(K)$, $T_{\mathcal{S}} = \mathfrak{T}_{\mathcal{S}}(K)$, $U_\alpha = \mathfrak{U}_\alpha(K)$, $x_\alpha : K \to U_\alpha \subset G_{\mathcal{S}}$, etc. We set also $\mathcal{U}_{\mathcal{S}K} = \mathcal{U}_{\mathcal{S}\mathbb{Z}} \otimes_{\mathbb{Z}} K, \cdots$. We shall sometimes forget the subscript $_{\mathcal{S}}$.

Definition 1.4. *cf.* [Ro06] A *root datum of type* a (real) root system Φ is a triple $(G, (U_\alpha)_{\alpha\in\Phi}, Z)$ where G is a group and Z, U_α (for $\alpha \in \Phi$) are subgroups of G, satisfying:

(RD1) For all $\alpha \in \Phi$, U_α is non trivial and normalized by Z.

(RD2) For each prenilpotent pair of roots $\{\alpha, \beta\}$, the commutator group $[U_\alpha, U_\beta]$ is included in the group generated by the groups U_γ for $\gamma = p\alpha + q\beta \in \Phi$ and $p, q \in \mathbb{Z}_{>0}$.

(there is a finite number of such roots γ, as $\{\alpha, \beta\}$ is supposed prenilpotent).

(RD3) If $\alpha \in \Phi$ and $2\alpha \in \Phi$, then $U_{2\alpha} \subsetneqq U_\alpha$.

(RD4) For all $\alpha \in \Phi$ and all $u \in U_\alpha \setminus \{1\}$, there exist $u', u'' \in U_{-\alpha}$ such that $m(u) := u'uu''$ conjugates U_β into $U_{s_\alpha(\beta)}$ for all $\beta \in \Phi$. Moreover, for all $u, v \in U_\alpha \setminus \{1\}$, $m(u)Z = m(v)Z$.

(RD5) If U^+ (resp. U^-) is the group generated by the groups U_α for $\alpha \in \Phi^+$ (resp. $\alpha \in \Phi^-$), then $ZU^+ \cap U^- = \{1\}$.

The root datum is called *generating* if moreover:

(GRD) The group G is generated by Z and the groups U_α for $\alpha \in \Phi$.

Remarks 1.5. a) This definition is given for a general (real) root system Φ. For the system Φ of 1.1 the axiom (RD3) is useless as Φ is reduced. In the classical case (*i.e.* for a finite root system) this is equivalent to the definition of "donnée radicielle de type Φ" in [BrT72]. In general a generating root datum is the same thing as a "donnée radicielle jumelée entière" as defined in [Re02, 6.2.5].

b) Actually Z has to be the intersection of the normalizers of the groups U_α: [Re02, 1.5.3], see also [AB08, 7.84]. So one may forget Z in the datum, as in [T92] or [Ch11, 10.1.1].

c) Even in the classical case, the notion of root datum (of type a root system) is more precise than the notion of RGD-system (of type a Coxeter system) defined in [AB08, 8.6.1] (which is the same thing as "donnée radicielle jumelée" defined in [Re02, 1.5.1], see also [T92]). The "roots" of (W^v,Σ) are in one to one correspondence with the non-divisible roots in Φ. So, if $(G,(U_\alpha)_{\alpha\in\Phi},Z)$ is a generating root datum, then $(G,(U_\alpha)_{\alpha\in\Phi_{nd}},Z)$ is a RGD-system; the difference is that axiom (RGD1) is less precise than (RD2): it allows p and q to be in $\mathbb{R}_{>0}$.

Root data describe more precisely the algebraic structure of reductive groups or Kac-Moody groups; with RGD systems one can describe more general actions of groups on (twin) buildings.

Consequences 1.6. Let $(G,(U_\alpha)_{\alpha\in\Phi},Z)$ be a generating root datum, then, by [Re02, chap. 1, 2] or [AB08], one has:

1) The group $B^\pm = ZU^\pm$ is called the *standard positive* (resp. *negative*) *Borel subgroup* or more generally *minimal parabolic subgroup*.

Let N be the group generated by Z and the $m(u)$ for $\alpha\in\Phi$ and $u\in U_\alpha\setminus\{1\}$. There is a surjective homomorphism $\nu^v : N\to W^v$ (where W^v is the Weyl group of the root system Φ) such that $\nu^v(m(u)) = s_\alpha$ and $\mathrm{Ker}(\nu^v) = Z$.

Then $B^\pm\cap N = Z$ and $(B^\pm,N)$ is a BN-pair in G. In particular we have two Bruhat decompositions: $G = \bigsqcup_{w\in W^v} B^\varepsilon w B^\varepsilon$ (for $\varepsilon = +$ or $-$).

Moreover $G = \bigsqcup_{w\in W^v} (\prod_{\beta\in\Phi^+\cap w\Phi^- w^{-1}} U_\beta).wZ.U^+$, with uniqueness of the decomposition (refined Bruhat decomposition). The same is also true when exchanging $+$ and $-$.

2) More precisely (B^+,B^-,N) is a twin BN-pair; in particular we have a Birkhoff decomposition: $G = \bigsqcup_{w\in W^v} B^+wB^-$. Moreover for $u,u'\in U^+$, $v,v'\in U^-$ and $z,z'\in Z$, if $uzv = u'z'v'$ then $u=u'$, $v=v'$ and $z=z'$.

3) Associated to the BN-pair (B^ε,N), there is a combinatorial building $\mathscr{I}_\varepsilon^{vc}$ (viewed as a simplicial complex) on which G acts strongly transitively (with preservation of the types of the facets). The group B^ε is the stabilizer and fixator of the fundamental chamber $C_\varepsilon^{vc}\subset\mathscr{I}_\varepsilon^{vc}$. The group N stabilizes the fundamental apartment $\mathbb{A}_\varepsilon^{vc}$ (which contains C_ε^{vc}); it is equal to the stabilizer in G of $\mathbb{A}_\varepsilon^{vc}$, as the BN-pair is saturated *i.e.* $Z = \bigcap_{w\in W^v} wB^\varepsilon w^{-1}$.

The Birkhoff decomposition gives a twinning between the buildings $\mathscr{I}_+^{vc}$ and $\mathscr{I}_-^{vc}$; we have $Z = B^+\cap B^-$.

4) As the facets of $\mathcal{T}_\varepsilon = \mathbb{A}_\varepsilon^v$ are in one to one, increasing and $N-$equivariant correspondence with those of the Coxeter complex $\mathbb{A}_\varepsilon^{vc}$, we can glue different apartments together to get a geometric realization $\mathscr{I}_\varepsilon^v = \mathscr{I}_\varepsilon^v(G,\mathbb{A}^v)$ of $\mathscr{I}_\varepsilon^{vc}$ (called vectorial or conical) in which the apartments and facets are cones. The different peculiar choices of $\mathbb{A}^v$ explained in 1.2 2), 3) give vectorial buildings $\mathscr{I}_\varepsilon^{vq}$, $\mathscr{I}_\varepsilon^{vx}$, $\mathscr{I}_\varepsilon^{vxl}$;

For any of these vectorial buildings, the vector space $V_0\subset\overrightarrow{\mathbb{A}_\varepsilon^v}$ acts $G-$equivariantly and stabilizes all facets or apartments. The essentialization of this building *i.e.* its quotient $\mathscr{I}_\varepsilon^{ve}(G,\mathbb{A}^v) = \mathscr{I}_\varepsilon^v(G,\mathbb{A}^v)/V_0$ by V_0 is canonically equal to $\mathscr{I}_\varepsilon^{vq} = \mathscr{I}_\varepsilon^v(G,\mathbb{A}^{vq})$.

5) There is a one to one decreasing correspondence between facets and parabolic subgroups: the stabilizer and fixator in G of a facet $F^v\subset\mathcal{T}_\varepsilon$ or of $F^v/V_0\subset\mathcal{T}_\varepsilon^q$ is a parabolic subgroup $P(F^v)$ of G (which is its own normalizer). We have a Levi decomposition $P(F^v) = M(F^v)\ltimes U(F^v)$. The group $P(F^v)$ (resp. $M(F^v)$) is generated by Z and the groups U_α for $\alpha\in\Phi(F^v)$

(resp. $\alpha \in \Phi^m(F^v)$) *i.e.* $\alpha \in \Phi$ and $\alpha(F^v) \geq 0$ (resp. $\alpha(F^v) = 0$). The subgroup $U(F^v)$ is normal in $P(F^v)$ and contains the groups U_α for $\alpha \in \Phi^u(F^v)$ *i.e.* $\alpha \in \Phi$ and $\alpha(F^v) > 0$ [Re02, 6.2]. We define $G(F^v) = P(F^v)/U(F^v) \simeq M(F^v)$ and $N(F^v) = N \cap P(F^v) \subset M(F^v)$.

If $F^v = F^v_\varepsilon(J)$ for $J \subset I$, then $P(F^v) = P^\varepsilon(J) = B^\varepsilon W^v(J) B^\varepsilon$, $\Phi^m(F^v) = \Phi^m(J)$ and $N(F^v)/Z = W^v(J)$. The group $G(J) = M(F^v)$ is endowed with the generating root datum $(G(J), (U_\alpha)_{\alpha \in \Phi^m(J)}, Z)$.

Theorem 1.7. *With the notations of 1.3, $(G_{\mathcal{S}}, (U_\alpha)_{\alpha\in\Phi}, T_{\mathcal{S}})$ is a root datum of type Φ. Moreover if $|K| \geq 4$, N is the normalizer of $T_{\mathcal{S}}$ in $G_{\mathcal{S}}$.*

Proof. This is essentially in [T87] and [T92]. See [Re02, 8.4.1] □

Remarks 1.8. 1) $B^\pm_{\mathcal{S}}$ (resp. $U^\pm_{\mathcal{S}}$) as defined in 1.3 coincide with $B^\pm$ defined in 1.6.1 (resp. $U^\pm$ defined in 1.4). The group N is $N_{\mathcal{S}} = \mathfrak{N}_{\mathcal{S}}(K)$, where $\mathfrak{N}_{\mathcal{S}}$ is a sub-group-functor of $\mathfrak{G}_{\mathcal{S}}$ normalizing $\mathfrak{T}_{\mathcal{S}}$. Moreover N is the normalizer in $G_{\mathcal{S}}$ of $\mathfrak{T}_{\mathcal{S}}$, but not always the normalizer of $T_{\mathcal{S}}$ (*e.g.* when $|K| = 2$, $T_{\mathcal{S}} = \{1\}$). The maximal split tori of $\mathfrak{G}_{\mathcal{S}}$ are conjugated by $G_{\mathcal{S}}$ to $\mathfrak{T}_{\mathcal{S}}$ [Re02, 12.5.3].

The Levi factor of $P^\varepsilon(J)$ is $G(J) = \mathfrak{G}_{\mathcal{S}(J)}(K)$ where $\mathfrak{G}_{\mathcal{S}(J)}$ is the split Kac-Moody group associated to the RGS $\mathcal{S}(J)$ of 1.1.2 [Ro12, 5.15.2].

2) The combinatorial buildings associated to this root datum are written $\mathscr{I}^{vc}_\varepsilon(\mathfrak{G}_{\mathcal{S}}, K)$ or $\mathscr{I}^{vc\mathbb{M}}_\varepsilon(K)$, as they depend only on the field K and the Kac-Moody matrix $\mathbb{M}$ (not of the SGR $\mathcal{S}$: [Ro12, 1.10]).

As N is the stabilizer of the fundamental apartment $\mathbb{A}^{vc}_\varepsilon$ in $\mathscr{I}^{vc\mathbb{M}}_\varepsilon(K)$ and the normalizer of $\mathfrak{T}_{\mathcal{S}}$, we get a one to one correspondence $\mathfrak{T} \mapsto A^{vc}_\varepsilon(\mathfrak{T})$ between the maximal split tori in $\mathfrak{G}_{\mathcal{S}}$ (or their points over K, if $|K| \geq 4$) and the apartments of $\mathscr{I}^{vc\mathbb{M}}_\varepsilon(K)$.

3) The geometric realization of $\mathscr{I}^{vc\mathbb{M}}_\varepsilon(K)$ introduced in 1.6.4 is named $\mathscr{I}^v_\varepsilon(\mathfrak{G}_{\mathcal{S}}, K, \mathbb{A}^v)$. If we use $\mathcal{T}^q_\varepsilon = \mathbb{A}^{vq}_\varepsilon$, we call it the *essential vectorial building* $\mathscr{I}^v_\varepsilon(\mathfrak{G}_{\mathcal{S}}, K, \mathbb{A}^{vq}) = \mathscr{I}^{vq}_\varepsilon(\mathfrak{G}_{\mathcal{S}}, K) = \mathscr{I}^{v\mathbb{M}}_\varepsilon(K)$ of $\mathfrak{G}_{\mathcal{S}}$ over K of sign $\varepsilon = \pm$. We have also *extended vectorial buildings* $\mathscr{I}^v_\varepsilon(\mathfrak{G}_{\mathcal{S}}, K, \mathbb{A}^{vxl}) = \mathscr{I}^{vxl}_\varepsilon(\mathfrak{G}_{\mathcal{S}}, K) = \mathscr{I}^{v\mathcal{S}^l}_\varepsilon(K)$ defined using $\mathcal{T}^{xl}_\varepsilon = \mathbb{A}^{vxl}_\varepsilon$ instead of $\mathcal{T}^q_\varepsilon$ in 1.6.4.

When $\mathcal{S}$ is free, we can also use $\mathcal{T}^x_\varepsilon = \mathbb{A}^{vx}_\varepsilon$ and define the *(normal) vectorial buildings* $\mathscr{I}^v_\varepsilon(\mathfrak{G}_{\mathcal{S}}, K, \mathbb{A}^{vx}) = \mathscr{I}^{vx}_\varepsilon(\mathfrak{G}_{\mathcal{S}}, K) = \mathscr{I}^{v\mathcal{S}}_\varepsilon(K)$.

To be short we omit often K and/or $\mathbb{M}$, $\mathcal{S}$, $\mathfrak{G}_{\mathcal{S}}$, $\mathbb{A}^v$ in the above notations.

4) $\mathscr{I}^{vc}_\varepsilon(\mathfrak{G}_{\mathcal{S}}, K)$ is clearly functorial in K. $\mathscr{I}^v_\varepsilon(\mathfrak{G}_{\mathcal{S}}, K, \mathbb{A}^v)$ is functorial in K and $\mathcal{S}$ (for commutative extensions).

1.9 Completions of $\mathfrak{G}_{\mathcal{S}}$

There is a *positive* (resp. *negative*) *completion* $\mathfrak{G}^{pma}_{\mathcal{S}}$ (resp. $\mathfrak{G}^{nma}_{\mathcal{S}}$) of $\mathfrak{G}_{\mathcal{S}}$ (defined in [Ma88], [Ma89]) which is used in [Ro12] to get better commutation relations. This is an ind-group-scheme which contains $\mathfrak{G}_{\mathcal{S}}$ but differs from it by its positive (resp. negative) maximal pro-unipotent subgroup: $\mathfrak{U}^+_{\mathcal{S}}$ (resp. $\mathfrak{U}^-_{\mathcal{S}}$) is replaced by a greater group scheme $\mathfrak{U}^{ma+}_{\mathcal{S}}$ (resp. $\mathfrak{U}^{ma-}_{\mathcal{S}}$) involving the full root system Δ of 1.1.3.

For a ring R, an element of $\mathfrak{U}^{ma\pm}_{\mathcal{S}}(R)$ can be written uniquely as an infinite product: $u = \prod_{\alpha\in\Delta^\pm} u_\alpha$ with $u_\alpha \in \mathfrak{U}_\alpha(R)$, for a given order on the roots $\alpha = \sum_{i\in I} n^\alpha_i \alpha_i \in \Delta^\pm$ (*e.g.* an order such that $|ht(\alpha)| = \sum_{i\in I} |n^\alpha_i|$ is increasing). For $\alpha \in \Phi$, u_α is written $u_\alpha = \mathfrak{x}_\alpha(r) = [exp] r e_\alpha$ for a unique $r \in R$ and e_α a fixed basis of $\mathfrak{g}_\alpha$. For $\alpha \in \Delta_{im}$, u_α is written $u_\alpha = \prod_{j=1}^{j=n_\alpha} [exp] r_{\alpha,j}.e_{\alpha,j}$ for unique $r_{\alpha,j} \in R$ and for $(e_{\alpha,j})_{j=1,n_\alpha}$ a fixed basis of $\mathfrak{g}_\alpha$. Moreover the conjugate of such an element $u \in \mathfrak{U}^{ma\pm}_{\mathcal{S}}(R)$ by $t \in \mathfrak{T}_{\mathcal{S}}(R)$ is given by the same formula: for $u'_\alpha = tu_\alpha t^{-1} \in \mathfrak{U}_\alpha(R)$, we just replace r by $\overline{\alpha}(t)r$ or each $r_{\alpha,j}$ by $\overline{\alpha}(t)r_{\alpha,j}$ [Ro12, 3.2, 3.5]. (Actually we often write $\alpha(t)$ for $\overline{\alpha}(t)$.)

The commutation relations between the u_α are deduced from the corresponding relations in the Lie algebra (or better in the Tits enveloping algebra $\mathcal{U}_{\mathcal{S}\mathbb{Z}}$). So we know well the structure of the Borel groups $\mathfrak{B}^{ma\pm}_{\mathcal{S}} = \mathfrak{T}_{\mathcal{S}} \ltimes \mathfrak{U}^{ma\pm}_{\mathcal{S}}$. For R a field there are Bruhat and Birkhoff decompositions of $\mathfrak{G}^{pma}_{\mathcal{S}}(R)$ and $\mathfrak{G}^{nma}_{\mathcal{S}}(R)$: $\mathfrak{G}^{pma}_{\mathcal{S}}(R) = \mathfrak{U}^{ma+}_{\mathcal{S}}(R).\mathfrak{N}_{\mathcal{S}}(R).\mathfrak{U}^{ma+}_{\mathcal{S}}(R) = \mathfrak{U}^{-}_{\mathcal{S}}(R).\mathfrak{N}_{\mathcal{S}}(R).\mathfrak{U}^{ma+}_{\mathcal{S}}(R)$ and $\mathfrak{G}^{nma}_{\mathcal{S}}(R) = \mathfrak{U}^{ma-}_{\mathcal{S}}(R).\mathfrak{N}_{\mathcal{S}}(R).\mathfrak{U}^{ma-}_{\mathcal{S}}(R) = \mathfrak{U}^{+}_{\mathcal{S}}(R).\mathfrak{N}_{\mathcal{S}}(R).\mathfrak{U}^{ma-}_{\mathcal{S}}(R)$.

1.10 centralizers of tori

Let $\mathfrak{T}'$ be a subtorus of $\mathfrak{T}_{\mathcal{S}}$ (over K_s). There is a linear map $Y(\mathfrak{T}')\otimes\mathbb{R} \hookrightarrow Y(\mathfrak{T}_{\mathcal{S}})\otimes\mathbb{R} \to V^q$ which sends $\lambda\otimes x$ to the map $\alpha \mapsto \overline{\alpha}(\lambda)x$. We write $V^q(\mathfrak{T}')$ its image. We say that $\mathfrak{T}'$ is *generic* (resp. *almost generic*) in $\mathfrak{T}_{\mathcal{S}}$ if $V^q(\mathfrak{T}')$ meets the interior of the Tits cone $\mathcal{T}^q_+$ (resp. if $V^q(\mathfrak{T}')\cap\mathcal{T}^q_+$ generates the vector space $V^q(\mathfrak{T}')$) *cf.* 1.2.1. Note that, if $\mathcal{S}$ is free, $\mathfrak{T}_{\mathcal{S}}$ is generic in $\mathfrak{T}_{\mathcal{S}}$, as the above map is then onto.

Proposition. *If $\mathfrak{T}'$ is generic, then, up to conjugacy, $V^q(\mathfrak{T}')$ is generated by $V^q(\mathfrak{T}')\cap\overline{C^{vq}_+}$ (which is convex) or more precisely by $V^q(\mathfrak{T}')\cap F^{vq}_+(J)$ where $F^{vq}_+(J)$ (with J spherical in I) is the greatest facet in $\overline{C^{vq}_+}$ meeting $V^q(\mathfrak{T}')$. Then the centralizer $\mathfrak{Z}(\mathfrak{T}')$ of $\mathfrak{T}'$ in $\mathfrak{G}_{\mathcal{S}}$ is $\mathfrak{G}_{\mathcal{S}(J)}$ (a reductive group).*

To be short, when $\mathfrak{T}'$ is generic, its centralizer $\mathfrak{Z}(\mathfrak{T}')$ is the group scheme $\mathfrak{Z}_g(\mathfrak{T}')$ generated by $\mathfrak{T}_{\mathcal{S}}$ and the $\mathfrak{U}_\alpha$ for $\alpha\in\Phi$ and $\overline{\alpha}_{|\mathfrak{T}'} = 1$ (called the generic centralizer of $\mathfrak{T}'$ in $\mathfrak{G}_{\mathcal{S}}$).

Proof. The reduction to $V^q(\mathfrak{T}')$ generated by $V^q(\mathfrak{T}')\cap F^{vq}_+(J)$ is clear as $V^q(\mathfrak{T}')\cap\mathcal{T}^q_+$ is convex and generates $V^q(\mathfrak{T}')$. We embed $\mathfrak{G}_{\mathcal{S}}$ in the ind-group-scheme $\mathfrak{G}^{pma}_{\mathcal{S}}$. Any element $g\in\mathfrak{G}^{pma}_{\mathcal{S}}(K_s)$ may be written uniquely as $g = (\prod_{\alpha\in\Phi^+\cap w\Phi^-} u_\alpha).t.\tilde{w}.(\prod_{\alpha\in\Delta^+} u_\alpha)$, where $t\in\mathfrak{T}_{\mathcal{S}}(K_s)$, $w\in W^v$, $\tilde{w}$ is its representant in a chosen system of representants $\widetilde{W^v}\subset\mathfrak{N}(\mathfrak{T}_{\mathcal{S}})(K_s)$ and each u_α is in $\mathfrak{U}_\alpha(K_s)$ *cf.* [Re02, 1.2.3] and [Ro12, 3.2]. If we conjugate by $s\in\mathfrak{T}'(K_s)$, t is fixed, each u_α is sent to $u'_\alpha\in\mathfrak{U}_\alpha(K_s)$. So g commutes with s if and only if $u'_\alpha = u_\alpha, \forall\alpha$ and $s\tilde{w}s^{-1} = \tilde{w}$. This last condition is $s = \tilde{w}s\tilde{w}^{-1} = w(s)$; as it must be true $\forall s\in\mathfrak{T}'(K_s)$, this means that $w\in W^v(J)$. Now for $\alpha\in\Phi$ and $u_\alpha = \mathfrak{x}_\alpha(r)$, $u'_\alpha = \mathfrak{x}_\alpha(\alpha(s).r)$; hence $u_\alpha = u'_\alpha, \forall s\in\mathfrak{T}'(K_s) \Rightarrow \overline{\alpha}_{|\mathfrak{T}'} = 1 \Rightarrow \alpha_{|V^q(\mathfrak{T}')} = 0 \Rightarrow \alpha\in Q(J)$. The same thing is true for $\alpha\in\Delta^+$ by the formulae in 1.9. Finally $g\in\mathfrak{Z}(\mathfrak{T}')(K_s) \iff g\in\mathfrak{G}^{pma}_{\mathcal{S}(J)}(K_s)$. But, as $F^{vq}_+(J)$ is in the interior of the Tits cone, J is spherical, $\Delta(J)$ is finite and $\mathfrak{G}^{pma}_{\mathcal{S}(J)} = \mathfrak{G}_{\mathcal{S}(J)}$ is a reductive group. □

Remarks 1.11. a) $\mathfrak{Z}(\mathfrak{T}')$ is the schematic centralizer of $\mathfrak{T}'$ or the centralizer of $\mathfrak{T}'(K_s)$. The centralizer of $\mathfrak{T}'(K)$ may be greater, *e.g.* if $|K| = 2$, $\mathfrak{T}_{\mathcal{S}}(K) = \{1\}$.

b) If $\mathfrak{T}'$ is almost generic, the above proof tells that $\mathfrak{Z}(\mathfrak{T}') = \mathfrak{G}^{pma}_{\mathcal{S}(J)}\cap\mathfrak{G}_{\mathcal{S}}$. But it is not clear that it is the Kac-Moody group $\mathfrak{G}_{\mathcal{S}(J)}$ *i.e.* that $\mathfrak{U}^{ma+}_{\mathcal{S}(J)}\cap\mathfrak{G}_{\mathcal{S}} = \mathfrak{U}^{ma+}_{\mathcal{S}(J)}\cap\mathfrak{U}^{+}_{\mathcal{S}}$ is $\mathfrak{U}^{+}_{\mathcal{S}(J)}$; *cf.* [Ro12, 3.17 and § 6].

c) In the affine case with $\mathcal{S}$ free, let δ be the smallest positive imaginary root. The torus $\mathfrak{T}'$ =Kerδ is not almost generic, there is no real root $\alpha\in\Phi$ with $\alpha_{|\mathfrak{T}'} = 1$ but $\mathfrak{Z}(\mathfrak{T}')$ is greater than $\mathfrak{T}_{\mathcal{S}}$: if $\mathfrak{G}_{\mathcal{S}}(K_s) = \mathfrak{G}^\circ(K_s[t,t^{-1}])\rtimes K_s^*$ for $\mathfrak{G}^\circ$ a semi-simple group with maximal torus $\mathfrak{T}^\circ$, then $\mathfrak{T}_{\mathcal{S}}(K_s) = \mathfrak{T}^\circ(K_s)\times K_s^*$, $\mathfrak{T}'(K_s) = \mathfrak{T}^\circ(K_s)$ and $\mathfrak{Z}(\mathfrak{T}')(K_s) = \mathfrak{T}^\circ(K_s[t,t^{-1}])\rtimes K_s^*$ is the subset of $\mathfrak{N}_{\mathcal{S}}(K_s)$ consisting of elements whose image in the affine Weyl group W^v are in the "translation group".

Otherwise said, when $\mathcal{S}$ is affine non free, $\overline{\delta} = 0$, $\mathfrak{T}' = \mathfrak{T}_{\mathcal{S}}$ and $\mathfrak{Z}(\mathfrak{T}_{\mathcal{S}})$ may be greater than $\mathfrak{T}_{\mathcal{S}}$: $\mathfrak{T}_{\mathcal{S}}$ is not almost generic in $\mathfrak{T}_{\mathcal{S}}$.

2 Almost split Kac-Moody groups

The reference for this section is B. Rémy's monograph [Re02].

2.1 Kac-Moody groups

1) A *Kac-Moody group* over the field K is a functor $\mathfrak{G} = \mathfrak{G}_K$ from the category $\mathcal{S}ep(K)$ to the category of groups such that there exist a RGS $\mathcal{S}$, a field $E \in \mathcal{S}ep(K)$ and a functorial isomorphism between the restrictions $\mathfrak{G}_E$ and $\mathfrak{G}_{\mathcal{S}E}$ of $\mathfrak{G}$ and $\mathfrak{G}_{\mathcal{S}}$ to $\mathcal{S}ep(E) = \{F \in \mathcal{S}ep(K) \mid E \subset F\}$. We say that $\mathfrak{G}$ is *split over* E, that $\mathfrak{G}$ is a $K-$*form of* $\mathfrak{G}_{\mathcal{S}}$ and we fix such a functorial isomorphism to identify $\mathfrak{G}_E$ and $\mathfrak{G}_{\mathcal{S}E}$.

The above condition is the most important but, to compensate the lack of a good notion of algebraicity, we need also a $K-$form $\mathcal{U} = \mathcal{U}_K$ of the Tits enveloping algebra $\mathcal{U}_{\mathcal{S}K_s}$ and some other technical conditions (PREALG1,2, SGR, ALG1,2) given in [Re02, chap. 11] and omitted here. We write only the following condition [*l.c.* 12.1.1] which makes more precise the functoriality:

(DCS2) For each extension L of K in $\mathcal{S}ep(K)$ the group $\mathfrak{G}(L)$ maps isomorphically to its canonical image in $\mathfrak{G}(K_s)$ which is the fixed-point-set $\mathfrak{G}(K_s)^{Gal(K_s/L)}$ of the Galois group.

We identify all these groups with their images in $\mathfrak{G}(K_s)$. We forget often the subscript $\mathcal{S}$ for subgroups of $\mathfrak{G}_{\mathcal{S}K_s}$ when we think of them as subgroups of $\mathfrak{G}_{K_s}$, *e.g.* $\mathfrak{B}^{\pm}{}_{\mathcal{S}K_s} = \mathfrak{B}^{\pm}{}_{K_s}$. Now the natural action of $\Gamma = Gal(K_s/K)$ on $\mathfrak{G}_{K_s}$ gives us a twisted action of Γ on $\mathfrak{G}_{\mathcal{S}K_s}$ such that $\mathfrak{G}_{\mathcal{S}}(K_s)^{Gal(K_s/L)} = \mathfrak{G}(L)$ for each $L \in \mathcal{S}ep(K)$ and $\mathfrak{G}(L) = \mathfrak{G}_{\mathcal{S}}(L)$ if $E \subset L$.

A subgroup H of $\mathfrak{G}(K_s)$ invariant under this twisted action of $Gal(K_s/L)$ defines a sub-group-functor $\mathfrak{H}_L$ on $\mathcal{S}ep(L)$; we say that H is $L-defined$ in $\mathfrak{G}(K_s)$ and that $\mathfrak{H}_L$ is a $L-$sub-group-functor of $\mathfrak{G}_L$.

2) We say that $\mathfrak{G}$ is *almost split* if the twisted action of each $\gamma \in \Gamma$ transforms $\mathfrak{B}^{\varepsilon}_{\mathcal{S}K_s}$ into a Borel subgroup in the same conjugacy class under $\mathfrak{G}_{\mathcal{S}}(K_s)$.

Let L be an infinite field in $\mathcal{S}ep(E)$, Galois over K, then there is a (twisted) action of $Gal(L/K)$ on the twin buildings $\mathscr{I}^{vc}_{\varepsilon}(L)$ such that the action of $\mathfrak{G}(L)$ on $\mathscr{I}^{vc}_{\varepsilon}(L)$ is $Gal(L/K)-$equivariant; this action permutes the types of the facets [*l.c.* 11.3.2]. One can extend affinely this action on the geometric realization $\mathscr{I}^{vq}_{\varepsilon}(L)$ [*l.c.* 12.1.2] or on the so-called "metric" realization, where the action is through a bounded group of isomorphisms; more precisely any point in this last realization has a finite orbit [*l.c.* 11.3.3, 11.3.4].

As a consequence any Borel subgroup of $\mathfrak{G}(K_s)$ is defined over a finite Galois extension of K; taking a greater extension K' such that $Gal(K_s/K')$ preserves the types, we see that the same thing is true for parabolic subgroups. A maximal torus of $\mathfrak{G}(K_s)$ is intersection of two opposite Borel subgroups, so it is also defined over a finite Galois extension of K.

3) If $\mathfrak{G}$ is almost split, the twisted action of Γ on $\mathfrak{G}_{\mathcal{S}}(K_s)$ and $\mathcal{U}_{\mathcal{S}K_s}$ is described through a "star action" [*l.c.* 11.2.2, 11.3.2]. More precisely there is a map $\gamma \mapsto g_\gamma$ from Γ to $\mathfrak{G}_{\mathcal{S}}(K_s)$ and for each $\gamma \in \Gamma$ an automorphism γ^* of $\mathfrak{G}_{\mathcal{S}}(K_s)$ and a $\gamma-$linear bijection γ^* of $\mathcal{U}_{\mathcal{S}K_s}$, such that the twisted actions are given by $\widetilde{\gamma} = \mathrm{Int}(g_\gamma) \circ \gamma^*$ on $\mathfrak{G}_{\mathcal{S}}(K_s)$ and $\widetilde{\gamma} = \mathrm{Ad}(g_\gamma) \circ \gamma^*$ on $\mathcal{U}_{\mathcal{S}K_s}$; moreover g_γ and γ^* are trivial for $\gamma \in Gal(K_s/E)$. On $\mathfrak{G}_{\mathcal{S}}(K_s)$, γ^* stabilizes $\mathfrak{T}_{\mathcal{S}}$ and $\mathfrak{B}^{\pm}{}_{\mathcal{S}}$; on $\mathcal{U}_{\mathcal{S}K_s}$, γ^* stabilizes $\mathcal{U}^0_{\mathcal{S}K_s}$ [*l.c.* 11.2.5(i)]. Actually g_γ is defined up to $\mathfrak{T}_{\mathcal{S}}(K_s)$ (but the map

$\gamma \mapsto g_\gamma$ may be chosen with a finite image, by 2.1.2 above). So γ^* is defined up to $\mathfrak{T}_{\mathcal{S}}(K_s)$. The "star action" is perhaps not an action on $\mathfrak{G}_{\mathcal{S}}(K_s)$ or $\mathcal{U}_{\mathcal{S}K_s}$, but it defines an action on $\mathcal{U}^0_{\mathcal{S}K_s}$, X, Δ, Φ, Φ^+, I or W^v.

Lemma 2.2. *Let $\mathfrak{G}$ be an almost split $K-$form of $\mathfrak{G}_{\mathcal{S}}$ as above.*

a) There is an almost split $K-$form $\mathfrak{G}^{xl}$ of $\mathfrak{G}_{\mathcal{S}^l}$ which is split over the same field E as $\mathfrak{G}$ and an homomorphism $\mathfrak{G} \to \mathfrak{G}^{xl}$ whose restriction to $\mathcal{S}ep(E)$ is the known homomorphism $\mathfrak{G}_{\mathcal{S}} \to \mathfrak{G}_{\mathcal{S}^l}$ [Ro12, 1.3d, 1.11].

b) Let L be an infinite field in $\mathcal{S}ep(E)$ Galois over K, then the action of $Gal(L/K)$ on the building $\mathscr{I}^{vc}_\varepsilon(L)$ may be extended linearly to $\mathscr{I}^{vxl}_\varepsilon(L)$. This action makes $\Gamma-$equivariant the action of $\mathfrak{G}^{xl}(K_s)$ $(=\mathfrak{G}(K_s))$ over this building and the essentialization map $\eta^v : \mathscr{I}^{vxl}_\varepsilon(L) \to \mathscr{I}^{vq}_\varepsilon(L)$.

c) When $\mathcal{S}$ is free, the same thing is true for $\mathscr{I}^{vx}_\varepsilon(L)$ and $\mathfrak{G}(K_s)$.

Remark. For $\mathbb{A}^v$ as in 1.2.4, let us suppose that the star action of Γ on I (hence on the α_i, $\alpha_i^\vee$) may be extended linearly to V. Then b) and c) above are also true for $\mathscr{I}^v(\mathfrak{G}_{\mathcal{S}}, K, \mathbb{A}^v)$.

Proof. a) We have to describe the form $\mathfrak{G}^{xl}$ and a $K-$form $\mathcal{U}^{xl}$ of the Tits enveloping algebra $\mathcal{U}_{\mathcal{S}^lK_s}$ through twisted actions of Γ on $\mathfrak{G}_{\mathcal{S}^l}(K_s)$ and $\mathcal{U}_{\mathcal{S}^lK_s}$. For the RGS $\mathcal{S}^l$, $Y^l = Y^{xl} = Y \oplus Q^*$, so, by [Ro12, 1.11], $\mathfrak{G}_{\mathcal{S}^lK_s}$ is the semi-direct product of $\mathfrak{G}_{\mathcal{S}K_s}$ by the torus $\mathfrak{T}_{QK_s} = \mathfrak{Spec}(K_s[Q])$. Clearly $\mathcal{U}_{\mathcal{S}^lK_s}$ is also a semi-direct product of $\mathcal{U}_{K_s} = \mathcal{U}_{\mathcal{S}K_s}$ and the "integral enveloping algebra" $\mathcal{U}_{QK_s}$ of the torus $\mathfrak{T}_{QK_s}$ (*i.e.* its algebra of distributions at the origin). Now the star action of Γ on Q gives a $\Gamma-$algebraic action on $\mathfrak{T}_{QK_s}$ and a $\Gamma-$linear action on $\mathcal{U}_{QK_s}$. This is compatible with the formulae defining semi-direct products and so we construct an automorphism γ^* of $\mathfrak{G}_{\mathcal{S}^l}(K_s)$ and a $\gamma-$linear bijection γ^* of $\mathcal{U}_{\mathcal{S}^lK_s}$. Now let $\widetilde{\gamma} =\text{Int}(g_\gamma)\circ\gamma^*$ or $\widetilde{\gamma} =\text{Ad}(g_\gamma)\circ\gamma^*$.

We have to prove that this defines actions of Γ. By definition $\text{Int}(g_{\gamma\gamma'})\circ(\gamma\gamma')^* = \widetilde{\gamma\gamma'}$ and $\widetilde{\gamma}\circ\widetilde{\gamma'} =\text{Int}(g_\gamma)\circ\gamma^*\circ\text{Int}(g_{\gamma'})\circ\gamma'^* =\text{Int}(g_\gamma.\gamma^*(g_{\gamma'}))\circ\gamma^*\circ\gamma'^*$. There is equality of these two expressions on $\mathfrak{G}_{\mathcal{S}}(K_s)$, moreover $(\gamma\gamma')^* = \gamma^*\circ\gamma'^*$ on $\mathfrak{T}_{\mathcal{S}K_s}$, hence $g_\gamma.\gamma^*(g_{\gamma'}) = g_{\gamma\gamma'}.t_{\gamma,\gamma'}$ with $t_{\gamma,\gamma'} \in \mathfrak{T}_{\mathcal{S}}(K_s)$. We have to verify that $\widetilde{\gamma\gamma'}(t) = \widetilde{\gamma}\circ\widetilde{\gamma'}(t)$ for $t \in \mathfrak{T}_Q(K_s)$. But $(\gamma\gamma')^*(t) = \gamma^*(\gamma'^*(t))$ is in $\mathfrak{T}_Q(K_s)$ hence centralized by $t_{\gamma,\gamma'}$; so the result follows. The same proof works also for $\mathcal{U}_{\mathcal{S}^lK_s}$.

We define $\mathcal{U}^{xl}$ as $(\mathcal{U}_{\mathcal{S}^lK_s})^{Gal(K_s/K)}$ and, for $L \in \mathcal{S}ep(K)$, $\mathfrak{G}^{xl}(L) = \mathfrak{G}_{\mathcal{S}^l}(K_s)^{Gal(K_s/K)}$ (fixed points for the twisted actions). We have now the two ingredients of the Kac-Moody group as defined above. We leave to the reader the verification of the technical conditions of [Re02] (PREALG, SGR, $\cdots$).

b,c) As the star action of Γ is well defined on $X^{xl} = X \oplus Q$ and on X, we just have to mimic the proof in the case $\mathscr{I}^{vq}_\varepsilon(L)$ (corresponding to $X^q = Q$) [*l.c.* 12.1.2].

□

2.3 Continuity of the actions of the Galois group

From now on in this section 2, we choose an almost split Kac-Moody group $\mathfrak{G}$ over K with Tits enveloping algebra $\mathcal{U}$ and keep the above notations. We forget now the (old) actions of $\Gamma = Gal(K_s/K)$ on $\mathfrak{G}_{\mathcal{S}K_s}$ or $\mathcal{U}_{\mathcal{S}K_s}$ and consider only the star action or the (twisted) action (which is the natural action on $\mathfrak{G}_{K_s}$ or $\mathcal{U}_{K_s}$).

1) By 2.1.2 above the orbits of Γ on the Borel subgroups of $\mathfrak{G}_{K_s}$ are finite. So the $g_\gamma \in \mathfrak{G}(K_s)$ (such that $\gamma^* =\text{Int}(g_\gamma)^{-1}\circ\gamma$ stabilizes $\mathfrak{T}_{K_s}$ and $\mathfrak{B}^\pm{}_{K_s}$) may be chosen in a finite set. In particular, if $\gamma^* = \mathrm{Ad}(g_\gamma)^{-1} \circ \gamma$ on $\mathcal{U}_{K_s}$, then $\{\gamma^* u \mid \gamma \in \Gamma\}$ is finite $\forall u \in \mathcal{U}_{K_s}$. So the star action of Γ has finite orbits on Φ [Re02, 11.2.5(iii)] and on Q. We know that this star action stabilizes the basis $\{\alpha_i \mid i \in I\}$ and acts on I by automorphisms of the Dynkin diagram.

2) The following extension of condition (ALG2) is implicit in *l.c.* starting *e.g.* from 11.3.2:

(ALG3) The star action of Γ on X or Y is continuous *i.e.* its orbits are finite.

As for (ALG2) this is useless if $X = Q$. Without it, only the description of the action of Γ on the center of $\mathfrak{G}(K_s)$ is less precise; in particular there is no problem for the buildings $\mathscr{I}_\varepsilon^{vq}(L)$. But the proof of [*l.c.* 12.5.1(i)] uses this property.

It would also be reasonable to ask:

(ALG3') The evident map $\varphi : Y \to Y \otimes K_s \subset \mathcal{U}^0_{K_s}$ is Γ^*-equivariant.

By [*l.c.* 11.2.5] the map $\varphi \circ bar : Q \to \mathcal{U}_{K_s}$ is Γ^*-equivariant. In characteristic 0 (ALG3) is a consequence of (ALG3').

In the following we add to the conditions of *l.c.* the condition (ALG3) but not (ALG3'). With these assumptions we get the good structure for $\mathfrak{G}$. But anybody interested in considering $\mathcal{U}$ as the good Tits enveloping algebra for $\mathfrak{G}$ should add (ALG3') and, in positive characteristic, even stronger conditions.

3) By 2.1.2 $\mathfrak{T}_{K_s}$ and $\mathfrak{B}^\pm{}_{K_s}$ are defined over a finite Galois extension L of K in $\mathcal{S}ep(K)$. Enlarging a little L we may suppose $\mathfrak{T}_{K_s}$ split over L (*i.e.* X or Y fixed pointwise under $Gal(K_s/L)$) and Q also fixed (2.3.1). Now we may modify each e_i in $K_s e_i = \mathcal{U}^+_{\alpha_i K s}$, so that e_i (and f_i) is fixed under Γ. By [*l.c.* 11.2.5(iii)] this proves that $\gamma(\mathfrak{x}_{\pm\alpha_i}(r)) = \mathfrak{x}_{\pm\alpha_i}(\gamma r)$ for $r \in K_s$ and $\gamma \in Gal(K_s/L)$. So the original action and the new twisted action of Γ on $\mathfrak{G}_\mathcal{S}(K_s)$ coincide on $\mathfrak{T}(K_s)$ and the groups $\mathfrak{U}_{\pm\alpha_i}(K_s)$. As these groups generate $\mathfrak{G}_\mathcal{S}(K_s)$ (see [Ro12, 1.6 KMT7]), the two actions coincide and $\mathfrak{G}$ is actually split over the finite galois extension L of K.

Now each of the above generators of $\mathfrak{G}(K_s)$ has a finite orbit under Γ, so this is also true for every element of $\mathfrak{G}(K_s)$: $\mathfrak{G}(K_s)$ is the union of the subgroups $\mathfrak{G}(L)$ for $L \in \mathcal{S}ep(K)$ with L/K finite.

4) We saw in 2.1.2 that the orbits of Γ on $\mathscr{I}_\varepsilon^{vc}(K_s)$ are finite. The stabilizer in Γ of a facet of $\mathscr{I}_\varepsilon^{vc}(K_s)$ acts on the corresponding facet of $\mathscr{I}_\varepsilon^{vq}(K_s)$, $\mathscr{I}_\varepsilon^{vxl}(K_s)$ or $\mathscr{I}_\varepsilon^{vx}(K_s)$ through a finite group (see 2) above and the definition of these actions). So the actions of Γ other these buildings have finite orbits.

If a Galois group $Gal(K_s/M)$ stabilizes a facet of one of these geometric buildings, then it has a fixed point in this facet (as this facet is a convex cone and the action is affine).

2.4 $K-$objects in the buildings

1) Let $E \in \mathcal{S}ep(K)$ be infinite, Galois over K and such that $\mathfrak{G}$ is split over E. By [Re02, 10.1.4 and 13.2.4] the buildings over E are the fixed point sets in the buildings over K_s of the Galois group $Gal(K_s/E)$. So we set $\Gamma = Gal(E/K)$ and we shall work over E (*cf. l.c.* 12.1.1(1)).

Let $\mathscr{I}^v = \mathscr{I}^v_+ \cup \mathscr{I}^v_-$ be the union $\mathscr{I}^v(\mathfrak{G}_\mathcal{S}, E, \mathbb{A}^v) = \mathscr{I}^v_+(\mathfrak{G}_\mathcal{S}, E, \mathbb{A}^v) \cup \mathscr{I}^v_-(\mathfrak{G}_\mathcal{S}, E, \mathbb{A}^v)$ as in remark 2.2 (*e.g.* $\mathbb{A}^v = \mathbb{A}^{vq}$, $\mathbb{A}^{vxl}$ or if $\mathcal{S}$ is free $\mathbb{A}^{vx}$). The essentialization $\mathscr{I}^{ve}$ of $\mathscr{I}^v$ is always $\mathscr{I}^{vq}(E) = \mathscr{I}^v(\mathfrak{G}_\mathcal{S}, E, \mathbb{A}^{vq})$ which is the building investigated in *l.c.* so one may use this reference.

2) Definitions. A $K-$*facet* (resp. *spherical* $K-$*facet*) in $\mathscr{I}^v$ is the fixed point set under Γ of a facet (resp. spherical facet) of $\mathscr{I}^v$ stable under Γ (by 2.3.4 the $K-$facet is non empty).

A $K-$*chamber* in $\mathscr{I}^v$ is a spherical $K-$facet with maximal closure.

A $K-$*apartment* in $\mathscr{I}^v$ is a generic subspace (of an apartment) of $\mathscr{I}^v$ which is (pointwise) fixed under Γ and maximal for these properties.

A (real) $K-$*wall* in a $K-$apartment ${}_KA^v$ is the intersection with ${}_KA^v$ of a wall in an apartment of $\mathscr{I}^v$ containing ${}_KA^v$, provided that this intersection contains a spherical $K-$facet. This $K-$wall divides ${}_KA^v$ into two (closed) $K-$*half-apartments*.

3) Properties. By definition $K-$facets (resp. spherical $K-$facets, $K-$chambers) correspond bijectively to $K-$defined parabolics (resp. $K-$defined spherical parabolics, minimal $K-$defined parabolics). The union of the $K-$facets is $(\mathscr{I}^v)^\Gamma$; their set is written ${}_K\mathscr{I}^{vc}$.

By [*l.c.* 12.2.4 and 12.3.1] two $K-$facets are always in a same $K-$apartment and there exists an integer $d = d(\mathscr{I}^{v\Gamma}) \geq 1$ such that each $K-$chamber or each $K-$apartment is of dimension d. One should notice that the different choices for $\mathscr{I}^v$ may give different integers d. The group $G = \mathfrak{G}(K)$ acts transitively on the pairs $({}_KC, {}_KA^v)$ of a $K-$chamber ${}_KC$ of given sign in a $K-$apartment ${}_KA^v$ (see also [*l.c.* 12.4.1]).

4) Standardizations. Any $K-$apartment ${}_KA^v$ in $\mathscr{I}^v$ is contained in a Galois stable apartment A^v of $\mathscr{I}^v$ (perhaps after enlarging a little E) [*l.c.* 12.3.2(1)]. We may choose moreover opposite chambers ${}_KC^v_+$, ${}_KC^v_-$ in ${}_KA^v$ and (non necessarily $\Gamma-$stable) opposite chambers C^v_+, C^v_- in A^v with ${}_KC^v_\pm \subset \overline{C^v_\pm}$. We say that $({}_KA^v, {}_KC^v_+, {}_KC^v_-)$ and (A^v, C^v_+, C^v_-) are *compatible standardizations* of $\mathscr{I}^{v\Gamma}$ and $\mathscr{I}^v$.

The apartment A^v determines a $K-$defined maximal torus $\mathfrak{T}_{K_s}$ (such that $A^v = A^v(\mathfrak{T}_{K_s})$). After enlarging a little E we may suppose $\mathfrak{T}_{K_s}$ split over E, and conjugated under $\mathfrak{G}(E)$ to the fundamental torus $\mathfrak{T}_{\mathcal{S}E}$ [*l.c.* 10.4.2]. So we may (and will) suppose $\mathfrak{T}_{K_s} = \mathfrak{T}_{\mathcal{S}K_s}$ and $C^v_\pm$ associated to the Borel subgroups $\mathfrak{B}^\pm_{\mathcal{S}E}$. Then the star action of Γ is defined by $\gamma^* = \mathrm{Int}(g_\gamma)^{-1} \circ \gamma$ with $g_\gamma \in \mathfrak{G}(E)$ normalizing $\mathfrak{T}_{\mathcal{S}}$ and fixing pointwise ${}_KC^v_+$, ${}_KC^v_-$ and ${}_KA^v$.

Let $I_0 = \{i \in I \mid \alpha_i({}_KA^v) = \{0\}\ \}$ and $A^{vI_0} = \{x \in A^v \mid \alpha_i(x) = 0, \forall i \in I_0\}$. Then I_0 is spherical (as ${}_KA^v$ meets spherical facets) and stable under Γ^*, the (normal twisted) action and the star action of Γ coincide on A^{vI_0} and ${}_KA^v = (A^{vI_0})^{\Gamma^*}$. The vector space generated by ${}_KA^v$ in $\overrightarrow{A^v}$ is $\overrightarrow{{}_KA^v} = \{v \in \overrightarrow{A^v} \mid \alpha_i(x) = 0, \forall i \in I_0\}^{\Gamma^*}$ [*l.c.* 12.6.1].

2.5 Maximal split tori and relative roots

We choose standardizations and identifications as in 2.4.4 above.

1) The maximal split subtorus $\mathfrak{S}$ of $\mathfrak{T}$ depends only on the $K-$apartment ${}_KA^v$ and is actually a maximal split torus in $\mathfrak{G}$. The maximal split tori are conjugated under $G = \mathfrak{G}(K)$ [Re02, 12.5.2, 12.5.3]. The dimension of a maximal split torus is the *reductive relative rank over* K of $\mathfrak{G}$, written $rrk_K(\mathfrak{G})$.

As a consequence of the lemma 2.6(ii) below, there is a bijection $\mathfrak{S} \mapsto {}_KA^v(\mathfrak{S})$ between maximal $K-$split tori in $\mathfrak{G}$ (or their points over K, if $|K| \geq 4$) and the $K-$apartments in $\mathscr{I}^v$.

2) Let ${}_KX$ (resp. ${}_KY$) be the group of characters (resp. cocharacters) of $\mathfrak{S}$. For each $\alpha \in Q$, let ${}_K\overline{\alpha} \in {}_KX$ be the restriction to $\mathfrak{S}$ of $\overline{\alpha} \in X$ and ${}_K\alpha$ be the restriction of α to $\overrightarrow{{}_KA^v}$. We define ${}_KQ$ as the image of Q by this restriction map $\alpha \mapsto {}_K\alpha$; the set of *relative* $K-$*roots* is ${}_K\Delta = \{{}_K\alpha \mid \alpha \in \Delta, {}_K\alpha \neq 0\}$; the set of *real relative* $K-$*roots* is ${}_K\Phi = {}_K\Delta^{re} = \{{}_K\alpha \in {}_K\Delta \mid {}_KA^v \cap \mathrm{Ker}\alpha$ is a (real) K-wall$\}$. Let ${}_KQ_{re}$ be the submodule of ${}_KQ$ generated by the real relative roots.

With the notations in 2.4.4, ${}_K\alpha_i$ is a root if and only if $i \notin I_0$; for $i, j \notin I_0$, ${}_K\alpha_i = {}_K\alpha_j$ if and only if i and j are in the same Γ^*-orbit; ${}_K\alpha_i$ is a real root if $I_0 \cup \Gamma^* i$ is spherical and different from I_0. Hence a basis of ${}_K\Delta$ (or ${}_KQ$) is given by $\{{}_K\alpha_i \mid i \in {}_KI\}$ where ${}_KI = (I \setminus I_0)/\Gamma^*$ and a basis of ${}_K\Phi$ (or ${}_KQ_{re}$) is given by $\{{}_K\alpha_i \mid i \in {}_KI_{re}\}$ where ${}_KI_{re} = \{i \in I \setminus I_0 \mid I_0 \cup \Gamma^* i \; spherical\}/\Gamma^*$: the basis of ${}_K\Delta$ may contain some imaginary relative roots. The set $\Phi_0 = \{\alpha \in \Phi \mid {}_K\alpha = 0\}$ is actually $\Phi \cap (\oplus_{i \in I_0} \mathbb{Z}\alpha_i)$. We say that $|{}_KI_{re}| = ssrk_K(\mathfrak{G})$ is the *semi-simple relative rank over* K of $\mathfrak{G}$.

For $i \in I_0$, α_i is trivial on $\mathfrak{S}$ (see 3) below); so the two actions of Γ (star or not) on $\mathfrak{T}$ coincide on $\mathfrak{S}$ and ${}_KY = \{y \in Y \mid \alpha_i(y) = 0, \forall i \in I_0\}^{\Gamma^*}$. Hence $\forall \alpha \in Q$, ${}_K\overline{\alpha}$ is the canonical image $\overline{{}_K\alpha}$ of ${}_K\alpha$ in ${}_KX$. It is now clear that $\{\overline{{}_K\alpha} \mid {}_K\alpha \in {}_K\Delta\}$ is the set of roots of $\mathfrak{S}$ for the adjoint representation on the Lie algebra $\mathfrak{g}_K \subset \mathcal{U}_K$.

When $\mathcal{S}$ is free and $\mathscr{I}^v = \mathscr{I}^{vx}(E)$ is the normal geometric realization, then $dim({}_KA^v) = dim(\mathfrak{S})$ is the reductive relative rank. Hence, when $\mathcal{S}$ is free, the reductive relative rank is at least 1: an almost split Kac-Moody group (with $\mathcal{S}$ free) cannot be anisotropic.

When $\mathscr{I}^v$ is the essential building $\mathscr{I}^{vq}(E)$, then $\dim({}_KA^v) = |{}_KI|$ may be greater than $|{}_KI_{re}| = ssrk_K(\mathfrak{G})$, so ${}_KA^v$ may be inessential.

3) Relative Weyl group. [*l.c.* 12.4.1, 12.4.2]

Let ${}_KN$ (resp. ${}_KZ$) be the stabilizer (resp. fixator) of ${}_KA^v$ in G; by 2.5.1 ${}_KN$ is the normalizer of $\mathfrak{S}$ in G and ${}_KZ$ centralizes $\mathfrak{S}$ (by definition of $\mathfrak{S}$ [*l.c.* 12.5.2]). Actually ${}_KZ$ is generated by T and the U_α for $\alpha \in \Phi_0$ [*l.c.* 6.4.1], hence $\alpha(\mathfrak{S}) = 1 \; \forall \alpha \in \Phi_0$. The quotient group ${}_KW^v = {}_KN/{}_KZ$ is the *relative Weyl group of* $\mathfrak{G}$ (associated to $\mathfrak{S}$ or ${}_KA^v$). It acts simply transitively on the $K-$chambers of fixed sign in ${}_KA^v$ and (as ${}_KN \subset N.{}_KZ$) is induced by the action of the subgroup of $W^v = W^v(A^v)$ stabilizing ${}_KA^v$.

To each real relative root ${}_K\alpha \in {}_K\Phi$ is associated an element $s_{{}_K\alpha} \in {}_KW^v$ of order 2 which fixes the wall $\mathrm{Ker}({}_K\alpha)$. The pair $({}_KW^v, \{s_{{}_K\alpha_i} \mid i \in {}_KI_{re}\})$ is a Coxeter system.

When $\mathcal{S}$ is free, the map $Y(\mathfrak{T}_\mathcal{S}) \otimes \mathbb{R} \to \overrightarrow{A^{vq}} = (Q \otimes \mathbb{R})^*$ is onto. But ${}_KA^{vq}$ is generic in A^{vq} (2.4.2) and, by 2.4.4 and 2.5.2, the same equations define $\overrightarrow{{}_KA^{vq}}$ in $\overrightarrow{A^{vq}}$ or $Y(\mathfrak{S}) \otimes \mathbb{R}$ in $Y(\mathfrak{T}_\mathcal{S}) \otimes \mathbb{R}$. So $\mathfrak{S}$ is generic in $\mathfrak{T}_\mathcal{S}$, $\mathfrak{Z}(\mathfrak{S}) = \mathfrak{Z}_g(\mathfrak{S})$ and ${}_KZ = \mathfrak{Z}(\mathfrak{S})(K)$ (1.10).

For $\mathcal{S}$ general ${}_KZ = \mathfrak{Z}_g(\mathfrak{S})(K)$ may be smaller than $\mathfrak{Z}(\mathfrak{S})(K)$, *cf.* 1.11c. The reductive group $\mathfrak{Z}_g(\mathfrak{S})$ is the *anisotropic kernel* [Re02, 12.3.2] associated to ${}_KA^v$ *i.e.* to $\mathfrak{S}$ (by 1) above).

4) The set ${}_K\Delta$ (resp. ${}_K\Phi = {}_K\Delta^{re}$) is a system of roots (resp. of real roots) in the sense of [Ba96] *cf.* [Re02, 12.6.2], [Ba96] or [B3R95]. If ${}_K\Delta^\pm = \pm({}_K\Delta \cap (\oplus_{i \in {}_KI} \mathbb{Z}_{\geq 0}.{}_K\alpha_i))$ (resp. ${}_K\Phi^\pm = \pm({}_K\Phi \cap (\oplus_{i \in {}_KI_{re}} \mathbb{Z}_{\geq 0}.{}_K\alpha_i))$ then ${}_K\Delta = {}_K\Delta^+ \sqcup {}_K\Delta^-$ and ${}_K\Phi = {}_K\Phi^+ \sqcup {}_K\Phi^-$. The system ${}_K\Delta$ or ${}_K\Phi$ is stable under ${}_KW^v$ and any real relative root is of the form $w.{}_K\alpha_i$ or $2w.{}_K\alpha_i$ with $w \in {}_KW^v$ and $i \in {}_KI_{re}$ (as the system may be unreduced).

It is not too hard to find a RGS $({}_K\mathbb{M}, {}_KY, ({}_K\overline{\alpha_i})_{i \in {}_KI_{re}}, ({}_K\alpha_i^\vee)_{i \in {}_KI_{re}})$ (in the sense of 1.1) with Weyl group ${}_KW^v$. But it is not sufficient to describe ${}_K\Delta$ (or even ${}_K\Phi$); one has to use a more complicated notion of RGS, see [Ba96], [B3R95] or [Re02, 12.6.2]. On the contrary the reduced system ${}_K\Phi_{red}$ is a system of real roots in the sense of [MoP89], [MoP95] and even of [K90] as its basis is free.

2.6 Relative root groups

For ${}_K\alpha \in {}_K\Phi$, we consider the finite set $({}_K\alpha) = \{\beta \in \Phi \mid {}_K\beta \in \mathbb{N}.{}_K\alpha\}$ and the unipotent group $\mathfrak{U}_{({}_K\alpha)K_s}$ generated by the $\mathfrak{U}_{\beta K_s}$ for $\beta \in ({}_K\alpha)$, it is defined over K. We set $V_{{}_K\alpha} = \mathfrak{U}_{({}_K\alpha)}(K)$.

The positive multiples of ${}_K\alpha$ in ${}_K\Delta$ are ${}_K\alpha$ and (eventually) $2{}_K\alpha$ ($\in {}_K\Phi$). If $2{}_K\alpha \notin {}_K\Delta$ we set $\mathfrak{U}_{(2{}_K\alpha)K_s} = \{1\}$ and $V_{2{}_K\alpha} = \{1\}$. So $\mathfrak{U}_{(2{}_K\alpha)K_s}$ (resp. $V_{2{}_K\alpha}$) is always a normal subgroup of $\mathfrak{U}_{({}_K\alpha)K_s}$ (resp. $V_{{}_K\alpha}$) [Re02, 12.5.4].

Lemma. *Let ${}_K\alpha \in {}_K\Phi$ be a real relative root.*

(i) $V_{{}_K\alpha}/V_{2{}_K\alpha}$ is isomorphic to a vector space over K on which $s \in \mathfrak{S}(K)$ acts by multiplication by ${}_K\overline{\alpha}(s) \in K^*$. Its dimension is $|({}_K\alpha)| - |(2{}_K\alpha)| > 0$.

(ii) The centralizer $Z_{V_{{}_K\alpha}}(\mathfrak{S}(K))$ of $\mathfrak{S}(K)$ in $V_{{}_K\alpha}$ is trivial if $|K| \geq 4$.

Proof. (suggested in [*l.c.* 12.5.3]) By [*l.c.* 12.5.4] there exists a reductive $K-$group $\mathfrak{H}$ of relative semi-simple rank 1 containing $\mathfrak{S}$ and $\mathfrak{U}_{({}_K\alpha)}$. Then (i) is classical *cf. e.g.* [BoT65, th. 3.17]. Now (for $\mathfrak{H}$) there exists a coroot ${}_K\alpha^\vee \in \mathrm{Hom}(\mathfrak{Mult}, \mathfrak{S})$ such that $2{}_K\alpha({}_K\alpha^\vee) = 2$ or ${}_K\alpha({}_K\alpha^\vee) = 2$ (if $2{}_K\alpha$ is not a root) (one may use the $K-$split reductive subgroup of $\mathfrak{H}$ constructed in [BoT65, 7.2]). So (ii) follows. □

Theorem 2.7. *([Re02, 12.6.3 and 12.4.4]) Let $\mathfrak{G}$ be an almost split Kac-Moody group over K, then,*

a) The triple $(\mathfrak{G}(K), (V_{{}_K\alpha})_{{}_K\alpha\in{}_K\Phi}, {}_KZ)$ is a generating root datum of type ${}_K\Phi$.

b) The fixed point set ${}_K\mathscr{I}^v = (\mathscr{I}^v)^\Gamma$ is a good geometrical representation of the combinatorial twin building ${}^K\mathscr{I}^{vc} = \mathscr{I}^{vc}(\mathfrak{G}, K)$ associated to this root datum: there are $\mathfrak{G}(K)-$equivariant bijections, between the $K-$apartments and the apartments of ${}^K\mathscr{I}^{vc}$, and between the $K-$chambers and the chambers of ${}^K\mathscr{I}^{vc}$; this last bijection is compatible with adjacency and opposition.

N.B. 1) When $\mathfrak{G}$ is already split over K, we see easily, using galleries, that ${}^K\mathscr{I}^{vc} = (\mathscr{I}^{vc})^\Gamma$ and ${}_K\mathscr{I}^v = \mathscr{I}^v(\mathfrak{G}, K, \mathbb{A}^v)$.

2) The group ${}_KB^+ = {}_KZU^+$ defined in 1.6.1 for this root datum is a minimal $K-$parabolic of $\mathfrak{G}$. It is a Borel subgroup if and only if there exist Borel subgroups defined over K (*i.e.* $\mathfrak{G}$ is *quasi split* over K); this is equivalent to $I_0 = \emptyset$ *i.e.* to $\mathfrak{Z}_g(\mathfrak{S})$ being a torus.

3) The objects defined in 1.4 to 1.6 for the above root datum will bear a left or right index $_K$, sometimes a left exponent ^{K}.

2.8 Comparison with a Weyl geometric realization

1) With the notations in 2.4.4, 2.5, we may describe the positive $K-$facets:

$C^v_+ = \{x \in \overrightarrow{A^b} \mid \alpha_i(x) > 0, \forall i \in I\}$

${}_KC^v_+ = \{x \in \overrightarrow{{}_KA^b} \mid {}_K\alpha_i(x) > 0, \forall i \in {}_KI\}$ (relative interior of $\overline{C^v_+} \cap \overrightarrow{{}_KA^b}$)

${}_KA^v = \cup_{w\in{}_KW^v} w.\overline{{}_KC^v_+}$

The $K-$facets in $\overline{{}_KC^v_+}$ correspond bijectively to subsets ${}_KJ$ of ${}_KI$ by setting:

${}_KF^v_+({}_KJ) = \{x \in \overrightarrow{{}_KA^b} \mid {}_K\alpha_i(x) > 0, \forall i \in {}_KI \setminus {}_KJ \, and \, {}_K\alpha_i(x) = 0, \forall i \in {}_KJ\}$.

so the definition of the $K-$facets uses the whole ${}_K\Delta$ (not only ${}_K\Phi$).

Moreover ${}_KF^v_+({}_KJ)$ is spherical if and only if ${}_K\underline{J} = I_0 \cup \{i \in I \mid \Gamma^* i \in {}_KJ\}$ is spherical, which is equivalent to ${}_KJ \subset {}_KI_{re}$ and ${}_KJ$ spherical in ${}_KI_{re}$ (as defined by the root system ${}_K\Phi$) *cf.* [Ba96, p 163, p 175].

2) A Weyl geometric realization ${}_K\mathscr{I}^v_+ = \mathscr{I}^v_+(\mathfrak{G}, K, {}^KA^v)$ of the combinatorial building ${}^K\mathscr{I}^{vc}_+$ can be constructed using, for fundamental apartment and facets, subcones of the vector space $\overrightarrow{{}_KA^b}$ defined using ${}_K\Phi$ (*i.e.* ${}_KW^v$). The corresponding Weyl facets in the closure of the positive fundamental chamber are defined as:

${}^KF^v_+({}_KJ) = \{x \in \overrightarrow{{}_KA^b} \mid {}_K\alpha_i(x) > 0, \forall i \in {}_KI_{re} \setminus {}_KJ \, and \, {}_K\alpha_i(x) = 0, \forall i \in {}_KJ\}$ for ${}_KJ \subset {}_KI_{re}$ and the positive fundamental Weyl$-K-$apartment is ${}^KA^v_+ = \bigcup_{w\in{}_KW^v, {}_KJ\subset{}_KI_{re}} w.{}^KF^v_+({}_KJ)$.

The building ${}^K\mathscr{I}^v$ is the disjoint union of the Weyl$-K-$facets associated to the parabolics of $(\mathfrak{G}(K),(V_{K\alpha})_{K\alpha\in_K\Phi},{}_KZ)$; it contains ${}_K\mathscr{I}^v$ by the following lemma. Its minimal facet is ${}^KF^v_+({}_KI_{re})=(\overrightarrow{{}_KA^v})_0$.

3) Lemma. *Let ${}_KJ\subset{}_KI_{re}$.*

a) The intersection ${}^KF^v_+({}_KJ)\cap{}_KA^v_+$ is the disjoint union of the $K-$facets ${}_KF^v_+({}_KJ')$ for ${}_KJ'\supset{}_KJ$ and ${}_KJ'\cap{}_KI_{re}={}_KJ$.

Among these $K-$facets the maximal one (for the inclusion of the closures) is ${}_KF^v_+({}_KJ)$, which is open in ${}^KF^v_+({}_KJ)$; moreover ${}^KF^v_+({}_KJ)={}_KF^v_+({}_KJ)+(\overrightarrow{{}_KA^v})_0$. The minimal one corresponds to ${}_KJ'={}_KJ\cup({}_KI\setminus{}_KI_{re})$.

b) The Weyl$-K-$facet ${}^KF^v_+({}_KJ)$ is spherical if and only if ${}_KF^v_+({}_KJ)$ is spherical and then this $K-$facet is the only spherical $K-$facet in ${}^KF^v_+({}_KJ)\cap{}_KA^v_+$.

c) The Weyl$-K-$facet ${}^KF^v_+({}_KJ)$ and all $K-$facets ${}_KF^v$ in ${}^KF^v_+({}_KJ)\cap{}_KA^v_+$ have the same fixator $P^+_K({}_KJ)=P_K({}_KF^v)$ in $\mathfrak{G}(K)$. Hence each $K-$facet of ${}_K\mathscr{I}^v_+$ is associated to a unique Weyl$-K-$facet in ${}^K\mathscr{I}^v_+$;

Proof. Let $w.\overline{{}_KC^v_+}$ (with $w\in{}_KW^v$) be a closed $K-$chamber meeting ${}^KF^v_+({}_KJ)$, then $w.\overline{{}^KC^v_+}$ meets ${}^KF^v_+({}_KJ)$; so $w\in{}_KW^v({}_KJ)$ which fixes (pointwise) ${}^KF^v_+({}_KJ)$. Hence $w.\overline{{}_KC^v_+}\cap{}^KF^v_+({}_KJ)\subset\overline{{}_KC^v_+}$ and ${}_KA^v_+\cap{}^KF^v_+({}_KJ)\subset\overline{{}_KC^v_+}$. Now a) and b) are clear.

The fixator in $\mathfrak{G}(K)$ of ${}_KF^v_+({}_KJ')$ contains the fixator P of ${}_KC^v_+$, hence it is a parabolic subgroup of the positive $BN-$pair associated to the root datum in $\mathfrak{G}(K)$ *i.e.* of the form $P.{}_KW^v({}_KJ'').P$ for some ${}_KJ''\subset{}_KI_{re}$. It is easy to check that ${}_KJ''$ has to be ${}_KJ$ and c) follows. □

4) So the Weyl$-K-$facets of ${}^K\mathscr{I}^{vc}_+$ correspond to some $K-$facets of $(\mathscr{I}^v_+)^\Gamma$ and there is a good correspondence between spherical Weyl$-K-$facets and spherical $K-$facets. But, if ${}_KI_{re}\neq{}_KI$, some non-spherical $K-$facets correspond to nothing in ${}^K\mathscr{I}^{vc}_+$. So $(\mathscr{I}^v)^\Gamma$ is only a geometric representation of ${}^K\mathscr{I}^{vc}_+$ in the sense of the theorem, it is not really a geometric realization of it. Note also that, if ${}_KI_{re}\neq{}_KI$, the Weyl geometric realization ${}^K\mathscr{I}^v_+$ of ${}^K\mathscr{I}^{vc}_+$, constructed in 2) above, is not essential, even if $\mathscr{I}^v=\mathscr{I}^{vq}$ is.

The above results (and those in 2.9) are well illustrated by example 13.4 in [Re02].

5) Remarks. a) In this example we see also that ${}_K\Phi$ may be a classical (finite) root system, even if Φ is infinite. It may also happen that ${}_K\Phi$ is empty (*i.e.* $ssrk_K(\mathfrak{G})=0$); this is always the case when Φ is infinite and $|{}_KI|=1$ (see examples for $K=\mathbb{R}$ in the tables of [B3R95]). Then $\mathfrak{G}(K)={}_KZ$ and $(\mathscr{I}^v)^\Gamma$ is reduced to one $K-$apartment and two $K-$chambers (one of each sign).

b) On the contrary, if Φ is infinite, ${}_K\Delta$ is always infinite and $|{}_KI|\geq 1$.

c) Actually some vectorial facets (*e.g.* the minimal one V_0) are positive and negative. So to associate a maximal $K-$facet to a Weyl$-K-$facet, we may have to make a choice of a sign, at least if ${}_KI_{re}\neq{}_KI$.

6) Lemma. *let $\mathfrak{G}$ be an almost split Kac-Moody group defined over K', with $K'\subset K\subset E$, E/K', K/K' Galois and $(\mathfrak{G}_K,E)$ as above (cf. 2.4.4).*

a) Let ${}_{K'}A^v\subset{}_KA^v\subset A^v$ be respectively a $K'-$apartment in ${}_{K'}\mathscr{I}^v$, a $K-$apartment in ${}_K\mathscr{I}^v$ and an apartment in $\mathscr{I}^v$ (stable under $Gal(E/K')$ or not). Then the $K-$facets or $K'-$facets are described in A^v as in 1) above with help of ${}_KI$ or ${}_{K'}I$.

b) The action of $Gal(E/K')$ on $\mathscr{I}^v$ induces an action of $Gal(K/K')$ on ${}_K\mathscr{I}^v$ which may be extended (linearly and uniquely) to ${}^K\mathscr{I}^v$.

Proof. There is a star-action of $Gal(E/K)$ (resp. $Gal(E/K')$) on A^v (and its vector space $\overrightarrow{A^v}$) and a subset I_0^K (resp. $I_0^{K'}$) of I which describes entirely ${}_KA^v$ (resp. ${}_{K'}A^v$); this is independent of the choice of A^v, as different choices are conjugated [Re02, prop. 6.2.3 (i)]. They describe also the $K-$facets (resp. $K'-$facets), so a) follows. The action of $Gal(K/K')$ on ${}_K\mathscr{I}^v = \mathfrak{G}(K).{}_KA^v$ is described through its action on ${}_K\mathscr{I}^{vc}$ and its star action on ${}_KA^v$ which may be extended (linearly and uniquely) to ${}^KA^v$. So b) is a consequence of 3)c above. □

2.9 Imaginary relative root groups

1) Let's consider ${}_K\alpha \in {}_K\Delta^{im}$. The sets $(\mathbb{Z}_{>0}.{}_K\alpha) \cap ({}_K\Delta)$ and $({}_K\alpha) = \{\beta \in \Delta \mid {}_K\beta \in \mathbb{Z}_{>0}.{}_K\alpha\}$ are infinite [B3R95, 3.3.2].

We saw in 1.9 that $\mathfrak{G}_E$ is embedded in some ind-group-scheme. If ${}_K\alpha$ is positive (resp. negative) we can define in the pro-unipotent group-scheme $\mathfrak{U}_E^{ma+}$ (resp. $\mathfrak{U}_E^{ma-}$) a pro-unipotent subgroup-scheme $\mathfrak{U}^{ma}_{({}_K\alpha)E}$ such that the elements of $U^{ma}_{({}_K\alpha)} = \mathfrak{U}^{ma}_{({}_K\alpha)E}(E)$ are written uniquely as infinite products: $u = \prod_{\beta\in({}_K\alpha)} \prod_{j=1}^{j=n_\beta} [exp]\lambda_{\beta,j}.e_{\beta,j}$ where $(e_{\beta,j})_{j=1,n_\beta}$ is a basis of $\mathfrak{g}_\beta$ ($n_\beta = 1$ for β real) and $\lambda_{\beta,j} \in E$. Moreover the conjugate of such an element $u \in U^{ma}_{({}_K\alpha)}$ by $s \in \mathfrak{S}(E)$ is $\prod_{\beta\in({}_K\alpha)} \prod_{j=1}^{j=n_\beta} [exp]\beta(s).\lambda_{\beta,j}.e_{\beta,j}$.

We define the root group corresponding to ${}_K\alpha$ as $V_{{}_K\alpha} = U^{ma}_{({}_K\alpha)} \cap \mathfrak{G}(K)$.

2) Lemma. *The group $G = \mathfrak{G}(K)$ has an extra large (abstract) center: it contains $S_Z = \{s \in \mathfrak{S}(K) \mid {}_K\alpha_i(s) = 1\,,\, \forall i \in {}_KI_{re}\}$.*

Proof. As $\mathfrak{S}(K)$ is in the center of ${}_KZ$ and G is generated by ${}_KZ$ and the groups $V_{{}_K\alpha}$ for ${}_K\alpha \in {}_K\Phi$, this result is a consequence of lemma 2.6 (i). □

3) Remarks and definition a) The schematic center of G, *i.e.* the centralizer in G of $\mathfrak{G}(K_s)$, is $\{s \in \mathfrak{T}(K) \mid \alpha_i(s) = 1\,,\, \forall i \in I\}$ [Re02, 9.6.2]. Hence its intersection with $\mathfrak{S}(K)$ is smaller than S_Z in general.

b) If ${}_K\alpha \in {}_K\Delta$, we can write uniquely ${}_K\alpha = \pm(\sum_{i\in {}_KI} n_i.{}_K\alpha_i)$ with $n_i \in \mathbb{Z}_{\geq 0}$. We shall say that ${}_K\alpha$ is *almost real* and write ${}_K\alpha \in {}_K\Delta^r$ if and only if $n_i = 0\ \forall i \in {}_KI \setminus {}_KI_{re}$. Hence ${}_K\Phi = {}_K\Delta_{re} \subset {}_K\Delta^r \subset {}_K\Delta$. This set ${}_K\Delta^r$ is a system of roots in the sense of [Ba96, 2.4.1].

c) By the following lemma the non trivial root groups $V_{{}_K\alpha}$ correspond to roots ${}_K\alpha \in {}_K\Delta^r$. So it is natural to abandon the $K-$facets (defined using ${}_K\Delta$) and to use the Weyl$-K-$facets of 2.8.2 (defined using ${}_K\Phi$ or ${}_K\Delta^r$).

We may define ${}^K\Delta = \{{}_K\alpha \in {}_K\Delta \mid V_{{}_K\alpha} \neq \{1\}\}$, so ${}_K\Phi \subset {}^K\Delta \subset {}_K\Delta^r$ (by the following).

4) Lemma. *If ${}_K\alpha \in {}_K\Delta \setminus {}_K\Delta^r$ (hence ${}_K\alpha \in {}_K\Delta_{im}$), then $V_{{}_K\alpha} = \{1\}$.*

Proof. Suppose $\mathcal{S}$ free, K infinite and ${}_K\alpha \in {}_K\Delta \setminus {}_K\Delta^r$, then $\forall n \in \mathbb{Z}_{>0}$ we have $(n.{}_K\alpha)(S_Z) \neq \{1\}$. But the conjugation by $s \in S_Z$ of an element of G (resp. $U^{ma}_{({}_K\alpha)}$) is trivial (resp. given by the formulae in 1) above). Hence $V_{{}_K\alpha} = \{1\}$.

When $\mathcal{S}$ is not free we obtain the same result by using $\mathfrak{G}^{xl}$ *cf.* 2.2a. When K is finite, the (schematic) centralizer $\mathfrak{Z}$ of $\mathfrak{S}$ is a $K-$quasi-split reductive group with $\mathfrak{S}$ as maximal $K-$split torus; so $\mathfrak{Z}$ is a torus. Now $\mathfrak{Z}$ splits over a Galois extension of degree D. If L is an infinite union of extensions of degree prime to D, $\mathfrak{S}$ is still maximal $K-$split over L and the wanted result is true over L. The result over K is then clear. □

3 Valuations and affine apartments

Definition 3.1. A *valuation* of a root datum $(G, (U_\alpha)_{\alpha\in\Phi}, Z)$ of type a real root system Φ is a family $(\varphi_\alpha)_{\alpha\in\Phi}$ of maps $\varphi_\alpha : U_\alpha \to \mathbb{R}\cup\{+\infty\}$ satisfying the following axioms:

(V0) $\forall\alpha\in\Phi$, $|\varphi_\alpha(U_\alpha)|\geq 3$

(V1) $\forall\alpha\in\Phi$, $\forall\lambda\in\mathbb{R}\cup\{+\infty\}$, $U_{\alpha,\lambda}=\varphi_\alpha^{-1}([\lambda,+\infty])$ is a subgroup of U_α and $U_{\alpha,\infty}=\{1\}$

(V2.1) $\forall\alpha,\beta\in\Phi$, $\forall u\in U_\alpha\setminus\{1\}$, $\forall v\in U_\beta\setminus\{1\}$,

$$\varphi_{r_\alpha(\beta)}(m(u)vm(u)^{-1})=\varphi_\beta(v)-\beta(\alpha^\vee)\varphi_\alpha(u)$$

(V2.2) $\forall\alpha\in\Phi$, $\forall t\in Z$, the map $v\mapsto\varphi_\alpha(v)-\varphi_\alpha(tvt^{-1})$ is constant on $U_\alpha\setminus\{1\}$

(V3) For each prenilpotent pair of roots $\{\alpha,\beta\}$ and all $\lambda,\mu\in\mathbb{R}$, the commutator group $[U_{\alpha,\lambda},U_{\beta,\mu}]$ is contained in the group generated by the groups $U_{p\alpha+q\beta,p\lambda+q\mu}$ for $p,q\in\mathbb{Z}_{>0}$ and $p\alpha+q\beta\in\Phi$

(V4) If $\alpha\in\Phi$ and $2\alpha\in\Phi$, then $\varphi_{2\alpha}$ is the restriction of $2\varphi_\alpha$ to $U_{2\alpha}$.

Remarks. 1) This definition appears in [Ch11, 10.2.1]. A weaker definition is given in [Ro06, 2.2]; there axiom (V2.1) is replaced by axioms named (V2a) and (V5). In the classical case, both definitions are equivalent to the original one of [BrT72, 6.2.1], *cf.* [Ch11, 10.2.3.2]. Actually (V2.1) is then the proposition 6.2.7 of [BrT72]. This definition may be extended to RGD-systems for a family $(\varphi_\alpha)_{\alpha\in\Phi_{nd}}$: in (V3) just allow p and q to be in $\mathbb{R}_{>0}$; in (V2.1) if $r_\alpha(\beta)=\lambda\gamma$ with $\lambda>0$ and $\gamma\in\Phi_{nd}$, replace $\varphi_{r_\alpha(\beta)}$ by $\lambda\varphi_\gamma$.

2)We define $\Lambda_\alpha=\varphi_\alpha(U_\alpha\setminus\{1\})\subset\mathbb{R}$. From (V2.1) with $\alpha=\beta$, $u=v$ we get $\Lambda_\alpha=-\Lambda_{-\alpha}$. For u,u',u'' as in 1.4 (RD4), we have $\varphi_{-\alpha}(u')=\varphi_{-\alpha}(u'')=-\varphi_\alpha(u)$, [Ch11, 11.1.11]. For $\lambda\in\mathbb{R}$, we set $U_{\alpha,\lambda+}=\varphi_\alpha^{-1}(]\lambda,+\infty])$

3) Let $Q=\mathbb{Z}\Phi$ be the $\mathbb{Z}-$module generated by Φ and $V^q=(Q\otimes\mathbb{R})^*$. Then using this (strong) definition one can build an action of the group N (defined in 1.6.1) over V^q (this seems impossible with the weaker definition of [Ro06]):

Proposition 3.2. *cf. [Ch11, prop. 11.1.9, 11.1.10] There exists a unique action ν^q of N over V^q by affine transformations such that:*

- $\forall t\in Z$, $\nu^q(t)$ is the translation by the vector $\overrightarrow{v_t}$ such that $\alpha(\overrightarrow{v_t})=\varphi_\alpha(u)-\varphi_\alpha(tut^{-1})$, $\forall\alpha\in\Phi$ and $\forall u\in U_\alpha\setminus\{1\}$,

- $\forall n\in N$, $\nu^q(n)$ is an affine automorphism with associated linear map $\overrightarrow{\nu^q(n)}=\nu^v(n)$.

3.3 Valuation for a split Kac-Moody group

Let $\mathfrak{G}=\mathfrak{G}_\mathcal{S}$ be a split Kac-Moody group over K, as in 1.3. We suppose the base field K endowed with a non trivial real valuation $\omega=\omega_K : K\to\mathbb{R}\cup\{+\infty\}$. Its *ring of integers* (resp. *maximal ideal*, *residue field*) is $\mathcal{O}=\mathcal{O}_K=\omega^{-1}([0,+\infty])$ (resp. $\mathfrak{m}=\mathfrak{m}_K=\omega^{-1}(]0,+\infty])$, $\kappa=\mathcal{O}/\mathfrak{m}$) and $\Lambda=\Lambda_K=\omega(K^*)$ is its *value group*. An important particular case (the *discrete case*) is when Λ is discrete in $\mathbb{R}$.

Let $u=x_\alpha(r)\in U_\alpha$ with $\alpha\in\Phi$ and $r\in K$, we set $\varphi_\alpha(u)=\omega(r)\in\mathbb{R}\cup\{+\infty\}$.

Proposition. *The family $(\varphi_\alpha)_{\alpha\in\Phi}$ is a valuation of the root datum $(G,(U_\alpha)_{\alpha\in\Phi},T)$.*

Proof. Clear except for (V2.1) proved in [Ch11, 10.2.3.1]. □

Remark. We have $\Lambda_\alpha=\Lambda$, $\forall\alpha\in\Phi$.

3.4 Affine apartments

We consider an abstract valuated root datum as in 3.1.

1) Let $V = \overrightarrow{\mathbb{A}}$ be a real vector space with $\Phi \subset Q \subset V^*$ and $(\alpha_i^\vee)_{i\in I} \subset V$ as in 1.2.4.

For $\lambda \in \mathbb{R}$ and $\alpha \in Q \setminus \{0\}$, we define the affine hyperplane $M(\alpha, \lambda) = \{x \in V \mid \alpha(x)+\lambda = 0\}$ of direction Kerα, the closed half-space $D(\alpha, \lambda) = \{x \in V \mid \alpha(x) + \lambda \geq 0\}$ and its interior $D^\circ(\alpha, \lambda) = \{x \in V \mid \alpha(x) + \lambda > 0\}$. For $\alpha \in \Phi$ the reflection $s_M = s_{\alpha,\lambda}$ with respect to $M = M(\alpha, \lambda)$ is the affine reflection with associated linear map $\overrightarrow{s_M} = s_\alpha$ and with fixed point set M.

We suppose V endowed with an action ν of N such that, $\forall n \in N$, $\nu(n)$ is an affine automorphism with associated linear map $\overrightarrow{\nu(n)} = \nu^v(n)$. We ask moreover that, for $\alpha \in \Phi$ and $u \in U_\alpha \setminus \{1\}$, $\nu(m(u))$ is the reflection $s_{\alpha,\varphi_\alpha(u)}$. We write $Z_0 =$Ker$\nu \subset Z$.

Then $t \in Z = \mathrm{Ker}(\nu^v)$ acts on V by a translation of vector $\overrightarrow{v_t}$. The action ν commutes with the translations by V_0 and the induced action on the essential quotient $V^q = V/V_0$ is ν^q as defined in proposition 3.2: as $m(tut^{-1}) = tm(u)t^{-1}$, we have clearly $\alpha(\overrightarrow{v_t}) = \varphi_\alpha(u) - \varphi_\alpha(tut^{-1})$.

As a consequence, $\forall n \in N$, $\forall \alpha \in \Phi$ and $\forall u \in U_\alpha \setminus \{1\}$ we have $\nu(n).D(\alpha, \varphi_\alpha(u)) = D(\nu^v(n).\alpha, \varphi_{\nu^v(n).\alpha}(nun^{-1}))$ and the same thing for the walls [Ch11, 11.1.10].

For $v \in V$, we may define a new valuation φ' (*equipollent* to φ) by $\varphi'_\alpha(u) = \varphi_\alpha(u) + \alpha(v)$ for $\alpha \in \Phi$. This corresponds to choosing for V a new origin $0_{\varphi'} = v$.

2) Definitions. a) A *wall* (resp. an *half-apartment*) in V is an hyperplane (resp. a closed half-space) of the form $M(\alpha, \varphi_\alpha(u)) = V^{m(u)}$ (fixed point set) (resp. $D(\alpha, \varphi_\alpha(u))$ with $\alpha \in \Phi$ and $u \in U_\alpha \setminus \{1\}$. The action ν of N permutes the walls and half-apartments.

b) The *affine apartment* $\mathbb{A}$ is V considered as an affine space and endowed with its family $\mathscr{M}$ of walls and the corresponding reflections. It is called *semi-discrete* if, $\forall \alpha \in \Phi$, the set of walls of direction Kerα is locally finite, *i.e.* if Λ_α is discrete in $\mathbb{R}$. Its essentialization is $\mathbb{A}^e = \mathbb{A}/V_0$ endowed with the image of the family $\mathcal{M}$.

A preorder is defined on $\mathbb{A}$ or $\mathbb{A}^e$ by $\quad x \leq y \iff y - x \in \mathcal{T}_+$.

c) An *automorphism* of $\mathbb{A}$ is an affine bijection $\varphi : \mathbb{A} \to \mathbb{A}$ stabilizing the family $\mathscr{M}$ of walls and conjugating the corresponding reflections. We ask also that its associated linear map $\overrightarrow{\varphi}$ stabilizes Φ (this is automatic in the semi-discrete case with Φ reduced and Λ_α independent of α) and the union $\mathcal{T}_+ \cup \mathcal{T}_-$ of the Tits cones (this is automatic in the classical case). Then $\overrightarrow{\varphi}$ normalizes the vectorial Weyl group W^v and transforms vectorial facets into vectorial facets.

d) We say that an automorphism φ is *positive* (or of first kind) (resp. *type-preserving*) if $\overrightarrow{\varphi}(\mathcal{T}_\pm) = \mathcal{T}_\pm$ (resp. $\overrightarrow{\varphi}$ preserves the types of the vectorial facets).

e) The *(affine) Weyl group* $W^a = W^a(\mathbb{A})$ of $\mathbb{A}$ is the subgroup of Aut($\mathbb{A}$) generated by the reflections s_M for $M \in \mathscr{M}$. Its elements are called *Weyl-automorphisms* of $\mathbb{A}$.

f) An *apartment of type* $\mathbb{A}$ is a set A endowed with a set $Isom_{W^a}(\mathbb{A}, A)$ of bijections $f : \mathbb{A} \to A$ (called *Weyl isomorphisms*) such that if $f_0 \in Isom_{W^a}(\mathbb{A}, A)$, then $f \in Isom_{W^a}(\mathbb{A}, A)$ if, and only if, there exists $w \in W^a$ such that $f = f_0 \circ w$.

g) An *isomorphism* between two apartments A and A' is a bijection $\varphi : A \to A'$ such that for some $f_0 \in Isom_{W^a}(\mathbb{A}, A)$ and $f'_0 \in Isom_{W^a}(\mathbb{A}, A')$ (the choices have no importance) the map $(f'_0)^{-1} \circ \varphi \circ f_0$ is an automorphism of $\mathbb{A}$. We say that this φ is *positive*, *type-preserving* or a *Weyl isomorphism* if $(f'_0)^{-1} \circ \varphi \circ f_0$ is positive, type-preserving or a Weyl automorphism; compare with [Ro11, 1.13].

3) Remarks. a) By definition N and W^a act on $\mathbb{A}$ by positive, type-preserving automorphisms. The Weyl group W^a is a normal subgroup of $\nu(N)$. They are not always equal.

b) If φ is an automorphism of $\mathbb{A}$, $\overrightarrow{\varphi}$ stabilizes Φ and $\mathcal{T}_+ \cup \mathcal{T}_-$; so, up to W^v, $\pm\overrightarrow{\varphi}$ stabilizes the basis $(\alpha_i)_{i\in I}$ and Σ or $(\alpha_i^\vee)_{i\in I}$. The induced action on I preserves the Kac-Moody matrix $\mathbb{M}$ and its Dynkin diagram (it is a diagram automorphism); it is trivial if φ is type-preserving. In particular when φ is positive and type-preserving we have $\overrightarrow{\varphi} \in W^v$.

c) We define $G^\emptyset$ (resp. $N^\emptyset$) as the subgroup of G (resp. N) generated by Z_0 and the groups U_α (resp. by Z_0 and the $m(u)$, $u \in U_\alpha \setminus \{1\}$) for $\alpha \in \Phi$. It is normal in G (resp. N) and $G = G^\emptyset.Z$ (resp. $N = N^\emptyset.Z$). By definition $\nu(N^\emptyset) = W^a$ and even $N^\emptyset = \nu^{-1}(W^a)$ is the group of Weyl automorphisms in N. We set $Z^\emptyset = N^\emptyset \cap Z$ which is normal in Z.

By [BrT72, 6.1.2 (12)] $(G^\emptyset, (U_\alpha)_{\alpha\in\Phi}, Z^\emptyset)$ is a generating root datum of type Φ. The associated group "N" (as in 1.6.1) is $N^\emptyset$. Comparing the refined Bruhat decompositions (1.6.1) of $G^\emptyset$ and G, we obtain $G^\emptyset \cap N = N^\emptyset$. Compare with [Ro11, 6.2].

4) Imaginary roots We consider moreover a set Δ_{im} in V^* of *imaginary roots* with $\Delta_{im} \cap (\cup_{\alpha\in\Phi} \mathbb{R}\alpha) = \emptyset$ and Δ_{im} W^v-stable; we write $\Delta_{re} = \Phi$ and $\Delta = \Phi \cup \Delta_{im}$. The best example for Δ is a root system as in [Ba96] with Φ as system of real roots (it can be *e.g.* the root system generated by Φ as in 1.1.3 or, if $\Phi = {}_K\Phi$, the system ${}_K\Delta$ as in 2.5). The *totally imaginary* choice Δ^{ti} for Δ corresponds to $\Delta^{ti}_{im} = V^* \setminus (\cup_{\alpha\in\Phi} \mathbb{R}\alpha)$.

We say that Δ is *tamely imaginary* [Ro11, 1.1] (resp. *relatively imaginary*) if $\Delta_{im} = \Delta^+_{im} \cup \Delta^-_{im}$ with W^v-stable sets $\Delta^\pm{}_{im} = \pm(\Delta \cap (\oplus_{i\in I} \mathbb{R}^+\alpha_i))$ (resp. $\Delta^\pm{}_{im} = \pm(\Delta \cap (\oplus_{i\in I_\Delta} \mathbb{R}^+\alpha_i))$, where $I_\Delta \supset I$ is finite and $(\alpha_i)_{i\in I_\Delta}$ is free). Remark that ${}_K\Delta$ (as defined in 2.5.2) is always relatively imaginary and is tamely imaginary if and only if it is equal to ${}_K\Delta^r$: 2.9.3b.

For all $\alpha \in \Delta_{im}$, we consider an infinite subset $\Lambda_\alpha = -\Lambda_{-\alpha}$ of $\mathbb{R}$. We define the system $\mathcal{M}^i$ of *imaginary walls* as the set of affine hyperplanes $M(\alpha, \lambda)$ for $\alpha \in \Delta_{im}$ and $\lambda \in \Lambda_\alpha$ (actually the real walls are given by the same formula for $\alpha \in \Phi$). We ask that these walls are permuted by $\nu(N) \supset W^a$, more precisely $\Lambda_{w\alpha} = \Lambda_\alpha$, $\forall w \in \nu(N)$.

For $\alpha \in \Delta$ and $k \in \mathbb{R}$, we sometimes say that $M(\alpha, k)$ (resp. $D(\alpha, k)$) is a *true* or *ghost* wall (resp. half-apartment), according to the fact that $k \in \Lambda_\alpha$ or $k \notin \Lambda_\alpha$.

5) Remarks. a) Actually these imaginary roots or walls will be used only to define enclosures, hence facets and chimneys (3.6). It would be possible to modify the vectorial facets (hence the sectors, facets, chimneys,...) with Δ_{im} (as in 2.8) in the relatively imaginary case (this changes nothing in the tamely imaginary case). But it seems useless for us: see 2.9.4 and section 6.

b) Let φ be an automorphism of $\mathbb{A}$. then $\overrightarrow{\varphi}$ stabilizes Δ^+_{im} and Δ^-_{im} or exchanges them if φ is a Weyl automorphism (by definition) or if Δ is generated by Φ [Ba96, 4.2.15, 4.2.20 and 2.4.1]. We say in general that φ is *imaginary-compatible* if $\overrightarrow{\varphi}(\Delta^+_{im}) = \Delta^\pm_{im}$ and φ permutes the imaginary walls (automatic *e.g.* if $\Lambda_\alpha = \mathbb{R}$, $\forall \alpha \in \Delta_{im}$).

3.5 Affine apartments for a split Kac-Moody group

We consider the group and valuation as in 3.3.

1) We can build easily examples of pairs (V, ν) as in 3.4.1. We choose a commutative extension of RGS $\varphi : \mathcal{S} \to \mathcal{S}' = (\mathbb{M}, Y', (\alpha'_i)_{i\in I}, (\alpha'^\vee_i)_{i\in I}$ with $\mathcal{S}'$ free and we set $V = Y' \otimes \mathbb{R}$.

There is an action ν_T of T over V by translations: for $t \in T$, $\nu_T(t)$ is the translation of vector $\nu_T(t)$ such that $\chi(\nu_T(t)) = -\omega(\overline{\chi}(t))$ for $\chi \in X'$ and $\overline{\chi} = \varphi^*(\chi) \in X$. In other words ν_T is the map $-(\varphi \otimes \omega)$ from $T = Y \otimes_{\mathbb{Z}} K^*$ to $V = Y' \otimes_{\mathbb{Z}} \mathbb{R}$. This action is W^v-equivariant.

By [Ro06, 2.9] there exists an affine action ν of N over V whose restriction to T is ν_T and satisfying the properties asked in 3.4.1. Actually $N/\mathrm{Ker}\nu_T$ is a semi-direct product by $\mathrm{Ker}\nu_T$ of a group isomorphic to W^v and generated by the images of $m(x_{\alpha_i}(\pm 1))$ for $i \in I$. This last group will fix the origin of V.

For $\mathcal{S}'$ we may choose $\mathcal{S}_{\mathbb{M}m}$, $\mathcal{S}^l$ or (if $\mathcal{S}$ is free) $\mathcal{S}$ itself. We get thus $V = V^q, V^{xl}$ or V^x and corresponding affine apartments $\mathbb{A} = \mathbb{A}^q, \mathbb{A}^{xl}$ or $\mathbb{A}^x$

2) Remarks. Suppose (V, ν) as in 1) above.

a) The kernel Z_0 =$\mathrm{Ker}\nu_T$ = $\mathrm{Ker}\nu$ of ν contains the group $\mathfrak{T}(\mathcal{O}) = Y \otimes \mathcal{O}^* \simeq (\mathcal{O}^*)^n$ of points of $\mathfrak{T}$ over $\mathcal{O}$. It is actually equal to it except when the image of the map $\varphi^* : X' \to X$, $\chi \mapsto \overline{\chi}$ has infinite index *i.e.* when φ is not injective.

b) We have $\nu(N) = W^v \ltimes (\overline{Y} \otimes_{\mathbb{Z}} \Lambda)$ and $W^a = W^v \ltimes (\overline{Q^\vee} \otimes_{\mathbb{Z}} \Lambda)$, where $\overline{Y}$ (resp. $\overline{Q^\vee}$) is the image by φ of Y (resp. $Q^\vee = \sum_{i \in I} \mathbb{Z}\alpha_i^\vee \subset Y$) in V. So there is equality in the simply connected case (in a strong sense: $Y = Q^\vee$) and only in this case when φ is injective (*e.g.* $V = V^{xl}$ or $V = V^x$) and ω discrete.

3) General affine apartments a) We consider now any pair (V, ν) as in 3.4.1. But we add the condition (useful in section 5) that the kernel $Z_0 = \mathrm{Ker}\nu$ contains $\mathfrak{T}(\mathcal{O})$. We speak then of a *suitable apartment* for $(\mathfrak{G}_{\mathcal{S}}, \mathfrak{T}_{\mathcal{S}})$; apartments defined in 1) are suitable.

Then $\nu_{|T}$ induces a $\mathbb{Z}-$linear map $\overline{\nu} : Y \otimes \Lambda \to V$ and this map sends $\alpha_i^\vee \otimes \lambda$ to $-\lambda\alpha_i^\vee$: $\alpha_i^\vee \otimes \lambda$ is the class modulo $\mathfrak{T}(\mathcal{O})$ of $\alpha_i^\vee(r) \in \mathfrak{T}(K)$ with $\omega(r) = \lambda$. But $\alpha_i^\vee(r) = m(x_{-\alpha_i}(1))^{-1}.m(x_{-\alpha_i}(r))$ by [Ro12, 1.5, 1.6], so by the hypothesis in 3.4.1, $\nu(\alpha_i^\vee(r)) = s_{-\alpha_i,0} \circ s_{-\alpha_i,\omega(r)}$ which is the translation of vector $-\lambda\alpha_i^\vee$. In particular the $\mathbb{Z}-$linear relations between the $\alpha_i^\vee$ in Y are also satisfied in V.

By 3.4.1 and 3.3, we have also $\alpha(\overline{\nu}(y \otimes \lambda)) = -\alpha(y).\lambda$.

b) We choose Δ_{im} as in 1.1.3 *i.e.* generated by Φ [K90, 5.4], [Ba96, 2.4.1]. We have $\Lambda_\alpha = \Lambda$, $\forall \alpha \in \Phi$ and we set $\Lambda_\alpha = \Lambda$, $\forall \alpha \in \Delta_{im}$. The system $\mathcal{M}$ of walls is discrete (resp. semi-discrete) if and only if we are in the classical discrete case (resp. if the valuation is discrete).

If φ is an automorphism of $\mathbb{A}$ and $\alpha \in \Delta$, $\lambda \in \Lambda$, then $x \in \varphi(M(\alpha, \lambda)) \iff 0 = \alpha(\varphi^{-1}(x)) + \lambda = \alpha(\varphi^{-1}(0)) + \overrightarrow{\varphi}(\alpha)(x) + \lambda$. So $\varphi(M(\alpha, \lambda))$ is a (real or imaginary) wall (of direction $\mathrm{Ker}\overrightarrow{\varphi}(\alpha)$) if and only if $\alpha(\varphi^{-1}(0)) \in \Lambda$. By hypothesis this is true for $\alpha \in \Phi$, so this is also true for $\alpha \in \Delta_{im} \subset Q$; hence φ permutes the imaginary walls. Therefore any automorphism of $\mathbb{A}$ is imaginary-compatible.

3.6 Enclosures, facets, sectors and chimneys

We come back to the general abstract case of 3.4; the following notions depend only on $\mathbb{A}$ (with $\mathcal{M}$) and $\mathcal{M}^i$.

We consider filters in $\mathbb{A}$ as in [GR08] or [Ro08], [Ro11], [Ro12]. The reference for the following is [Ro12] or [Ro11]. The *support* of a filter in $\mathbb{A}$ is the smallest affine subspace in $\mathbb{A}$ containing it. We identify a subset in $\mathbb{A}$ to the filter whose elements are the subsets of $\mathbb{A}$ containing this subset. We use definitions for filters (inclusion, union, closure, (pointwise) fixation or stabilization by a group) which coincide with the usual ones for sets when these filters are associated to subsets.

1) If F is a filter in $\mathbb{A}$, we define several types of *enclosures* for F (corresponding to different choices for the family of real or imaginary walls) *cf.* [Ro12, 4.2.5]: if $\mathcal{P} \subset \Delta$ and $\forall \alpha \in \mathcal{P}$ $\Lambda_\alpha \subset \Lambda'_\alpha \subset \mathbb{R}$, $cl^{\mathcal{P}}_{\Lambda'}(F)$ is the filter made of the subsets of $\mathbb{A}$ containing an element of F of the form $\cap_{\alpha \in \mathcal{P}} D(\alpha, \lambda_\alpha)$ with, for each $\alpha \in \mathcal{P}$, $\lambda_\alpha \in \Lambda'_\alpha \cup \{+\infty\}$; in particular each

$D(\alpha,\lambda_\alpha)$ contains the filter F *i.e.* is an element of this filter. When $\Lambda'_\alpha=\Lambda_\alpha$ (resp. $\Lambda'_\alpha=\mathbb{R}$) $\forall\alpha$, we write $cl^{\mathcal{P}}:=cl^{\mathcal{P}}_{\Lambda}$ (resp. $cl^{\mathcal{P}}_{\mathbb{R}}:=cl^{\mathcal{P}}_{\Lambda'}$); when $\Lambda'_\alpha=\Lambda_\alpha$ $\forall\alpha\in\Phi$ and $\Lambda'_\alpha=\mathbb{R}$ $\forall\alpha\in\Delta^{im}$ we write $cl^{\mathcal{P}}_{ma}:=cl^{\mathcal{P}}_{\Lambda'}$.

We define $cl^{\#}(F)$ (resp. $cl^{\#}_{\mathbb{R}}(F)$) as the filter made of the subsets of $\mathbb{A}$ containing an element of F of the form $\cap_{j=1}^{k} D(\beta_j,\lambda_j)$ for $\beta_j\in\Phi$ and $\lambda_j\in\Lambda_{\beta_j}\cup\{+\infty\}$ (resp. $\lambda_j\in\mathbb{R}\cup\{+\infty\}$); $cl^{\#}$ is the enclosure map used by Charignon [Ch11, sec. 11.1.3].

In [GR08] (resp. [Ro11] or [Ro12]) one uses cl^{Δ} (resp. cl^{Δ}_{ma}, $cl^{\Delta}_{\mathbb{R}}$, cl^{Φ}, $cl^{\Phi}_{\mathbb{R}}$ or cl^{Δ}, cl^{Φ}, $cl^{\#}$) under the names cl (resp. cl, $cl_{\mathbb{R}}$, cl^{si}, $cl^{si}_{\mathbb{R}}$ or cl, cl^{si}, $cl^{\#}$).

One has: $cl^{\#}(F)\supset cl^{\Phi}(F)\supset cl^{\Delta}(F)\supset cl^{\Delta}_{ma}(F)\supset cl^{\Delta}_{\mathbb{R}}(F)\supset cl^{\Delta^{ti}}(F)=\overline{conv}(F)$ (closed convex hull), $cl^{\Phi}(F)\supset cl^{\Phi}_{\mathbb{R}}(F)\supset cl^{\Delta}_{\mathbb{R}}(F)$ and some other clear inclusions.

The maps $cl^{\Delta^{ti}}_{\mathbb{R}}=\overline{conv}$, cl^{Φ},$cl^{\Phi}_{\mathbb{R}}$, $cl^{\#}$, $cl^{\#}_{\mathbb{R}}$ (resp. cl^{Δ}, $cl^{\Delta^{ti}}$, cl^{Δ}_{ma}, $cl^{\Delta}_{\mathbb{R}}$) are equivariant with respect to automorphisms (resp. imaginary-compatible automorphisms) of $\mathbb{A}$.

In the following, we choose one of these enclosure maps which we call cl. We say that F is *enclosed* or *cl−enclosed* if $F=cl(F)$.

2) A *local-facet* is associated to a point x in $\mathbb{A}$ and a vectorial facet F^v in $\overrightarrow{\mathbb{A}}$; it is the filter $F^l(x,F^v)=germ_x(x+F^v)$ intersection of $x+F^v$ with the filter of neighbourhoods of x in $\mathbb{A}$.

The *facet* or *cl−facet* associated to $F^l(x,F^v)$ and the enclosure map $cl=cl^{\mathcal{P}}_{\Lambda'}$ (resp. $cl=cl^{\#}$ or $cl=cl^{\#}_{\mathbb{R}}$) is the filter $F(x,F^v)=F^{\mathcal{P}}_{\Lambda'}(x,F^v)$ (resp. $F^{\#}(x,F^v)$ or $F^{\#}_{\mathbb{R}}(x,F^v)$) made of the subsets containing an intersection (resp. a finite intersection) of half spaces $D(\alpha,\lambda_\alpha)$ or $D^\circ(\alpha,\lambda_\alpha)$ (at most one $\lambda_\alpha\in\Lambda'_\alpha$ for each $\alpha\in\mathcal{P}$) (resp. with $\alpha\in\Phi$ and $\lambda_\alpha\in\Lambda_\alpha$ or $\lambda\in\mathbb{R}$) such that this intersection contains $F^l(x,F^v)$ *i.e.* a neighbourhood of x in $x+F^v$.

The *closed-facet* $\overline{F}(x,F^v)$ is the closure of $F(x,F^v)$, also $\overline{F}(x,F^v)=cl(F(x,F^v))=cl(F^l(x,F^v))$. Note that $F^l=F^{\Delta^{ti}}_{\mathbb{R}}\subset F^{\Delta}_{\mathbb{R}}\subset F^{\Phi}_{\mathbb{R}}=F^{\#}_{\mathbb{R}}=F^l+V_0$ and $\overline{F}^l=\overline{F}^{\Delta^{ti}}_{\mathbb{R}}\subset\overline{F}^{\Delta}_{\mathbb{R}}\subset\overline{F}^{\Phi}_{\mathbb{R}}=\overline{F}^{\#}_{\mathbb{R}}=\overline{F}^l+V_0$, where V_0 is as defined in 1.2.4.

These facets are called *spherical* (resp. *positive*, *negative*) if F^v is. When F^v is a vectorial chamber, these facets are *chambers* hence spherical.

3) A *sector* (resp. *sector-face*) is a $V-$translate $\mathfrak{q}=x+C^v$ (resp. $\mathfrak{f}=x+F^v$) of a vectorial chamber C^v (resp. vectorial facet F^v). A *shortening* of a sector or sector-face $\mathfrak{f}=x+F^v$ is a sector or sector-face $\mathfrak{f}'=x'+F^v$ included in $\mathfrak{f}$. The *germ* of a sector $\mathfrak{q}=x+C^v$ (resp. sector-face $\mathfrak{f}=x+F^v$) is the filter $\mathfrak{Q}=germ_\infty(\mathfrak{q})$ (resp. $\mathfrak{F}=germ_\infty(\mathfrak{f})$) made of the subsets containing shortenings of $\mathfrak{q}$ (resp. $\mathfrak{f}$). The *direction* of $\mathfrak{f}=x+F^v$ or of its germ is F^v, its *sign* is the sign of F^v. When F^v is spherical, we say that $\mathfrak{f}$ and $\mathfrak{F}$ are spherical or *splayed* ("évasé" in [Ro11] or [Ro12]). The *vertex* x of $\mathfrak{f}=x+F^v$ is well defined by $\mathfrak{f}$ when $\mathbb{A}$ is essential.

4) A *chimney* or *cl−chimney* is associated to a facet $F=F(x,F^v_0)$ (its *base*) and a vectorial facet F^v; it is the filter $\mathfrak{r}(F,F^v):=cl(F+F^v)=cl(F^l(x,F^v_0)+F^v)=:\mathfrak{r}(F^l,F^v)$ (containing $cl(F)+\overline{F^v}=\overline{F}+\overline{F^v}$). If $cl=cl^{\mathcal{P}}_{\Lambda'}$, we write $\mathfrak{r}^{\mathcal{P}}_{\Lambda'}(F,F^v)=\mathfrak{r}(F,F^v)$.

A *shortening* of $\mathfrak{r}(F,F^v)$ (with $F=F(x,F^v_0)$) is defined by $\xi\in\overline{F^v}$, it is the chimney $\mathfrak{r}(F(x+\xi,F^v_0),F^v)$. The germ of this chimney is the filter $\mathfrak{R}(F,F^v)$ made of the subsets containing a shortening of $\mathfrak{r}(F,F^v)$. The *direction* of $\mathfrak{r}(F,F^v)$ or $\mathfrak{R}(F,F^v)$ is F^v, its *sign* is the sign of F^v, it is said *splayed* if F^v is spherical and *solid* (resp. *full*) if the direction of its support has a finite fixator in W^v (resp. if its support is $\mathbb{A}$).

For example the enclosure $cl(\mathfrak{f})$ of a sector-face $\mathfrak{f}=x+F^v$ is a chimney of direction F^v; its germ is splayed if and only if $\mathfrak{f}$ is spherical, it is full if (but not only if) it is a sector. A facet is a chimney and a chimney germ with direction the minimal vectorial facet $V_0=F^v_{\pm}(I)$; it is splayed or solid if and only if it is spherical, it is full if and only if it is a chamber.

N.B. In [Ro77] a chimney is a specific set among the sets of the chimney as defined above, the chimney germs are the same. In [CaL11] P.E. Caprace and J. Lecureux introduce (generalized) sectors in any combinatorial building; in the classical discrete case for $\mathbb{A}$, these sectors are the enclosures of a facet and a chimney germ.

3.7 Bordered apartments

Following [Ch11], we shall add to $\mathbb{A}$ some other apartments at infinity, see also [Ro11].

1) Façades : For F^v a vectorial facet in V, we consider the sets $\Delta^m(F^v) = \{\alpha \in \Delta \mid \alpha(F^v) = 0\}$ and $\Phi^m(F^v) = \Phi \cap \Delta^m(F^v)$ of roots. They are clearly systems of roots: if $F^v = F^v(J)$, then $\Delta^m(F^v) = \Delta^m(J)$ and $\Phi^m(F^v) = \Phi^m(J)$.

We define $\mathbb{A}^{ne}_{F^v}$ as the affine space V endowed with the set $\mathscr{M}(F^v) = \{M(\alpha,\lambda) \mid \alpha \in \Phi(F^v), \lambda \in \Lambda\}$ of walls, the corresponding reflections and $\mathscr{M}^i(F^v)$ defined similarly using $\Delta(F^v)$. Its points are written (x, F^v) with $x \in V$. The essentialization $\mathbb{A}^e_{F^v}$ of $\mathbb{A}^{ne}_{F^v}$ is the quotient of V by the vector space $\overrightarrow{F^v}$ generated by F^v (with the corresponding walls and reflections); the class of $x \in V$ in $\mathbb{A}^e_{F^v}$ is written $[x + F^v]$.

When $F^v_1 \in F^{v*}$ *i.e.* $F^v \subset \overline{F^v_1}$, we have $\overrightarrow{F^v} \subset \overrightarrow{F^v_1}$; so there is a projection $pr_{F^v_1}$ of $\mathbb{A}^e_{F^v}$ onto $\mathbb{A}^e_{F^v_1}$: $pr_{F^v_1}([x + F^v]) = [x + F^v_1]$. We also write $pr_{F^v_1}$ the evident map from $\mathbb{A}^{ne}_{F^v}$ onto $\mathbb{A}^{ne}_{F^v_1}$ or $\mathbb{A}^e_{F^v_1}$.

Following [Ch11], we say that $\mathbb{A}^e_{F^v}$ (resp. $\mathbb{A}^{ne}_{F^v}$) is the (essential) *façade* (resp. *non essential façade*) of $\mathbb{A}$ in the *direction* F^v. A façade is called *spherical* (resp. *positive* or *negative*) if its direction is spherical (resp. positive, or negative). The same things as in 3.6 may be defined in each façade.

2) Bordered apartments : Let $\overline{\overline{\mathbb{A}}}$ (resp. $\overline{\mathbb{A}}^e$) be the disjoint union of all $\mathbb{A}^{ne}_{F^v}$ (resp. $\mathbb{A}^e_{F^v}$) for F^v a vectorial facet in V and let $\overline{\mathbb{A}}^i$ be the disjoint union of $\mathbb{A}$ and all $\mathbb{A}^e_{F^v}$ for F^v a non trivial vectorial facet in V. Then $\overline{\overline{\mathbb{A}}}$ (resp. $\overline{\mathbb{A}}^e$, $\overline{\mathbb{A}}^i$) is the *strong* (resp. *essential*, *injective*) *bordered apartment* associated to $\mathbb{A}$; its *main façade* is $\mathbb{A}_0 = \mathbb{A}$ (resp. $\mathbb{A}^e$, $\mathbb{A}$) of direction the trivial vectorial facet $V_0 = F^v_\pm(I)$.

In the following we set $\overline{\mathbb{A}} = \overline{\overline{\mathbb{A}}}$ (resp. $\overline{\mathbb{A}}^e$, $\overline{\mathbb{A}}^i$), $\mathbb{A}_{F^v} = \mathbb{A}^{ne}_{F^v}$ (resp. $\mathbb{A}^e_{F^v}$, $\mathbb{A}^{ne}_{F^v}$), *etc.*

For $x \in \overline{\mathbb{A}}$, we write $F^v(x)$ the direction of the façade containing x. For $\varepsilon = \pm$, $\overline{\mathbb{A}}^\varepsilon$ (resp. $\overline{\mathbb{A}}_{sph}$) is the union of the façades of sign $\pm$ (resp. the spherical façades) in $\overline{\mathbb{A}}$ and $\overline{\mathbb{A}}^\varepsilon_{sph} = \overline{\mathbb{A}}^\varepsilon \cap \overline{\mathbb{A}}_{sph}$.

To each wall $M(\alpha,\lambda)$ or half-apartment $D(\alpha,\lambda)$ is associated a wall $\overline{M}(\alpha,\lambda)$ or half-apartment $\overline{D}(\alpha,\lambda)$ of $\overline{\mathbb{A}}$: $\forall F^v$, $\overline{M}(\alpha,\lambda) \cap \mathbb{A}_{F^v}$ (resp. $\overline{D}(\alpha,\lambda) \cap \mathbb{A}_{F^v}$) is the projection of $M(\alpha,\lambda)$ (resp. $D(\alpha,\lambda)$) on $\mathbb{A}_{F^v}$ if $\alpha(F^v) = 0$, the empty set if $\alpha(F^v) < 0$ and the empty set (resp. $\mathbb{A}_{F^v}$) if $\alpha(F^v) > 0$. With these definitions we may define enclosures $cl(\Omega)$ in $\overline{\mathbb{A}}$.

The essentialization of $\overline{\mathbb{A}}$ is $\overline{\mathbb{A}}^e$, which is the bordered apartment defined in [Ch11]. We shall focus on $\overline{\mathbb{A}}^i$, as $\overline{\mathbb{A}}^e$ is $\overline{\mathbb{A}}^i$ if we choose $V = V^q$.

The set $\overline{\mathbb{A}}^{i\varepsilon}_{sph}$ is the microaffine apartment of sign ε as in [Ro06] (in its Satake realization). The corresponding object $\overline{\overline{\mathbb{A}}}^\varepsilon_{sph}$ is closer to the apartments of [Ro06, 2.3].

3) Links with sector-face germs and chimney germs : There is a one to one correspondence between the points of $\overline{\mathbb{A}}^e$ and the sector-face germs in $\mathbb{A}$. To $\mathfrak{F} = germ_\infty(x + F^v)$ corresponds the class $[x + F^v]$ of x modulo $\overrightarrow{F^v}$ in $\mathbb{A}^e_{F^v}$, also written $[\mathfrak{F}]$. When $\mathbb{A} = \mathbb{A}^q$ is essential, the points in $\overline{\overline{\mathbb{A}}}$ correspond bijectively to the sector-faces in $\mathbb{A}$.

By definition $\mathbb{A}_{F^v}$ itself is an affine apartment with walls defined using $\Phi^m(F^v)$. The closed-facets in $\mathbb{A}_{F^v}$ correspond bijectively with the chimney germs of $\mathbb{A}$ of direction F^v. To $\mathfrak{R} = \mathfrak{R}(F, F^v)$ corresponds the closed facet $[\mathfrak{R}]$ which is the filter made of the subsets in $\mathbb{A}_{F^v}$ containing $\{ [\mathfrak{F}] \mid \mathfrak{F} \subset \Sigma\}$ for some subset Σ of $\mathbb{A}$ containing $\mathfrak{R}$.

Actually $\mathfrak{R}$ is splayed (resp. solid, full) if and only if $[\mathfrak{R}]$ is in a spherical façade (resp. is spherical in its façade, is a chamber in its façade).

4) Topology : On $\overline{\mathbb{A}}^i$ (or $\overline{\mathbb{A}}^e$) one can define a topology inducing the affine topology on each façade and such that $\overline{\mathbb{A}}$ (or $\overline{\mathbb{A}}^e$) is the closure of $\mathbb{A}_0 = \mathbb{A}$ (or $\mathbb{A}^e$) which is open in $\overline{\mathbb{A}}$ (or $\overline{\mathbb{A}}^e$) [Ch11, 11.1.1]:

For a non trivial vectorial facet F^v, $x \in \mathbb{A}$ and U an open subset of $\mathbb{A}$ containing x, we set $\mathcal{V}(U, F^v) = (U + F^v) \cup \{ [\mathfrak{F}] \mid \mathfrak{F} \subset U + F^v \}$. When x, U vary but F^v and $germ_\infty(x + F^v)$ are fixed, we get a fundamental system of neighborhoods of $[x + F^v]$ in $\overline{\mathbb{A}}^i$ (or $\mathbb{A}^e$).

For this topology the closure $\overline{\mathbb{A}}_{F^v}$ of a façade $\mathbb{A}_{F^v}$ (with F^v non trivial) is the union of the façades $\mathbb{A}_{F_1^v}$ for $F_1^v \in F^{v*}$ *i.e.* $F^v \subset \overline{F_1^v}$; we take this for definition of $\overline{\mathbb{A}}_{F^v}$ when $\overline{\mathbb{A}} = \overline{\overline{\mathbb{A}}}$. In the classical case, $\overline{\mathbb{A}}^i$ is a compactification of $\mathbb{A}$ called the Satake or polyhedral compactification, see *e.g.* [Ch08].

5) Automorphisms : Any automorphism φ of $\mathbb{A}$ may be extended to an automorphism $\overline{\varphi}$ of $\overline{\mathbb{A}}$. For $\overline{\mathbb{A}}^i$ or $\overline{\mathbb{A}}^e$ the image of $[x + F^v] \in \mathbb{A}_{F^v}$ is $[\varphi(x) + \overrightarrow{\varphi}(F^v)] \in \mathbb{A}_{\overrightarrow{\varphi}(F^v)}$ and $\overline{\varphi}$ is continuous. For $\overline{\overline{\mathbb{A}}}$, $\overline{\varphi}(x, F^v) = (\varphi(x), \overrightarrow{\varphi}(F^v))$. Automorphisms permute the façades.

In particular, the action ν of N on $\mathbb{A}$ may be extended as an action on $\overline{\mathbb{A}}$ which is also written ν.

4 Hovels and bordered hovels

4.1 Wanted

Let G be a group, N a subgroup and ν an action of N over some space A. We want a space $\mathscr{I}$ containing A as a subset and an action of G on $\mathscr{I}$ such that $G.A = \mathscr{I}$, A is stable under N and the induced action is ν.

Following F. Bruhat and J. Tits [BrT72, 7.4.1], a good way to get it, is to define $\mathscr{I}$ as a quotient:

$\mathscr{I} = G \times A / \sim_{\mathcal{Q}}$ with $(g, x) \sim_{\mathcal{Q}} (h, y) \iff \exists n \in N$ such that $y = \nu(n).x$ and $g^{-1}hn \in Q(x)$
where $\mathcal{Q} = (Q(x))_{x \in A}$ is a family of subgroups of G.

The action of G on $\mathscr{I}$ is induced by the left multiplication on G, we have a map $i : A \to \mathscr{I}$: for $x \in A$, $i(x)$ is the class of $(1, x)$.

We are interested in the groups G and N as in 3.1 and an action ν as in 3.4.1 or 3.7.5; as in [Ch10], [Ch11] we skip a possible generalization to RGD-systems. We shall now precise the conditions on the family $(Q(x))_{x \in A}$, following [Ch11, 11.2.1 and 11.3].

4.2 Families of parahoric subgroups

1) Let $A = \overline{\mathbb{A}}$ be $\overline{\overline{\mathbb{A}}}$, $\overline{\mathbb{A}}^e$ or $\overline{\mathbb{A}}^i$ and ν the corresponding action of N. For a family $\mathcal{Q} = (Q(x))_{x \in A}$, we then write $\overline{\mathscr{I}} = G \times \overline{\mathbb{A}} / \sim_{\mathcal{Q}}$, it is the *bordered hovel* associated to the situation. The *(bordered) apartments* of $\overline{\mathscr{I}}$ are the sets $g.i(\overline{\mathbb{A}})$ for $g \in G$.

For a subset or a filter Ω in $\overline{\mathbb{A}}$ (resp. $\mathbb{A}$), $\alpha \in \Phi$ and $\Psi \subset \Phi$, we define $\overline{D}(\alpha, \Omega) = \overline{D}(\alpha, \sup(-\alpha(\Omega)))$ (resp. $D(\alpha, \Omega) = \overline{D}(\alpha, \Omega) \cap \mathbb{A}$), $U_\alpha(\Omega) = \{u \in U_\alpha \mid \Omega \subset \overline{D}(\alpha, \varphi_\alpha(u))\}$

(hence $U_\alpha(y) = U_{\alpha,-\alpha(y)}$), $N(\Omega) = \{n \in N \mid n \text{ fixes } \Omega\}$, $G(\Psi,\Omega) = \langle U_\alpha(\Omega) \mid \alpha \in \Psi\rangle$ and $G(\Omega) = G(\Phi,\Omega)$, written U_Ω in [GR08, § 3.2] or [Ro12, 4.6a]. As in these references we write $U_\Omega^{++} := G(\Phi^+,\Omega) \subset U_\Omega^+ := U^+ \cap G(\Omega)$ and the same things with $-$. It often happens that $U_\Omega^{++} \neq U_\Omega^+$ [Ro12, 4.12.3a], see also 5.11.4 below.

It is clear that $N(\Omega) \subset N(F^v)$ normalizes $G(\Omega)$ and $U(F^v)$ and that $G(\Omega) \subset G(\Phi(F^v),\Omega) \subset P(F^v)$ normalizes $U(F^v)$, if $\Omega \cap \mathbb{A}_{F^v} \neq \emptyset$. We always have $G(\Psi,\Omega) = G(\Psi, cl^\#(\Omega))$. When $\Omega = \emptyset$, we have $G^\emptyset = Z_0.G(\emptyset)$. For $\Omega \neq \emptyset$, the group $N_\Omega^{min} = N \cap G(\Omega)$ is normal in $N(\Omega)$. Its image W_Ω^{min} by ν is in W^a and generated by the reflections with respect to the (true) walls of $\mathbb{A}$ containing Ω. This group W_Ω^{min} is isomorphic to its image W_Ω^v in W^v [GR08, § 3.2].

2) Definition. A family $\mathcal{Q} = (Q(x))_{x\in\overline{\mathbb{A}}}$ of subgroups of G is a *family of parahoric subgroups* if it satisfies the following axioms:
(For the convenience of the reader, we give to axioms the names in [Ch11] and shorter names.)

(P1) (para 0.1) For all $x \in \overline{\mathbb{A}}$, $U(F^v(x)) \subset Q(x) \subset P(F^v(x))$

(P2) (para 0.2) For all $x \in \overline{\mathbb{A}}$, $N(x) \subset Q(x)$

(P3) (para 0.3) For all $x \in \overline{\mathbb{A}}$, for all $\alpha \in \Phi$, for all $\lambda \in \mathbb{R}$, if $x \in \overline{D}(\alpha,\lambda)$, then $U_{\alpha,\lambda} \subset Q(x)$

(P4) (para 0.4) For all $x \in \overline{\mathbb{A}}$, for all $n \in N$, $nQ(x)n^{-1} = Q(\nu(n).x)$.

If Ω is a subset of $\overline{\mathbb{A}}$, we define $Q(\Omega) = \bigcap_{x\in\Omega} Q(x)$. If Ω is a filter in $\overline{\mathbb{A}}$, we define $Q(\Omega) = \bigcup_{\Omega'\in\Omega} Q(\Omega')$.

3) Easy consequences. a) Axiom (P4) tells that $\sim_\mathcal{Q}$ is an equivalence relation [*l.c.* , 11.3.2]. By axiom (P3), $U_{\alpha,\lambda}$ fixes (pointwise) $\overline{D}(\alpha,\lambda)$. Axiom (P2) tells that the map $i : \overline{\mathbb{A}} \to \overline{\mathscr{I}}$ is $N-$equivariant, but it is not clearly one to one, *cf.* 4.3.2 below.

b) [*l.c.* , 11.3.8] The fixator of $i(x) \in i(\overline{\mathbb{A}})$ in G is $G_x = Q(x)$. More generally for a subset or filter Ω in $g.i(\overline{\mathbb{A}}) \subset \overline{\mathscr{I}}$, we define $G_\Omega = Q(\Omega)$ as the fixator $g.Q(g^{-1}.\Omega).g^{-1}$ of Ω.

For $x \in \overline{\mathbb{A}}$ and $g \in G$, if $g.i(x) \in i(\overline{\mathbb{A}})$, then there exists $n \in N$ with $g.i(x) = n.i(x)$. For a subset or filter Ω in $\overline{\mathbb{A}}$, the set $G(\Omega \subset \overline{\mathbb{A}}) = \{g \in G \mid G.i(\Omega) \subset i(\overline{\mathbb{A}})\}$ is equal to $\bigcap_{x\in\Omega} NQ(x)$ (if Ω is a set) or $\bigcup_{\Omega'\in\Omega} G(\Omega' \subset \overline{\mathbb{A}})$ (if Ω is a filter).

For all $x \in \overline{\mathscr{I}}$, $Q(x)$ is transitive on the apartments containing x.

c) If F^v is a vectorial facet of $\mathbb{A}^v$, axiom (P1) tells that the map $P(F^v) \times \mathbb{A}_{F^v} \to \overline{\mathscr{I}}$ induces a map $G(F^v) \times \mathbb{A}_{F^v}/\sim_{F^v} \to \overline{\mathscr{I}}$, where $\sim_{F^v}$ is the equivalence relation defined using $\mathcal{Q}_{|\mathbb{A}_{F^v}}$ and $N(F^v)$. This map is one to one, as $y = \nu(n).x$ with $x, y \in \mathbb{A}_{F^v}$ and $n \in N$ implies $n \in N(F^v) = N \cap P(F^v)$.

The image of this map is the *façade* $\mathscr{I}_{F^v}$ of $\overline{\mathscr{I}}$ in the direction F^v. In particular the *main façade* of $\overline{\mathscr{I}}$ is the *hovel* $\mathscr{I} = G \times \mathbb{A}/\sim$ where $\sim$ is defined using $\mathcal{Q}_{|\mathbb{A}}$ and N. Actually each façade $\mathscr{I}_{F^v}$ is an hovel, the main façade of $\overline{\mathscr{I}}_{F^v} = \bigcup_{F_1^v \in F^{v*}} \mathscr{I}_{F_1^v}$ associated to $\overline{\mathbb{A}}_{F^v}$, $\mathcal{Q}_{|\overline{\mathbb{A}}_{F^v}}$ and the valuated root datum $(G(F^v), (U_\alpha)_{\alpha\in\Phi^m(F^v)}, Z, (\varphi_\alpha)_{\alpha\in\Phi^m(F^v)})$.

d) By (P1) and (P3), if $\Omega \subset \mathbb{A}_{F^v}$ is non empty, then $G(\Phi^m(F^v),\Omega) \subset G(\Omega) \subset U(F^v) \rtimes G(\Phi^m(F^v),\Omega) \subset Q(\Omega) = U(F^v) \rtimes (M(F^v) \cap Q(\Omega))$.

e) If $\mathcal{Q}$ is a family of parahorics and $x \in \overline{\mathbb{A}}$, then $Q(x) \supset P(x) := \langle N(x), G(x), U(F^v(x))\rangle = N(x).G(x).U(F^v(x))$. So it is clear that $\mathcal{P} = (P(x))_{x\in\overline{\mathbb{A}}}$ is the *minimal family of parahorics.* In the classical (= spherical) case it is the right family; this is the reason for axiom (P6) below. But it is not clear in general that $\mathcal{P}$ satisfies axiom (P5) below. Note that, even for $x \in \mathbb{A}$, $P(x)$ is seldom equal to P_x, as defined in [Ro12, 5.14] or [GR08, 3.12].

4) Definition. A *good family of parahorics* is a family $\mathcal{Q}$ of parahorics satisfying moreover:

(P5) (para inj) For all $x \in \overline{\mathbb{A}}$, $N(x) = Q(x) \cap N$

(P6) (para sph) For all $x \in \overline{\mathbb{A}}$, if $F^v(x)$ is spherical, $Q(x) = P(x)$ *i.e.* $\mathcal{Q}_{|\overline{\mathbb{A}}_{sph}} = \mathcal{P}_{|\overline{\mathbb{A}}_{sph}}$.

(P7) (para 2.2)(sph) If F^v is a spherical facet, $F_1^v \subset \overline{F^v}$ and $x \in \mathbb{A}_{F_1^v}$ then $NQ(x) \cap NP(F^v) = NQ(\{x, pr_{F^v}(x)\})$.

5) Definition. A *very good family of parahorics* is a good family $\mathcal{Q}$ of parahorics satisfying moreover:

(P8) (para dec) For all $x \in \overline{\mathbb{A}}$, for all chamber $C^v \in F^v(x)^*$,
$$Q(x) = (Q(x) \cap U(C^v)).(Q(x) \cap U(-C^v)).N(x)$$

(P9) (para 2.1+)(sph) If F^v is a spherical facet, $F_1^v \subset \overline{F^v}$ and $x \in \mathbb{A}_{F_1^v}$ then $Q(x) \cap P(F^v) = Q(\overline{x + F^v})$

where $x + F^v = pr_{F_1^v}(x_1 + F^v)$ for any $x_1 \in \mathbb{A}$ with $pr_{F_1^v}(x_1) = x$ and $\overline{x + F^v}$ is the union of the sets $pr_{F_2^v}(x + \overline{F^v} + \overrightarrow{F_2^v})$ for $F_1^v \subset \overline{F_2^v} \subset \overline{F^v}$ (it is the closure of $x + F^v$ when $\overline{\mathbb{A}} = \overline{\mathbb{A}}^e$ or $\overline{\mathbb{A}}^i$).

(P10) If $x < y$ or $y < x$ in $\mathbb{A}_{F^v}$, then $Q(]x, y]) \subset Q(x)$ *i.e.* $Q(]x, y]) = Q([x, y])$ where $]x, y] = [x, y] \setminus \{x\}$ is an half-open-segment.

6) Remarks. a) (P8) is an important tool for calculations. (P7) and (P9) give links between $\mathcal{Q}$ and $\mathcal{Q}_{|\overline{\mathbb{A}}_{sph}}$ which is well known by (P6).

b) By [*l.c.* , 11.9.2] a consequence of (P9) is the following condition:

(P9-) (para 2.1+-)(sph) If F^v is a vectorial facet and $g \in U(F^v)$, there exists $x \in \mathbb{A}$ such that $g \in Q(\overline{x + C^v})$ for all chamber $C^v \in F^{v*}$.

c) For x, $F^v = F^v(x)$ and C^v as in (P8), suppose $x \notin \mathbb{A}$ *i.e.* F^v non trivial. Then we have $Q(x) \cap U(C^v) = (Q(x) \cap M(F^v) \cap U(C^v)) \ltimes U(F^v)$. Now, by the uniqueness in Birkhoff decomposition (1.6.2), $P(F^v) \cap U(-C^v) = M(F^v) \cap U(-C^v) = M(-F^v) \cap U(-C^v)$ which is a "maximal unipotent" subgroup (opposite $M(F^v) \cap U(C^v)$) in $M(F^v) = M(-F^v)$; hence $Q(x) \cap U(-C^v) = Q(x) \cap M(-F^v) \cap U(-C^v)$. Now, if F^v is spherical, we can give another explanation: $M(F^v) \cap U(-C^v) = M(F^v) \cap U(C_-^v)$ where C_-^v is the chamber opposite to C^v in F^{v*}; so $Q(x) \cap U(-C^v) = Q(x) \cap M(F^v) \cap U(C_-^v)$.

d) Except (P9) all axioms impose relations between a single façade $\mathbb{A}_{F^v}$ and the spherical façades $\mathbb{A}_{F_1^v}$ for $F_1^v \in F^{v*}$. We may fix F_0^v and take $x \in \mathbb{A}_{F_0^v} \cup \overline{\mathbb{A}}_{sph}$ in the axioms, then we get the same results. So, starting with 4.4, we shall use actually a family $\mathcal{Q}$ of groups $Q(x)$ for $x \in \mathbb{A} \cup \overline{\mathbb{A}}_{sph}$ with the corresponding axioms.

e) A priori a good family of parahorics has no property of continuity. This is the reason of the (weak) axiom (P10). But without it everything in this section is still true (except when the contrary is explicitly told). This axiom (P10) is satisfied by the minimal family $\mathcal{P}$.

f) If $\mathcal{Q}$ is a very good family for $(G, (U_\alpha)_{\alpha\in\Phi}, Z, (\varphi_\alpha)_{\alpha\in\Phi})$ then we define $Q^\emptyset(x) = Q(x) \cap G^\emptyset$. We have $Q(x) = Q^\emptyset(x).N(x)$ (by (P8)) and $\mathcal{Q}^\emptyset$ is a very good family of parahorics for $(G^\emptyset, (U_\alpha)_{\alpha\in\Phi}, Z^\emptyset, (\varphi_\alpha)_{\alpha\in\Phi})$ (defined in 3.4.3c). The two bordered hovels associated to $\overline{\mathbb{A}}$ and $(G, \mathcal{Q})$ or $(G^\emptyset, \mathcal{Q}^\emptyset)$ are canonically isomorphic.

4.3 Bordered hovels associated to good families

We explain now some of the abstract results of [Ch11] (or [Ch10]). So let $\mathcal{Q}$ be a good family of parahorics (if it exists) and $\overline{\mathscr{I}}$ be the associated bordered hovel.

1) By Bruhat-Tits theory and (P6) $\mathcal{Q}$ is well known on the spherical façades [*l.c.* , 11.2.3]: *e.g.* the results of (P8) and (P9) are true when $F^v(x)$ is spherical, for Ω in a spherical façade $\mathbb{A}_{F^v}$, $Q(\Omega) = U(F^v) \rtimes (N(\Omega).G(\Phi^m(F^v), \Omega)) = N(\Omega).Q(cl^\#(\Omega))$ and $G(\Omega \subset \overline{\mathbb{A}}) = N.Q(\Omega)$.

Actually for F^v spherical $\mathscr{I}_{F^v}$ is the Bruhat-Tits building of the classical valuated root datum $(G(F^v),(U_\alpha)_{\alpha\in\Phi^m(F^v)},Z,(\varphi_\alpha)_{\alpha\in\Phi^m(F^v)})$ (with the facets associated to cl).

2) The minimal family $\mathcal{P}$ satisfies also (P5) and (P6). Axiom (P5) tells us that $i:\overline{\mathbb{A}}\to\overline{\mathscr{I}}$ is one to one [*l.c.* , 11.3.4]; we identify $\overline{\mathbb{A}}$ and $i(\overline{\mathbb{A}})$. The stabilizer of $\overline{\mathbb{A}}$ in G is N [*l.c.* , 11.3.5].

3) Iwasawa decomposition [*l.c.* , 11.4.2]: For a chamber C^v or a facet F^v in $\mathbb{A}^v$ and a facet $F\subset\overline{\mathbb{A}}$, we have $G=U(C^v).N.G(F)=U(F^v).N.Q(cl^\#(F))=U(F^v).N.Q(F)$.

4) Bruhat-Birkhoff-Iwasawa decomposition [*l.c.* , 11.5] : Let $F_1\subset\mathbb{A}_{F_1^v}$ and $F_2\subset\mathbb{A}_{F_2^v}$ be two facets with F_1^v or F_2^v spherical. Then $G=U(F_1^v).G(\Phi^m(F_1^v),F_1).N.G(\Phi^m(F_2^v),F_2).U(F_2^v)=Q(cl^\#(F_1)).N.Q(cl^\#(F_2))=Q(F_1).N.Q(F_2)$. If F_1 and F_2 are facets in $\overline{\mathscr{I}}$ and F_1 or F_2 is in a spherical façade, then there is an apartment of $\overline{\mathscr{I}}$ containing F_1 and F_2 (even $cl^\#(F_1)$ and $cl^\#(F_2)$).

5) Projection : Let F^v be a spherical facet and $F_1^v\subset\overline{F^v}$. Then, by (P7), the projection pr_{F^v} of $\mathbb{A}_{F_1^v}$ onto $\mathbb{A}_{F^v}$ extends to a well defined map $pr_{F^v}:\mathscr{I}_{F_1^v}\to\mathscr{I}_{F^v}$ between the corresponding façades. For each $g\in G$, $g.pr_{F^v}(x)=pr_{gF^v}(gx)$ [*l.c.* , prop. 11.7.3].

6) If $\overline{\mathbb{A}}=\overline{\mathbb{A}}^e$, then (P8) is satisfied by any good family of parahorics [*l.c.* , 11.7.5].

7) Let $\mathcal{Q}$ be a good family of parahorics for $\overline{\mathbb{A}}=\overline{\mathbb{A}}^e$, satisfying moreover (P9) or (P9-). Suppose $\Omega\subset\overline{\mathbb{A}}$ is in $\overline{\mathbb{A}}^\varepsilon$ and intersects non trivially $\overline{\mathbb{A}}^\varepsilon_{sph}$ or intersects non trivially $\overline{\mathbb{A}}^+_{sph}$ and $\overline{\mathbb{A}}^-_{sph}$. Then $G(\Omega\subset\overline{\mathbb{A}})=N.Q(\Omega)$, hence $Q(\Omega)$ is transitive on the apartments containing Ω. If Ω intersects non trivially $\overline{\mathbb{A}}^+_{sph}$ and $\overline{\mathbb{A}}^-_{sph}$, then $Q(\Omega)=N(\Omega).Q(cl^\#(\Omega))$: $cl^\#(\Omega)$(and also $cl(\Omega)$,...) is well defined in $\overline{\mathscr{I}}$ independently of the apartment containing Ω. [*l.c.* , section 11.9.2]

8) One can find in *loc. cit.* many other implications between the various axioms. Actually Charignon introduces also useful notions of functoriality *i.e.* the possibility of embedding the valuated root datum in greater ones, with arbitrarily large subsets Λ_α of $\mathbb{R}$ and various good compatibilities. We shall not explain this, as it is more natural in the framework of split Kac-Moody groups over valuated fields on which we shall concentrate in the next section.

Proposition 4.4. *Let $\mathcal{Q}$ be a good family of parahorics satisfying (P9) and Ω be a non empty subset or filter in $\mathbb{A}$.*

a) Let $F^v\subset\overline{C^v}\subset\mathbb{A}^v$ be a spherical vectorial facet in the closure of a chamber and $\alpha_1,\cdots,\alpha_n\in\Phi$ be the non divisible roots such that $\alpha(C^v)>0$ and $\alpha(F^v)=0$. Then:

$$Q(\Omega)\cap P(C^v)=(Q(\Omega)\cap U(C^v))\rtimes Z_0$$

$$Q(\Omega)\cap U(C^v)=(Q(\Omega)\cap U(F^v))\rtimes(Q(\Omega)\cap U(C^v)\cap M(F^v))$$

and $Q(\Omega)\cap U(C^v)\cap M(F^v)=U_{\alpha_1}(\Omega).\cdots.U_{\alpha_n}(\Omega)$ with uniqueness of the decomposition.

b) Let $C^v\subset\mathbb{A}^v$ be a chamber. Then the set $Q^{dec}(\Omega,C^v)=(Q(\Omega)\cap U(C^v)).(Q(\Omega)\cap U(-C^v)).N(\Omega)$ depends only of the sign of the chamber C^v.

N.B. 1) So we define $Q^{dec}(\Omega,\varepsilon)=Q^{dec}(\Omega,C^v)$ if C^v is of sign ε.

2) By (P9) $Q(\Omega)\cap U(F^v)\subset Q(\Omega+\overline{C^v})$ for all $C^v\in F^{v*}$.

Proof. a) We have $P(F^v)=U(C^v)\rtimes Z$, $U(C^v)=U(F^v)\rtimes(U(C^v)\cap M(F^v))$ and, by Bruhat-Tits theory (4.3.1), $U(C^v)\cap M(F^v)=U_{\alpha_1}.\cdots.U_{\alpha_n}$ (unique). Using these uniqueness results, we have just to prove a) for $\Omega=\{x\}$ and $x\in\mathbb{A}$. We write $x'=pr_{F^v}(x)$.

By (P9) and 4.3.1 $Q(x)\cap P(F^v)=Q(\overline{x+F^v})\subset Q(x')=U(F^v)\rtimes(Q(x')\cap M(F^v))$ and $Q(x')\cap M(F^v)=U_{\alpha_1}(x).\cdots.U_{\alpha_n}(x).U_{-\alpha_1}(x).\cdots.U_{-\alpha_n}(x).N(x')$ [BrT72, 7.1.8]. So $Q(x')\cap M(F^v)\cap U(C^v)=U_{\alpha_1}(x).\cdots.U_{\alpha_n}(x)$ and $Q(x')\cap M(F^v)\cap P(C^v)=U_{\alpha_1}(x).\cdots.U_{\alpha_n}(x).Z_0$

(by uniqueness in the Birkhoff decomposition 1.6.2.). And, as each $U_{\alpha_i}(x)$ is in $Q(x)$, we get what we wanted.

b) Any two chambers of sign ε are connected by a gallery of chambers of sign ε. So one has only to show that $Q^{dec}(\Omega, C^v) = Q^{dec}(\Omega, r_\alpha(C^v))$ when $\alpha \in \Phi$ is simple with respect to $\Phi^+(C^v)$. We consider $F^v = \overline{C^v} \cap \mathrm{Ker}\alpha$ and apply a). But $U_\alpha(\Omega).U_{-\alpha}(\Omega).N(\Omega{+}\mathrm{Ker}\alpha) = U_{-\alpha}(\Omega).U_\alpha(\Omega).N(\Omega{+}\mathrm{Ker}\alpha)$ by [BrT72, 6.4.7]; so the same proof as in [GR08, 3.4a] applies. □

4.5 Good fixators

1) We consider now a very good family of parahorics $\mathcal{Q} = (Q(x))_{x\in\mathbb{A}\cup\overline{\mathbb{A}}_{sph}}$ and we want to define the same notions as in [GR08, def. 4.1], using the axioms and proposition 4.4; this is suggested in the beginning of [Ro12, sec. 5].

2) Definition. Consider the following conditions for a subset or filter Ω in $\mathbb{A}$:

(GFε) $Q(\Omega) = Q^{dec}(\Omega, \varepsilon)$ for $\varepsilon = +$ or $-$

(TF) $G(\Omega \subset \overline{\mathbb{A}}) = NQ(\Omega)$ (where $G(\Omega \subset \overline{\mathbb{A}})$ is defined in 4.2.3b)

We say that Ω has a *good fixator* if it satisfies these three conditions.
We say that Ω has an *half-good fixator* if it satisfies (TF) and (GF+) or (GF-).
We say that Ω has a *transitive fixator* if it satisfies (TF).

3) Consequences. We get the following results by mimicking the proofs in [GR08, sec. 4.1]. The ingredients are proposition 4.4 and the facts that $Q(\Omega) \cap U(\pm C^v) = Q(\Omega \pm \overline{C^v}) \cap U(\pm C^v)$, $\bigcap_\Omega Q(\Omega) \cap U(\pm C^v) = Q(\cup_\Omega \Omega) \cap U(\pm C^v)$ for a family Ω of filters, *etc.*

a) By (P8) a point has a good fixator. The group N permutes the filters with good fixators and the corresponding fixators.

If Ω has a transitive fixator, then $Q(\Omega)$ acts transitively on the apartments containing Ω. Hence the "shape" of Ω doesn't depend of the apartment containing it. As a consequence of the many examples below of filters with (half) good fixators, we may define in $\mathscr{I}$ (independently of the apartment containing it) what is a preordered segment, preordered segment-germ, generic ray, closed (local) facet, spherical sector face, solid chimney *etc.*

b) In the classical case every filter has a good fixator.

c) Let $\mathscr{F}$ be a family of filters with good (or half-good) fixators such that the family Ω of the sets belonging to one of these filters is a filter. Then Ω has a good (or half-good) fixator $Q(\Omega) = \bigcup_{F\in\mathscr{F}} Q(F)$.

d) Suppose the filter Ω is the union of an increasing sequence $(F_i)_{i\in\mathbb{N}}$ of filters with good (or half-good) fixators and that, for some i, the support of F_i has a finite fixator in $\nu(N)$, then Ω has a good (or half-good) fixator $Q(\Omega) = \bigcap_{i\in\mathbb{N}} Q(F_i)$.

e) Let Ω and Ω' be two filters in $\mathbb{A}$ and $C_1^v, \cdots, C_n^v$ be positive vectorial chambers. If Ω' satisfies (GF+) and (TF) and $\Omega \subset \bigcup_{i=1}^n (\Omega' + \overline{C_i^v})$, then $\Omega \cup \Omega'$ satisfies (GF+) and (TF) with $Q(\Omega \cup \Omega') = Q(\Omega) \cap Q(\Omega')$. If moreover Ω (resp. Ω') satisfies (GF-) and $\Omega' \subset \bigcup_{i=1}^n (\Omega - \overline{C_i^v})$ (resp. $\Omega \subset \bigcup_{i=1}^n (\Omega' - \overline{C_i^v})$), then $\Omega \cup \Omega'$ has a good fixator.

4) Remarks. a) Let Ω in $\mathbb{A}$ be a filter with good (or half-good) fixator and F^v be a spherical vectorial facet. We write $\Theta = \bigcup_{C^v \in F^{v*}} \overline{C^v}$ and $\Omega' = (\Omega+\Theta)\cap(\Omega-\Theta)\cap(\bigcap_{\alpha\in\Phi^m(F^v)} D(\alpha,\Omega))$ (which is in $cl_{\mathbb{R}}^{\Phi}(\Omega)$), then, by 4.4a and 4.5.3e, any Ω'' with $\Omega \subset \Omega'' \subset \Omega'$ has a good (or half-good) fixator; moreover $Q(\Omega) = Q(\Omega'')N(\Omega)$. In particular any apartment A of $\mathscr{I}$ containing Ω contains Ω' and is conjugated to $\mathbb{A}$ by $Q(\Omega')$.

b) By 4.2.6f, for every filter Ω, we have $Q^\emptyset(\Omega) = Q(\Omega)\cap G^\emptyset$, $Q(\Omega)\cap U(C^v) = Q^\emptyset(\Omega)\cap U(C^v)$, and $N^\emptyset(\Omega) = N(\Omega) \cap G^\emptyset$. Hence if Ω has a (half) good fixator for $\mathcal{Q}$, $N.Q(\Omega) = N.Q^\emptyset(\Omega)$, $N.Q(\Omega) \cap G^\emptyset = N^\emptyset.Q^\emptyset(\Omega)$ and Ω has a (half) good fixator for $\mathcal{Q}^\emptyset$.

4.6 Examples of filters with good fixators

1) If $x \leq y$ or $y \leq x$ in $\mathbb{A}$ (*preordered* situation), then $\{x, y\}$ and the segment $[x, y]$ have good fixators $Q(\{x, y\}) = Q([x, y])$ (apply 4.5.3e); in particular any apartment A of $\mathscr{I}$ containing $\{x, y\}$ contains $[x, y]$ and is conjugated to $\mathbb{A}$ by $Q([x, y])$. If moreover $x \neq y$ the segment germ $[x, y) = germ_x([x, y])$ has a good fixator (4.5.3c).

2) If $x, y \in \mathbb{A}$ and $\xi = y - x \neq 0$ is in a spherical vectorial facet F^v (*generic* situation), then the half-open segment $]x, y] = [x, y] \setminus \{x\}$, the line (x, y) and the ray $\delta = x + [0, +\infty[.\xi$ of origin x containing y (or the open ray $\delta^\circ = \delta \setminus \{x\}$) have good fixators (4.5.3d). Using now 4.5.3c the germs $]x, y) = germ_x(]x, y])$ and $germ_\infty(\delta)$ have good fixators.

3) A closed local facet $\overline{F^l(x + F^v)}$ has a good fixator: choose $\xi \in F^v$ and $\lambda > 0$ then the intersection $\Omega_{\lambda,\xi}$ of $(x + \overline{F^v}) \cap (x + \lambda\xi - \overline{F^v})$ with a ball of radius $\|\lambda\xi\|$ and center x (for any norm) has a good fixator (4.5.3e with $\Omega' = [x, x + \lambda\xi]$) and $\overline{F^l(x + F^v)}$ is as described in (4.5.3c) using the family $\Omega_{\lambda,\xi}$ (when λ varies).

If the local facet is spherical, then it has a good fixator. We just have to use above $(x + \varepsilon\xi + \overline{F^v}) \cap (x + \lambda\xi - \overline{F^v})$ for $0 < \varepsilon < \lambda$ and 4.5.3c,d,e.

4) By similar arguments we see that a spherical sector face or its closure or its germ has a good fixator. The apartment $\mathbb{A}$ has a good fixator $Q(\mathbb{A}) = Z_0$, so the stabilizer of $\mathbb{A}$ is N. An half-apartment $D(\alpha, k)$ has a good fixator $Z_0.U_{\alpha,k}$ *cf.* [Ro12, 5.7.7].

5) It is important in this paragraph that the family $\mathcal{Q}$ satisfies axiom (P10). If $y < x$ or $x < y$ in $\mathbb{A}$, then the half-open segment $]x, y]$ (resp. the open-segment-germ $]x, y)$) has a good fixator $Q(]x, y]) = Q([x, y])$ (resp. $Q(]x, y)) = Q([x, y))$), even if $F^v(y - x)$ is not spherical. By arguments as in 3) above (using $x + F^v =]x, x + \lambda\xi] + \overline{F^v}$ instead of $x + \overline{F^v}$) we deduce that any local facet $F^l = F^l(x, F^v)$ has a good fixator and $Q(F^l) = Q(\overline{F^l})$.

Proposition 4.7. *Let $\mathcal{Q}$ be a very good family of parahorics, $\xi \neq 0$ a vector in a spherical vectorial facet F^v and $x \in \mathbb{A}$. We consider the ray $\delta = x + [0, +\infty[\xi$, then $Q(\delta) \subset Q(germ_\infty(\delta)) \subset P(F^v)$.*

N.B. 1) This is a kind of reciprocity for axiom (P9). We have $Q(x) \cap P(F^v) = Q(\overline{x + F^v}) = Q(x + \overline{F^v})$ with $x + \overline{F^v}$ in $\mathbb{A}$.

2) We see thus directly that $Q(\mathbb{A})$ fixes $\overline{\mathbb{A}}_{sph}$.

Proof. It is sufficient to prove that $Q(\delta) \subset P(F^v)$. Let C^v be a chamber in F^{v*}, then by 4.6.2, $Q(\delta) = (Q(\delta) \cap U(C^v)).(Q(\delta) \cap U(-C^v)).N(\delta)$. As $N(\delta)$ and $U(C^v)$ are in $P(F^v)$, we have only to prove $Q(\delta) \cap U(-C^v) \subset P(F^v)$. By (P9) and 4.3.1, for $\lambda \geq 0$, $Q(x + \lambda\xi) \cap U(-C^v) \subset Q(x + \lambda\xi) \cap P(-F^v) \subset Q(pr_{-F^v}(x)) = N(pr_{-F^v}(x)).G(\Phi^m(F^v), pr_{-F^v}(x)).U(-F^v)$. So $Q(\delta) \cap U(-C^v) = (G(\Phi^m(F^v), pr_{-F^v}(x)) \cap U(-C^v)).(U(-F^v) \cap Q(\delta))$ as $G(\Phi^m(F^v), pr_{-F^v}(x))$ fixes pointwise $x + \mathbb{R}\xi \supset \delta$.

Now $U(-F^v) \cap Q(\delta) = \bigcap_{C^v \in F^{v*}, \lambda \geq 0} U(-C^v) \cap Q(x + \lambda\xi) \subset Q(\bigcup_{C^v \in F^{v*}, \lambda \geq 0} (x + \lambda\xi - \overline{C^v}))$. But this last union is actually $\mathbb{A}$, so $U(-F^v) \cap Q(\delta) = U(-F^v) \cap Q(\mathbb{A}) = U(-F^v) \cap Z_0 = \{1\}$ and $U(-C^v) \cap Q(\delta) \subset G(\Phi^m(F^v), pr_{-F^v}(x)) \subset P(F^v)$. □

Corollary 4.8. *Let $F \subset \mathbb{A}_{F^v}$ be a facet in a façade and $\mathfrak{R} \subset \mathbb{A}$ be the corresponding chimney germ (cf. 3.7.3). Then $U(F^v).G(\Phi^m(F^v), F).N(\mathfrak{R}) \subset Q(F) \cap Q(\mathfrak{R})$. If F^v is spherical $Q(F) = U(F^v).G(\Phi^m(F^v), F).N(F) \supset Q(\mathfrak{R}) = U(F^v).G(\Phi^m(F^v), F).N(\mathfrak{R})$.*

N.B. We sometimes say that $Q(\mathfrak{R})$ is the *strong fixator* of F.

Proof. For $x \in \mathbb{A}$, it is clear that $\mathfrak{R}$ is in the union of all $\overline{x + C^v}$ for $C^v \in F^{v*}$. So the first result is due to (P9-). For F^v spherical $Q(F)$ is given in 4.3.1. By proposition 4.7 $Q(\mathfrak{R}) \subset P(F^v)$ and $Q(\mathfrak{R}) \subset Q(F)$ by (P9), hence the result. □

4.9 Properties specific to $cl_{\mathbb{R}}$

We are interested here in the cases $cl = cl_{\mathbb{R}}$ *i.e.* $cl = cl_{\mathbb{R}}^{\Phi}$, $cl_{\mathbb{R}}^{\#}$, $cl_{\mathbb{R}}^{\Delta}$ or $cl_{\mathbb{R}}^{\Delta^{ti}}$.

1) For a filter Ω in $\mathbb{A}$, $\Omega \subset \Omega + V_0 \subset (\Omega + \overline{C^v}) \cap (\Omega - \overline{C^v})$ for all vectorial chamber C^v. So, by 4.5.4, if Ω has a good or half-good fixator, it is also true for any Ω' with $\Omega \subset \Omega' \subset \Omega + V_0$. Moreover $Q(\Omega) = Q(\Omega')$: $N(\Omega) = N(\Omega + V_0)$ as W^v fixes V_0.

2) For a local facet F^l, we saw that $\overline{F}^l = \overline{F}_{\mathbb{R}}^{\Delta^{ti}} \subset \overline{F}_{\mathbb{R}}^{\Delta} \subset \overline{F}_{\mathbb{R}}^{\Phi} = \overline{F}_{\mathbb{R}}^{\#} = \overline{F}^l + V_0$, hence the closed $cl_{\mathbb{R}}-$facet associated to F^l has a good fixator by 1) above.

We saw also that the $cl_{\mathbb{R}}-$facet associated to F^l is between F^l and $F^l + V_0$. If the family $\mathcal{Q}$ satisfies (P10), then this $cl_{\mathbb{R}}-$facet has a good fixator (4.6.5 and 1) above); by 4.5.4 any apartment containing F^l contains $cl_{\mathbb{R}}(F^l)$ and is conjugated to $\mathbb{A}$ by $Q(F^l) = Q(\overline{F^l}) = Q(\overline{F^l} + V_0)$.

3) Let Ω be a point, preordered segment, preordered segment-germ, generic ray, generic ray-germ or generic line (resp. preordered half-open segment, preordered open-segment-germ or generic open ray if $\mathcal{Q}$ satisfies (P10)) as in 4.6 and let $\Omega \subset \Omega'' \subset cl_{\mathbb{R}}(\Omega)$. Then $Q(\Omega) = Q(\Omega'')$ by 4.5.4, as $cl_{\mathbb{R}}(\Omega) \subset (\Omega + \overline{F^v}) \cap (\Omega - \overline{F^v})$ (resp. $cl_{\mathbb{R}}(\Omega) \subset (\overline{\Omega} + \overline{F^v}) \cap (\overline{\Omega} - \overline{F^v})$) for some facet F^v pointwise fixed by $\nu^v(N(\Omega))$. Hence any apartment containing Ω contains Ω'' and is conjugated to $\mathbb{A}$ by $Q(\Omega'')$.

We may choose $cl_{\mathbb{R}} = cl_{\mathbb{R}}^{\Phi}$. So, for Ω a preordered segment-germ, generic ray or generic ray-germ, we may choose above Ω'' equal to its $cl_{\mathbb{R}}^{\Phi}-$enclosure *i.e.* the corresponding closed-local-facet, spherical sector-face-closure or spherical sector-face-germ. For Ω a preordered open segment-germ (if $\mathcal{Q}$ satisfies (P10)) (resp. a generic open ray) the same result is true with Ω'' the corresponding local facet (resp. corresponding spherical sector-face).

4) Let $\overline{F^l} = \overline{F}^l(x, F^v)$ be a closed local facet in $\mathbb{A}$ and F_1^v a vectorial facet. Then $\mathfrak{r} = \overline{F^l} + \overline{F_1^v}$ is closed convex *i.e.* $cl_{\mathbb{R}}^{\Delta^{ti}}-$enclosed; hence it is the $cl_{\mathbb{R}}^{\Delta^{ti}}-$chimney $\mathfrak{r}_{\mathbb{R}}^{\Delta^{ti}}(F^l, F_1^v)$; note that this is not always true for $cl_{\mathbb{R}}^{\Phi}$, $cl_{\mathbb{R}}^{\#}$ or $cl_{\mathbb{R}}^{\Delta}$.

Suppose $\mathfrak{r}$ solid *i.e.* the fixator in $\nu(N)$ of its support finite. Then $\mathfrak{r}$ and its germ $\mathfrak{R}$ have good fixators: we apply 4.5.3e to $\overline{F^l}$ and $\overline{F^l} + \lambda\xi$ (with $\lambda > 0$, $\xi \in F_1^v$), then 4.5.4 and 4.5.3d to see that $\mathfrak{r}$ has a good fixator; now the result for $\mathfrak{R}$ is a consequence of 4.5.3c.

5) Remark. Suppose F^v and F_1^v as above and of the same sign. Then $\overline{F^v} + \overline{F_1^v}$ meets a vectorial facet F_2^v with $F^v \subset \overline{F_2^v}$ and $F_2^v \cap \langle F^v, F_1^v \rangle$ open in the vector space $\langle F^v, F_1^v \rangle$ (F_2^v is the projection of F_1^v in F^{v*}). By 4.9.3 any apartment containing $\overline{F}^l(x, F^v)$ and $x + F_1^v$ (or $F^l(x, F_1^v)$) contains $\overline{F}^l(x, F_2^v)$. Suppose F_2^v spherical (*e.g.* if $\mathfrak{r}$ is solid) then, by using a few more times the same argument, we see that any apartment containing $\mathfrak{r}$ contains the $cl_{\mathbb{R}}^{\Phi}-$enclosure Ω of $\overline{F}^l(x, F^v)$ and $F^l(x, F_1^v)$ and also $\Omega + \overline{F_1^v}$ which is the $cl_{\mathbb{R}}^{\Phi}-$chimney $\mathfrak{r}_{\mathbb{R}}^{\Phi}(F^l, F_1^v)$. So one could use this $cl_{\mathbb{R}}^{\Phi}-$chimney. But unfortunately it is not clear that Ω or $\Omega + \overline{F_1^v}$ has a good fixator. Moreover the following proposition seems difficult to prove for $cl_{\mathbb{R}}^{\Phi}$. So we shall concentrate on $cl_{\mathbb{R}}^{\Delta^{ti}}$.

6) Proposition. *Let $\mathfrak{R}_1$ be the germ of a splayed $cl_{\mathbb{R}}^{\Delta^{ti}}$−chimney $\mathfrak{r}_1 = \overline{F_1^l} + \overline{F_3^v}$ and $\mathfrak{R}_2$ be either a closed local facet $\overline{F_2^l}$ or the germ of a solid $cl_{\mathbb{R}}^{\Delta^{ti}}$−chimney $\mathfrak{r}_2 = \overline{F_2^l} + \overline{F_4^v}$. Then $\Omega = \mathfrak{R}_1 \cup \mathfrak{R}_2$ has a half-good fixator and $Q(\Omega) = Q(cl_{\mathbb{R}}^{\Delta^{ti}}(\Omega)).N(\Omega)$. In particular any apartment of $\mathscr{I}$ containing Ω also contains $cl_{\mathbb{R}}^{\Delta^{ti}}(\Omega)$ and is conjugated to $\mathbb{A}$ by $Q(cl_{\mathbb{R}}^{\Delta^{ti}}(\Omega))$.*

N.B. Actually if $\mathfrak{R}_2$ is the germ of a splayed $cl_{\mathbb{R}}^{\Delta^{ti}}$−chimney (*i.e.* F_4^v is spherical), then Ω has a good fixator [Ro11, 6.10].

Proof. We may replace Ω by $\Omega = \mathfrak{r}_1 \cup \mathfrak{r}_2$ with $\mathfrak{r}_1$ and $\mathfrak{r}_2$ sufficiently small. Consider $\Theta = \bigcup_{C^v \in F_3^{v*}} \overline{C^v}$; by shortening $\mathfrak{r}_1$ we may assume $\mathfrak{r}_1 \subset \mathfrak{r}_2 + \Theta$. So, by 4.5.3e and 4.6.3 or 4) above, Ω has a half good fixator. We use 4.5.4 with Ω and F_3^v: as $\mathfrak{r}_1 - \Theta = \mathbb{A}$, Ω' is actually equal to $\Omega' = (\mathfrak{r}_2 + \Theta) \cap (\bigcap_{\alpha \in \Phi^m(F_3^v)} D(\alpha, \Omega))$ which is convex and closed. So $\Omega \subset \Omega'' = cl_{\mathbb{R}}^{\Delta^{ti}}(\Omega) = \overline{conv}(\Omega) \subset \Omega'$ and $Q(\Omega) = Q(\Omega'').N(\Omega)$. □

4.10 (Generalized) affine hovels

Definitions. An *affine hovel* of type $(\mathbb{A}, cl)$ is a set $\mathcal{I}$ endowed with a covering $\mathcal{A}$ by subsets called apartments such that:

(MA1) Every $A \in \mathcal{A}$ is an apartment of type $\mathbb{A}$.

(MA2) If F is a point, a preordered open-segment-germ, a generic ray or a solid chimney in an apartment A and if A' is another apartment containing F, then $A \cap A'$ contains the enclosure $cl(F)$ of F in A and there exists a Weyl isomorphism from A to A' fixing (pointwise) this enclosure.

(MA3) If $\mathfrak{R}$ is a splayed chimney-germ, if F is a facet or a solid chimney-germ, then $\mathfrak{R}$ and F are always contained in a same apartment.

(MA4) If two apartments A, A' contain $\mathfrak{R}$ and F as in (MA3), then their intersection $A \cap A'$ contains the enclosure $cl(\mathfrak{R} \cup F)$ of $\mathfrak{R} \cup F$ in A and there exists a Weyl isomorphism from A to A' fixing (pointwise) this enclosure.

This affine hovel is told *ordered* if it satisfies moreover:

(MAO) Let x, y be two points in $\mathcal{I}$ and A, A' be two apartments containing them; if $x \le y$ in A, then the segments $[x, y]_A$ and $[x, y]_{A'}$ defined by x, y in A and A' are equal.

An automorphism of the hovel $\mathcal{I}$ is a bijection $\varphi : \mathcal{I} \to \mathcal{I}$ such that, for every apartment A, $\varphi(A)$ is an apartment and $\varphi_{|A}$ an isomorphism. We say that φ is *positive*, *type-preserving* or a *Weyl automorphism*, if each $\varphi_{|A}$ is positive, type-preserving or a Weyl isomorphism.

We say that a group G acting on $\mathcal{I}$ acts *strongly transitively* if it acts by automorphisms of $\mathcal{I}$ and moreover the Weyl isomorphisms between apartments involved in the axioms (MA2) or (MA4) are induced by elements of G. (In the classical case of thick discrete affine buildings and type preserving groups, this is equivalent to the known definition, *cf.* 4.13.1 below and *e.g.* [AB08, prop. 6.6].)

So G acts strongly transitively if, and only if, the subgroup G^w of Weyl automorphisms acts strongly transitively.

Generalizations. Unfortunately this definition of affine hovel in [Ro11] is still not general enough *e.g.* too restrictive for cl. We shall need three generalizations, that may be considered independently:

1) The enclosure map considered in *loc. cit.* is cl^{Δ}_{ma} or (after changing Δ or $\Lambda = (\Lambda_\alpha)_{\alpha\in\Phi}$) cl^{Φ}, $cl^{\Phi}_{\mathbb{R}}$, $cl^{\Delta}_{\mathbb{R}}$. As suggested in [*l.c.* 1.6] the enclosure map cl^{Δ} is often not so different from cl^{Δ}_{ma}. The results of *loc. cit.* are true for cl^{Δ} without changing anything. We may also enlarge as we want the family Λ to a family Λ'.

2) In *loc. cit.* (except in section 1) the root system Δ is asked to be tamely imaginary. This excludes in particular the totally imaginary case Δ^{ti}.

When Δ is not tamely imaginary, the axioms of affine hovels of type $(\mathbb{A}, cl^{\Delta}_{\Lambda'}))$ have to be modified as follows:

We must add to the list of the filters involved in (MA2) the local facets and the spherical sector faces. Moreover in (MA3) and (MA4) we must add the possibilities that F is a point or a preordered segment germ and that $\mathfrak{R}$ or F is a generic ray germ.

Then all results of *loc. cit.* are true up to section 4 (except the last sentence of [*l.c.* 4.8.2]). In section 5 (specially 5.2 N.B.) we must add (MA2) for F a segment germ and $cl^{\Phi}_{\mathbb{R}}$ *i.e.* :

For $]x,y) \subset F^l(x,F^v)$, any apartment containing $[x,y)$ contains $\overline{F}^l(x,F^v)$
(We can restrict to the case where F^v is a chamber.)

3) The third generalization is necessary when we don't suppose axiom (P10). To get nevertheless an affine hovel in this case, we change the definition as follows:

In the list of axiom (MA2) or in (MA3), (MA4), we replace preordered open-segment-germ by preordered segment-germ, facet by closed facet and (if the generalization 2) above is also used) local facet by closed local facet, spherical sector-face by spherical sector-face closure. Then all results in *loc. cit.* are true if we make the same replacements.

Theorem 4.11. *Let $\mathcal{Q}$ be a very good family of parahorics in G.*

1) Then $\mathscr{I}$ with its family of apartments is an ordered affine hovel of type $(\mathbb{A}, cl^{\Delta^{ti}}_{\mathbb{R}})$. The group G acts strongly transitively on $\mathscr{I}$.

2) The twin buildings $\mathscr{I}^{\pm\infty}$ constructed at infinity of $\mathscr{I}$ in [Ro11, sec. 3] are $G-$equivariantly isomorphic to the combinatorial twin buildings $\mathscr{I}^{vc}_{\pm}$ of 1.6.3 (restricted to their spherical facets). This isomorphism associates to each spherical sector-face-direction $\mathfrak{F}^\infty$ a spherical vectorial facet $F^v \in \mathscr{I}^{vc}_{\pm}$.

3) If F^v spherical corresponds to $\mathfrak{F}^\infty$, then there is a $P(F^v)-$equivariant isomorphism between the affine building $\mathcal{I}(\mathfrak{F}^\infty)$ of [l.c. 4.2] and the (essentialization of the) façade $\mathscr{I}^e_{F^v}$ of $\overline{\mathscr{I}}$.

N.B. a) Of course in this theorem affine hovel must be understood in the generalized sense of 4.10.2 above and also of 4.10.3 if $\mathcal{Q}$ doesn't satisfy axiom (P10).

b) We may replace Δ^{ti} by the non essential system Δ^{tine} with $\Delta^{tine}_{im} = (\sum_{\alpha\in\Phi}\mathbb{R}\alpha) \setminus (\cup_{\alpha\in\Phi}\mathbb{R}\alpha)$ *cf.* 4.9.1.

c) As we chose $cl = cl^{\Delta^{ti}}_{\mathbb{R}}$ (or $cl = cl^{\Delta^{tine}}_{\mathbb{R}}$), the Bruhat-Tits building $\mathscr{I}^e_{F^v}$ in 3) above is endowed with its $\mathbb{R}-$structure.

d) The hovel $\mathscr{I}$ inherits all properties proved in *loc. cit.* . In particular it is endowed with a preorder relation $\leq$ inducing on each apartment the known relation associated to the Tits cone *cf.* 3.4.2b.

e) If a wall $M(\alpha,k)$ contains a panel of a chamber $C \subset D(\alpha,k) \subset \mathbb{A}$, then the chambers adjacent to C along this panel are in one to one correspondence with $U_{\alpha,k}/U_{\alpha,k+}$ (*cf.* [Ro11, 2.9.1] and 4.6.4). In particular $\mathscr{I}$ is thick (3.4.2a).

f) As $G = G^\emptyset.N$ and $N^\emptyset = G^\emptyset \cap N$ is the group of Weyl automorphisms of $\mathbb{A}$, the following proof tells that $G^\emptyset = G^w$ is the subgroup of Weyl automorphisms in G.

Proof. 1) It is sufficient to use the family $\mathcal{Q}^\emptyset$ in $G^\emptyset$. Axiom (MA1) is then clear by definition and all the properties asked for axioms (MA2), (MA4) and (MAO) are proved in 4.5, 4.6 or 4.9. If F and $\mathfrak{R}$ in $\mathbb{A}$ are as in (MA3), then the Bruhat-Birkhoff-Iwasawa decomposition 4.3.4 and corollary 4.8 prove that $G = Q(F).N.Q(\mathfrak{R})$; it is classical that this proves (MA3).

2) The fixator $Q(\mathfrak{f})$ of a spherical sector-face $\mathfrak{f} = x + F^v$ in $\mathbb{A}$ is in $P(F^v)$ (4.7). So the map $\mathfrak{f} \mapsto F^v$ is well defined and onto the spherical facets of $\mathscr{I}^{vc}_\pm$. Consider $\mathfrak{f}_1$ and $\mathfrak{f}_2$, after shortening they are in a same apartment and then, by definition, they are parallel if and only if they correspond to the same F^v. So we have got the desired bijection. Now this bijection is clearly compatible with domination and opposition *cf.* [Ro11, 3.1]: it is an isomorphism of the twin buildings.

3) $\mathcal{I}(\mathfrak{F}^\infty)$ is the set of sector-face-germs with direction $\mathfrak{F}^\infty$. Now in $\mathbb{A}$ we saw (3.7.3) that the map $\mathfrak{F} = germ_\infty(x + F^v) \mapsto [x + F^v]$ identifies the apartment $\mathbb{A}(\mathfrak{F}^\infty)$ in $\mathcal{I}(\mathfrak{F}^\infty)$ with $\mathbb{A}^e_{F^v}$. By 4.8 $Q([x + F^v]) = Q(\mathfrak{F}).N([x + F^v])$; but in $\mathbb{A}$ it is clear that $N([x + F^v]) = N(\mathfrak{F})$, so $Q([x+F^v]) = Q(\mathfrak{F})$. The identification of $\mathcal{I}(\mathfrak{F}^\infty)$ and $\mathscr{I}^e_{F^v}$ is now clear, through a construction as in 4.1. □

4.12 Compatibility with enclosure maps

We have proved good properties with respect to $cl_{\mathbb{R}}^{\Delta^{ti}}$. But the example of Kac-Moody groups ([GR08] or 5 below) proves that we may hope the following strong compatibility property.

1) Definition. The family $\mathcal{Q}$ of parahorics is *compatible with the enclosure map cl* if for all non empty filter Ω in $\mathbb{A}_{F^v}$ and all vectorial chamber $C^v \in F^{v*}$, we have: $Q(\Omega) \cap U(\pm C^v) \subset Q(cl(\Omega))$.

2) Remarks. a) Combined with (P9) and 4.4 this implies $Q(\Omega) \cap P(C^v) \subset Q(cl(\Omega + C^v))$.

b) Even for $cl = cl_{\mathbb{R}}^{\Delta^{ti}}$ this is stronger than (P9), *e.g.* if $\overline{\Omega} + \overline{C^v}$ is not closed in $\mathbb{A}$ or Ω not convex. It implies always (P10).

c) The most important case is when Ω has an (half) good fixator. Then $Q(\Omega) = Q(cl(\Omega)).N(\Omega)$, more precisely we may generalize [GR08, prop. 4.3] :

3) Lemma. *Suppose $\mathcal{Q}$ very good, compatible with cl and $\Omega \subset \Omega' \subset cl(\Omega) \subset \mathbb{A}$.*

If Ω has a good (or half good) fixator, then this is also true for Ω' and $Q(\Omega) = Q(\Omega').N(\Omega)$, $Q(\Omega).N = Q(\Omega').N$. In particular any apartment containing Ω contains its enclosure $cl(\Omega)$ and is conjugated to $\mathbb{A}$ by $Q(cl(\Omega))$.

Conversely, if $supp(\Omega) = \mathbb{A}$ (or $supp(\Omega') = supp(\Omega)$, hence $N(\Omega') = N(\Omega)$), Ω has an half good fixator and Ω' has a good fixator, then Ω has a good fixator.

4) Consequences. All the results proved in [GR08, sec. 4] are then true. For example the results in 4.9 above for $cl_{\mathbb{R}}$ or $cl_{\mathbb{R}}^{\Delta^{ti}}$ are true for cl; hence:

5) Theorem. *If $\mathcal{Q}$ is a very good family of parahorics compatible with cl, then theorem 4.11 is true with type $(\mathbb{A}, cl)$ instead of $(\mathbb{A}, cl_{\mathbb{R}}^{\Delta^{ti}})$. If $cl = cl_{\Lambda'}^{\mathcal{P}}$ and $\mathcal{P} \subset \Delta$ is tamely imaginary, there is no need of generalization in the notion of affine hovel (except perhaps generalization 1).*

4.13 Backwards constructions

1) Lemma. *Let $\mathcal{I}$ be an affine hovel of type $(\mathbb{A}, cl)$ with a group G acting on it strongly transitively. Then G acts transitively on the apartments and the stabilizer N of an apartment A induces in $\mathbb{A}$ a group $\nu(N)$ containing the group W^{ath} generated by the reflections along the thick (hence true) walls.*

N.B. The subgroup W^{ath} of W^a is equal to it when $\mathcal{I}$ is thick.

Proof. Let $\mathfrak{S}_1 \subset A_1$, $\mathfrak{S}_2 \subset A_2$ be sector germs in apartments. By (MA3) there exists an apartment A_3 containing $\mathfrak{S}_1$ and $\mathfrak{S}_2$. By (MA2) there exists $g_1, g_2 \in G$ with $A_1 = g_1 A_3$ and $A_2 = g_2 A_3$, so A_1 and A_2 are conjugated by G.

If now M is a thick wall in A, we write D_1, D_2 the half-apartments in A limited by M. By [Ro11, 2.9] there is a third half-apartment D_3 in $\mathcal{I}$ limited by M such that for $i \neq j$, $D_i \cap D_j = M$ and $D_i \cup D_j$ is an apartment A_{ij}. By (M4) applied to a sector-panel-germ $\mathfrak{F}$ in M and a sector-germ in D_i (dominating the opposite in M of $\mathfrak{F}$) there exists $g_{ijk} \in G$ with $g_{ijk}.A_{ij} = A_{ik}$ (where $\{1,2,3\} = \{i,j,k\}$). Now $A = A_{12}$ and $g_{142}.g_{231}.g_{123}$ (where $D_4 = g_{231}.D_1$) stabilizes A and exchanges D_1 and D_2: it is the reflection with respect to M. □

2) Let $\mathcal{I}$ and G be as in the lemma. Then G acts nicely on the twin buildings $\mathcal{I}^{\pm\infty}$ and we saw in [Ro11, 3.8] following [T92], that G is often endowed with a RGD system.

3) Suppose G endowed with a root group datum such that the corresponding twin buildings $\mathscr{I}^{vc}_{\pm}$ are identified with $\mathcal{I}^{\pm\infty}$, in particular G acts via positive, type-preserving automorphisms. Then the action of G on the affine buildings $\mathcal{I}(\mathfrak{F}^\infty)$ (for $\mathfrak{F}^\infty$ a panel in $\mathcal{I}^{\pm\infty}$) should endow the root group datum with a valuation as in the classical case [Ro11, 4.12].

4) Suppose now the existence of a valuation of the root group datum which gives the affine buildings $\mathcal{I}(F^v)$ on which $P(F^v)$ acts through $P(F^v)/U(F^v)$ (for any spherical vectorial facet F^v). Then $\mathcal{I}$ is constructed as in 4.1 with a family $\mathcal{Q} = (Q(x))_{x\in\mathbb{A}}$ of parahorics. We define also $\mathcal{Q}$ on $\overline{\mathbb{A}}^e_{sph}$ by the action of G on the buildings $\mathcal{I}(F^v)$. Let's look to the properties satisfied by $\mathcal{Q}$:

(P1), (P2), (P4), (P5) and (P6) are clear by definition and hypothesis.

By [Ro11, 4.7] $x \in \mathcal{I}$ and $F^v \in \mathcal{I}^{\pm\infty}$ (hence spherical) determine a unique sector face $x + F^v$ so (P9) is satisfied: $Q(x) \cap P(F^v)$ stabilizes $x + F^v$ and, up to elements fixing $x + \overline{F^v}$, it stabilizes $\mathbb{A}$ and is type preserving, hence fixes $x + \overline{F^v}$. As $\mathcal{Q}$ is well known on $\overline{\mathbb{A}}^e_{sph}$, $Q(x) \cap P(F^v)$ fixes $(x + \overline{F^v}) \cap \overline{\mathbb{A}}^e_{sph}$.

Now let $u \in U_{\alpha,\lambda}$ and F^v a panel in $\text{Ker}\alpha = M^\infty$. Then by [*l.c.* sec.4] $u\mathbb{A}$ is an apartment of the building $\mathcal{I}(M^\infty) \simeq \mathcal{I}(F^v)$ (which is a tree) and its intersection with $\mathbb{A}$ is an half-apartment $D(\alpha, \mu)$. But by definition of the valuation u fixes $pr_{F^v}(D(\alpha, \lambda)) \subset \mathbb{A}^e_{F^v}$; so $\mathbb{A} \cap u\mathbb{A} \supset D(\alpha, \lambda)$ hence u fixes $D(\alpha, \lambda)$. So (P3) is satisfied.

For $x \in \mathbb{A}$ and $g \in G$, suppose $g \in Q(x).N \cap P(F^v).N$ then $g\mathbb{A} \ni x$, $g\mathbb{A}^v \supset F^v$ and $g\mathbb{A}$ contains the sector face $x + F^v$ [Ro11, 4.7]. So by (MA2) $g \in Q(x + \overline{F^v}).N$. But $Q(x + \overline{F^v}) \supset Q(x)$ and $Q(x + \overline{F^v}) \supset Q(pr_{F^v}(x))$ as $pr_{F^v}(x) \in \mathbb{A}_{F^v}$ is the class of $x + \overline{F^v}$. Hence (P7) is satisfied.

When $\overline{\mathbb{A}} = \overline{\mathbb{A}}^e$, Charignon proved that (P8) is satisfied for every good family of parahorics (4.3.6). We may also use a geometrical translation of (P8) for good families satisfying (P9): let $x \in \mathbb{A}$ and $\mathfrak{s}$ a sector of origin x in $\mathbb{A}$, then any apartment A' containing x contains also a sector $\mathfrak{s}_1$ of origin x opposite $\mathfrak{s}$ (in an apartment containing them both).

So if $\overline{\mathbb{A}} = \overline{\mathbb{A}}^e$ is essential, we know that the family $\mathcal{Q}$ (defined on $\mathbb{A} \cup \overline{\mathbb{A}}_{sph}$) is very good. (P10) is satisfied if $\mathcal{I}$ is an hovel in the sense of [Ro11] (without generalization 4.10.3).

5) These sketchy constructions reduce more or less the classification problem for affine hovels with a good group of automorphisms to the problem of existence (or uniqueness ?) of very good (excellent ?) families of parahorics associated to valuated RGD systems.

Proposition 4.14. *We consider a group G (resp. G') acting strongly transitively on an ordered affine hovel $\mathcal{I}$ (resp. $\mathcal{I}'$) and a map $j : \mathcal{I} \to \mathcal{I}'$ which is $G-$equivariant with respect to an homomorphism $\varphi : G \to G'$. We suppose that:*

1) There exist apartments $A \subset \mathcal{I}$ and $A' \subset \mathcal{I}'$ such that $j_{|A}$ is injective affine from A to A'.

2) There exists a sector germ $\mathfrak{S}$ in A such that the direction of the cone $j(\mathfrak{S})$ meets the interior of the Tits cone $\mathcal{T}'_\pm$ in $\overrightarrow{A'}$.

Then j is injective.

N.B. We exclude here for $\mathcal{I}$ the generalization 4.10.3 in the definition of hovels. For buildings the proof is easier, as two points are in a same apartment.

Proof. Let $x_1, x_2 \in \mathcal{I}$ such that $j(x_1) = j(x_2)$. There is an apartment $A_i = g_i A$ containing x_i and $\mathfrak{S}$, with g_i fixing pointwise a sector $\mathfrak{s}$ in $\mathfrak{S}$. Then $A'_i = \varphi(g_i)A'$ is an apartment containing $j(x_i)$ and $j(\mathfrak{s})$ with $\varphi(g_i)$ fixing pointwise $j(\mathfrak{s})$. Let's consider $y \in \mathfrak{s}$ sufficiently far away; then $[y, x_i]$ and $j([y, x_i]) = [j(y), j(x_i)]$ are preordered (even generic) segments in A_i and A'_i. But $j(x_1) = j(x_2)$, so $[j(y), j(x_1)] = [j(y), j(x_2)]$ (axiom (MAO)). As $g = g_2 g_1^{-1}$ fixes pointwise the segment germs $[y, x_1)$ and $[y, x_2)$, $\varphi(g)$ fixes pointwise $[j(y), j(x_1)) = [j(y), j(x_2))$ and, as $\varphi(g)$ is an affine isomorphism from A'_1 to A'_2, it fixes pointwise the whole segment $[j(y), j(x_1)] = [j(y), j(x_2)]$. Then $g[y, x_1]$ and $[y, x_2]$ are two segments in A_2 with the same image $[j(y), j(x_2)]$ in A'_2 by j (injective on the apartments). So these segments are equal; in particular $[y, x_1) = [y, x_2)$.

Now $j([y, x_1]) = j([y, x_2])$ and $[y, x_1) = [y, x_2)$. Then $[y, x_1] \cap [y, x_2]$ is a segment $[y, z]$ (*cf.* (MA2) for open-segment-germs, as we avoid 4.10.3) with $z \neq y$. We are done if $z = x_1$ or $z = x_2$. Otherwise $[z, x_1)$ and $[z, x_2)$ are distinct segment germs in a same apartment [Ro11, 5.1] with the same image by j, contrary to the hypothesis. □

5 Hovels and bordered hovels for split Kac-Moody groups

We consider now the situation of 3.3 and 3.5 and shall build a very good family $\widehat{\mathcal{P}}$ of parahorics following [Ro12]. We choose the enclosure map $cl = cl^\Delta$.

5.1 The parahoric subgroup associated to $y \in \mathbb{A}$

1) The free case with $V = V^x$: In [Ro12] the RGS $\mathcal{S}$ is supposed free and the affine apartment $\mathbb{A}$ is equal to $\mathbb{A}^x$ with associated vector space $V^x = Y \otimes_{\mathbb{Z}} \mathbb{R}$. Then for $y \in \mathbb{A}$, one defines the group $\widehat{P}(y) = U_y^{pm+}.U_y^{nm-}.N(y) = U_y^{nm-}.U_y^{pm+}.N(y)$ where $N(y)$ is the fixator of y in N and U_y^{pm+} (resp. U_y^{nm-}) is the intersection with G or U^+ (resp. U^-) of a group $U_y^{ma+} = \prod_{\alpha \in \Delta^+} U_\alpha(y)$ (resp. $U_y^{ma-} = \prod_{\alpha \in \Delta^-} U_\alpha(y)$) which exists in a suitable completion G^{pma} (resp. G^{nma}) of the Kac-Moody group G [*l.c.* 4.5, 4.14]; actually one has to define suitably $U_\alpha(y)$ for $\alpha \in \Delta_{im}$: $U_\alpha(y) = U_{\alpha,-\alpha(y)} := U_{\{y\}}^{ma}(\{\alpha\})$ in the notations of [*l.c.* 4.5.2].

The group ${U^\pm}_y = U^\pm \cap G(y)$ of 4.2.1 is clearly included in $\widehat{P}(y)$. As $U_y^{pm+} = U^+ \cap \widehat{P}(y)$, we have $U_y^+ \subset U_y^{pm+}$ and, similarly, $U_y^- \subset U_y^{nm-}$. Moreover we know that $\widehat{P}(y) = U_y^{pm+}.U_y^-.N(y) = U_y^{nm-}.U_y^+.N(y)$ [*l.c.* 4.14];

The interesting point for us is that U_y^{ma+}, U_y^{ma-}, U_y^{pm+}, U_y^{nm-}, U_y^+ or U_y^- depend only of the true half-apartments (imaginary or not) containing y. In particular they depend only of the class $\overline{y}$ of y in the essentialization $\mathbb{A}^q = \mathbb{A}^x/V_0$.

In the classical case where Φ is finite (and Δ_{im} empty) the group U_y^{pm+} (resp. U_y^{nm-}) is the group U_y^{++} (resp. U_y^{--}) generated by the groups $U_\alpha(y)$ for $\alpha \in \Phi^+$ (resp. $\alpha \in \Phi^-$).

2) Consider now any RGS $\mathcal{S}$, any affine apartment $\mathbb{A}$ as in 3.5 for the root datum in $G = \mathfrak{G}_{\mathcal{S}}(K)$ and any $y \in \mathbb{A}$. By [Ro12, 1.3, 1.11] there is an injective homomorphism $\varphi : G \hookrightarrow G^{xl} = \mathfrak{G}_{\mathcal{S}^l}(K)$ where $\mathcal{S}^l$ is a free RGS. The affine apartment associated to it is $\mathbb{A}^{xl}$ and we know that the essentializations of $\mathbb{A}$ and $\mathbb{A}^{xl}$ are equal: $\mathbb{A}/V_0 = \mathbb{A}^{xl}/V_0^{xl} = \mathbb{A}^q$.

To $\overline{y} \in \mathbb{A}^q$ we associated above some subgroups of G^{xl}. By [*l.c.* 1.9.2, 3.19.3] the groups $U^\pm$ in G and G^{xl} are isomorphic by φ, so $U_{\overline{y}}^{pm+}$, $U_{\overline{y}}^{nm-}$, $U_{\overline{y}}^+$ and $U_{\overline{y}}^-$ are actually in G (and if $\mathcal{S}$ is free, they are as defined in 1) above). We define the group $\widehat{P}^m(\overline{y})$ as generated by $U_{\overline{y}}^{pm+}$, $U_{\overline{y}}^{nm-}$ and $\mathfrak{T}_{\mathcal{S}}(\mathcal{O}) \subset Z_0 = \mathrm{Ker}\nu$. We define $N(y)$ the fixator of y in N and $\widehat{P}(y) = \widehat{P}^m(\overline{y}).N(y)$ which is called the *fixator group* associated to y in $\mathbb{A}$ (*cf.* 5.2b below).

Lemma 5.2. *a) We have* $\widehat{P}^m(\overline{y}) = U_{\overline{y}}^{pm+}.U_{\overline{y}}^{nm-}.N^m(\overline{y}) = U_{\overline{y}}^{nm-}.U_{\overline{y}}^{pm+}.N^m(\overline{y}) = U_{\overline{y}}^{pm+}.U_{\overline{y}}^-.N^m(\overline{y}) = U_{\overline{y}}^{nm-}.U_{\overline{y}}^+.N^m(\overline{y})$ *where* $N^m(\overline{y})$ *is a subgroup of* $N(y)$*, hence fixing pointwise* $y + V_0 \subset \mathbb{A}$.

b) Moreover $\widehat{P}^m(\overline{y})$ *does not change when one changes* Φ^+ *by* W^v*, hence it is normalized by* $N(y)$ *and* $\widehat{P}(y)$ *is a group.*

Proof. a) We identify G and $\varphi(G) \subset G^{xl}$. We choose an origin in $\mathbb{A}$ (resp. $\mathbb{A}^{xl}$) fixed by $\nu(m(x_{\alpha_i}(1)))$, $\forall i \in I$; hence $\mathbb{A}$ (resp. $\mathbb{A}^{xl}$) is identified with V (resp. V^{xl}) and $\nu(N)$ (resp. $\nu^{xl}(N^{xl})$ where $N^{xl} = N_{\mathcal{S}^l}(K)$) with $\nu(T) \rtimes W^v$ (resp. $\nu^{xl}(T^{xl}) \rtimes W^v$) where W^v acts linearly via ν^v. Actually $\nu : T \to V$ factorizes through $T/\mathfrak{T}(\mathcal{O}) = Y \otimes \Lambda$: $\nu(t) = \overline{\nu}(\overline{t})$ where $\overline{t}$ is the class of t modulo $\mathfrak{T}(\mathcal{O})$; and the same thing for ν^{xl}. We consider $z \in V^{xl}$ such that $\overline{z} = \overline{y} \in V^q$.

By [*l.c.* 4.6, 4.14] we have $U_{\overline{y}}^{pm+}.U_{\overline{y}}^{nm-}.N^{min}(\overline{y}) \subset \widehat{P}^m(\overline{y}) \subset \widehat{P}^{xl}(z) = U_{\overline{y}}^{pm+}.U_{\overline{y}}^{nm-}.N^{xl}(z) = U_{\overline{y}}^{pm+}.U_{\overline{y}}^-.N^{xl}(z)$ where $N^{min}(\overline{y})$ is a subgroup of N and $N^{xl}(z)$ the fixator in N^{xl} of z. Moreover $N^{xl}(z) = N^{xl} \cap \widehat{P}^{xl}(z)$. It is now clear that $\widehat{P}^m(\overline{y}) = U_{\overline{y}}^{pm+}.U_{\overline{y}}^{nm-}.N^m(\overline{y}) = U_{\overline{y}}^{pm+}.U_{\overline{y}}^-.N^m(\overline{y})$ with $N^m(\overline{y}) = \widehat{P}^m(\overline{y}) \cap N^{xl}(z) = \widehat{P}^m(\overline{y}) \cap N \subset N \cap N^{xl}(z)$. The same thing is clearly true when exchanging $U_{\overline{y}}^{pm+}$, $U_{\overline{y}}^+$ and $U_{\overline{y}}^{nm-}$, $U_{\overline{y}}^-$.

Let $n = tw \in N \cap N^{xl}(z)$ with $t \in T$ and $w \in W^v$ (fixing 0). We have $z = nz = \nu^{xl}(t) + w(z)$. But, if $w = s_{i_1}.\cdots.s_{i_n} \in W^v$, $z - w(z) = \sum_{j=1}^n (s_{i_{j+1}}.\cdots.s_{i_n}(z) - s_{i_j}.\cdots.s_{i_n}(z)) = \sum_{j=1}^n \alpha_{i_j}(s_{i_{j+1}}.\cdots.s_{i_n}(z)).\alpha_{i_j}^\vee =: \partial(\overline{z}, V^{xl})$ an element of V^{xl} depending only of $\overline{z} = \overline{y}$. Hence $\overline{\nu}^{xl}(\overline{t}) = \partial(\overline{z}, V^{xl})$; but $\overline{\nu}^{xl}$ is one to one, so $\overline{t} \in (\sum_{i\in I} \mathbb{R}\alpha_i^\vee \otimes 1) \cap Y \otimes \Lambda$. By 3.5.3a, there exists $r \in \mathbb{Z}_{>0}$ with $r\overline{t} = -\sum_{i\in I} \alpha_i^\vee \otimes \lambda_i$ with $\lambda_i \in \Lambda$ a suitable $\mathbb{Z}-$linear combination of the coefficients $r.\alpha_{i_j}(s_{i_{j+1}}.\cdots.s_{i_n}(z)) \in \mathbb{R}$ (as the relations between the $\alpha_i^\vee$ in $Y \subset Y^{xl}$ have coefficients in $\mathbb{Q}$). Now $\overline{\nu}(r\overline{t}) = \sum_{i\in I} \lambda_i\alpha_i^\vee \in V$ and, by the expression of the λ_i, $r\overline{\nu}(\overline{t}) = r\partial(\overline{z}, V)$ (as the $\alpha_i^\vee$ in V satisfy the $\mathbb{Z}-$linear relations between the $\alpha_i^\vee$ in Y). In V we may divide by r, so $\overline{\nu}(\overline{t}) = \partial(\overline{z}, V)$. By the same calculations as above $\nu(n)$ fixes any element y with $\overline{y} = \overline{z}$.

b) It is proved in [Ro12, 4.6c] that $U_{\overline{y}}^{pm+}.U_{\overline{y}}^{nm-}.N^{min}(\overline{y})$ doesn't change when one changes Φ^+ by W^v. So it is the same for the subgroup $\widehat{P}^m(\overline{y})$ it generates. □

5.3 The fixator group associated to $y \in \overline{\mathbb{A}} \setminus \mathbb{A}$

For F^v a non minimal vectorial facet, the façade $\mathbb{A}_{F^v}$ is an affine apartment for the group $P(F^v)/U(F^v) = G(F^v) \simeq M(F^v)$ endowed with the generating root datum $(G(F^v), (U_\alpha)_{\alpha\in\Phi^m(F^v)}, Z)$ *cf.* 1.6.5 and 3.4. Moreover $G(F^v)$ is actually the group of $K-$points of a Kac-Moody group: if $F^v = F^v_\varepsilon(J)$ then $G(F^v) = G(J) = \mathfrak{G}_{\mathcal{S}(J)}(K)$.

So for $y \in \mathbb{A}_{F^v}$ we may define $\widehat{P}(y)$ as the subgroup of $P(F^v)$ inverse image of the subgroup $\widehat{P}_{F^v}(y)$ constructed inside $G(F^v) \simeq M(F^v)$ as above. We have $\widehat{P}(y) = \widehat{P}_{F^v}(y) \ltimes U(F^v) = U^{pm+}_{F^v y}.U^{nm-}_{F^v y}.N(y).U(F^v) = U^{nm-}_{F^v y}.U^{pm+}_{F^v y}.N(y).U(F^v) = U^{pm+}_{F^v y}.U^{-}_{F^v y}.N(y).U(F^v) = U^{nm-}_{F^v y}.U^{+}_{F^v y}.N(y).U(F^v)$ where $+$ and $-$ refer to the choice of a chamber $C^v \in F^{v*}$.

Remark. Even if $F^v = V_0$ is minimal, a point $\overline{y} \in \mathbb{A}^e_{F^v} = \mathbb{A}_{F^v}/\overrightarrow{F^b}$ corresponds to a collection $y + \overrightarrow{F^b}$ of points $y \in \mathbb{A}^{ne}_{F^v} = \mathbb{A}$. So we have two parahoric subgroups $\widehat{P}(y) \subset \widehat{P}(\overline{y}) = \widehat{P}(y).N(F^v)(\overline{y})$ and $N(F^v)(\overline{y})$ acts by translations on $y + \overrightarrow{F^b}$. We say that $\widehat{P}(y)$ (resp. $\widehat{P}(\overline{y})$) is the strong (resp. weak) fixator of y or $\overline{y}$.

Definition 5.4. We define $\widehat{\mathcal{P}}$ as the family $(\widehat{P}(y))_{y\in\overline{\mathbb{A}}}$. By construction it is a family of parahorics. The corresponding hovel (resp. bordered hovel) will be written $\mathscr{I} = \mathscr{I}(\mathfrak{G}_{\mathcal{S}}, K, \mathbb{A})$ (resp. $\overline{\mathscr{I}} = \overline{\mathscr{I}}(\mathfrak{G}_{\mathcal{S}}, K, \overline{\mathbb{A}})$) and called the *affine hovel* (resp. *affine bordered hovel*) of $\mathfrak{G}_{\mathcal{S}}$ over K with model apartment $\mathbb{A}$ (resp. $\overline{\mathbb{A}}$). When we add the adjective essential we mean that $\mathbb{A} = \mathbb{A}^q$ (resp. $\overline{\mathbb{A}} = \overline{\mathbb{A}}^e$).

It is perhaps possible that $\widehat{\mathcal{P}} = \mathcal{P}$ [Ro12, 4.13.5], see also 5.11.4c.

Lemma 5.5. *Let $x \in \overline{\mathbb{A}}$, $F^v = F^v(x)$ and C^v a chamber in F^{v*}; then:*
$\widehat{P}(x) \cap N = N(x)$; $\widehat{P}(x) \cap N.U(C^v) = N(x).U^{pm+}_{F^v x}.U(F^v)$; $\widehat{P}(x) \cap U(C^v) = U^{pm+}_{F^v x}.U(F^v)$
$\widehat{P}(x) \cap N.U(-C^v) = N(x).U^{nm-}_{F^v x}$ *and* $\widehat{P}(x) \cap U(-C^v) = U^{nm-}_{F^v x}$.

Proof. Let $g \in \widehat{P}(x) \cap N.U(C^v)$. So $g = nu^+ = n'v^-v^+u_{F^v}$ with $n \in N$, $u^+ \in U(C^v)$, $n' \in N(x)$, $v^- \in U^{nm-}_{F^v x}$, $v^+ \in U^{pm+}_{F^v x}$ and $u_{F^v} \in U(F^v)$. Hence $(n'^{-1}n)(u^+u^{-1}_{F^v}(v^+)^{-1}) = v^- \in N.U(C^v) \cap U(-C^v)$. By the uniqueness in the Birkhoff decomposition (1.6.2) we have $v^- = 1$, $n = n'$ and $u^+ = v^+u_{F^v}$ so $g \in N(x).U^{pm+}_{F^v x}.U(F^v)$. If moreover $g \in N$ (resp. $g \in U(C^v)$) we have $u^+ = 1$ (resp. $n = 1$) hence $g = n' \in N(x)$ (resp. $g = v^+u_{F^v} \in U^{pm+}_{F^v x}.U(F^v)$).

Now let $g \in \widehat{P}(x) \cap N.U(-C^v)$. We write $g = nu^- = n'v^+v^-u_{F^v} = n'v^+u'_{F^v}v^-$ (with obvious notations). Hence $(n'^{-1}n)(u^-(v^-)^{-1}) = v^+u'_{F^v} \in N.U(-C^v) \cap U(C^v)$. So $v^+u'_{F^v} = 1$ (hence $v^+ = u'_{F^v} = 1$, as $P(F^v) = M(F^v) \ltimes U(F^v)$), $n = n'$ and $u^- = v^-$. We have $g = n'v^- \in N(x).U^{nm-}_{F^v x}$ and, if $g \in U(-C^v)$, $n = n' = 1$ so $g = v^- \in U^{nm-}_{F^v x}$. □

Proposition 5.6. *The family $\widehat{\mathcal{P}}$ is a very good family of parahorics. It is compatible with the enclosure map cl^Δ. Hence $\mathscr{I}(\mathfrak{G}_{\mathcal{S}}, K, \mathbb{A})$ is a thick ordered affine hovel of type $(\mathbb{A}, cl^\Delta)$ and $G = \mathfrak{G}(K)$ acts strongly transitively on it.*

Proof. We proved above (P5). If $F^v(x)$ is spherical, $U^{pm+}_{F^v x}$ and $U^{nm-}_{F^v x}$ are generated by the groups $U_\alpha(x)$ for $\alpha \in \Phi^{m\pm}(F^v)$, so (P6) holds. By definition $\widehat{P}(x) = U^{pm+}_{F^v x}.U^{nm-}_{F^v x}.N(x).U(F^v) = U^{nm-}_{F^v x}.U^{pm+}_{F^v x}.N(x).U(F^v)$, so (P8) is a consequence of lemma 5.5. We have also $\widehat{P}(x) = (\widehat{P}(x) \cap U(-C^v)).(\widehat{P}(x) \cap U(C^v)).N(x)$.

Consider now the situation of (P7) or (P9). We have to prove $N.\widehat{P}(x) \cap P(F^v) \subset N.\widehat{P}(\{x, pr_{F^v}(x)\})$ and $\widehat{P}(x) \cap P(F^v) \subset \widehat{P}(\overline{x + F^v})$. These relations are in $P(F^v_1)$ and each side contains $U(F^v_1)$, so we may argue in $G(F^v_1) = P(F^v_1)/U(F^v_1)$. Actually we shall suppose

$x \in \mathbb{A}$. Consider a chamber $C^v \in F^{v*}$, we have $P(F^v) = U(F^v) \rtimes M(F^v)$ and (by Iwasawa) $M(F^v) = (U(C^v) \cap M(F^v)).N(F^v).G(\Phi^m(F^v), x)$ with $G(\Phi^m(F^v), x) \subset \widehat{P}(\overline{x + F^v})$. Let $g \in N.\widehat{P}(x) \cap P(F^v)$ (resp. $g \in \widehat{P}(x) \cap P(F^v)$). We write $g = n'q = u_{F^v} v^+ nq'$ with $n' \in N$ (resp. $n' = 1$), $q \in \widehat{P}(x)$, $u_{F^v} \in U(F^v)$, $v^+ \in U(C^v) \cap M(F^v)$, $n \in N(F^v)$, $q' \in \widehat{P}(\overline{x + F^v})$ and we want to prove that $g \in N.\widehat{P}(\overline{x + F^v})$ (resp. $g \in \widehat{P}(\overline{x + F^v})$). So one may suppose $q' = 1$, then $g \in nU(n^{-1}C^v)$ and $q \in n'^{-1}nU(n^{-1}C^v)$. By the proof of 5.5 $q \in n'^{-1}nU_x^{pm}(n^{-1}C^v)$ with $n'^{-1}n \in N(x)$ and, as $n \in N(F^v)$, $U_x^{pm}(n^{-1}C^v) \subset \widehat{P}(\overline{x + F^v})$. So $g = n'q \in N.\widehat{P}(\overline{x + F^v})$ (resp. $g = q \in \widehat{P}(\overline{x + F^v})$, as $n = n'^{-1}n \in N(x) \cap N(F^v) \subset N(\overline{x + F^v})$).

By 5.5 $\widehat{P}(x) \cap U(C^v) = U_{F^v x}^{pm+} \ltimes U(F^v)$ and $\widehat{P}(x) \cap U(-C^v) = U_{F^v x}^{nm-}$. So $\widehat{P}(\Omega) \cap U(C^v) = U_{F^v \Omega}^{pm+} \ltimes U(F^v)$ and $\widehat{P}(\Omega) \cap U(-C^v) = U_{F^v \Omega}^{nm-}$ and these groups depend only of $cl^\Delta(\Omega)$ [Ro12, 4.5.4f]. We have proved that $\widehat{\mathcal{P}}$ is compatible with cl^Δ. □

5.7 Remarks

1) So we get for $\mathscr{I}$ and $\overline{\mathscr{I}}$ all the properties proved in section 4. The map $g\mathfrak{T}g^{-1} \mapsto A(g\mathfrak{T}g^{-1}) = g\mathbb{A}$ (resp. $g\mathfrak{T}g^{-1} \mapsto \overline{A}(g\mathfrak{T}g^{-1}) = g\overline{\mathbb{A}}$) is a bijection between the split maximal tori in $\mathfrak{G}_\mathcal{S}$ and the apartments in $\mathscr{I}$ (resp. the bordered apartments in $\overline{\mathscr{I}}$) *cf.* 4.6.4 and 1.8.1.

2) Actually we proved (P7) and (P9) even when F^v is non spherical. So one may define a projection $pr_{F^v} : \mathscr{I}_{F_1^v} \to \mathscr{I}_{F^v}$ even if $F^v \in F_1^{v*}$ is non spherical [Ch11, 11.7.3]. This gives stronger links between the hovel $\mathscr{I}$ and its non spherical façades.

3) For (P8) we proved also $\widehat{P}(x) = (\widehat{P}(x) \cap U(-C^v)).(\widehat{P}(x) \cap U(C^v)).N(x)$ which improves (P8) essentially when $F^v(x)$ has a well defined sign.

4) If we choose $\mathbb{A}$ as in [Ro12, 4.2] (which implies $\mathcal{S}$ free) then $\mathscr{I}(\mathfrak{G}_\mathcal{S}, K, \mathbb{A})$ is the affine hovel $\mathscr{I}(\mathfrak{G}_\mathcal{S}, K)$ defined in [*l.c.* 5.1], with the same action of $G = \mathfrak{G}_\mathcal{S}(K)$, the same apartments, the same enclosure map, the same facets, ...). By lemma 5.5 the notions of (half) good fixators for filters in $\mathbb{A}$ are the same. Note however that, when (and only when) Ω has not an (half) good fixator, $\widehat{P}(\Omega)$ may be different from $\widehat{P}_\Omega$ as defined in [Ro12].

The group $G^\emptyset$ of Weyl automorphisms in G (3.4.3c, 4.11f) is equal to $\psi(G^A).Z_0$ defined in [Ro12, 5.13.2 or] (as $\psi^{-1}(N) = N^A$ and $\nu(\psi(N^A)) = W^a$).

5) A point $\overline{x} \in \mathscr{I}_{F^v}$ determines a sector-face-germ $\mathfrak{F} = germ_\infty(x + F^v)$ of direction F^v in $\mathscr{I}$ and the correspondence is one to one if $\overline{\mathbb{A}} = \overline{\mathbb{A}}^e$ (or $\overline{\mathbb{A}} = \overline{\mathbb{A}}^i$ and F^v non trivial) *cf.* 4.11. The strong (resp. weak) fixator of $\overline{x}$ (*cf.* 5.3) is the set of g in G which fix pointwise an element (resp. which induce a bijection between the sets which are elements) of the filter $\mathfrak{F}$.

5.8 Functoriality

1) Changing the group, commutative extensions: Let's consider a commutative extension of RGS $\varphi : \mathcal{S} \to \mathcal{S}'$ [Ro12, 1.1]. We then get an homomorphism $\mathfrak{G}_\varphi : \mathfrak{G}_\mathcal{S} \to \mathfrak{G}_{\mathcal{S}'}$ inducing homomorphisms $\mathfrak{T}_\varphi : \mathfrak{T}_\mathcal{S} \to \mathfrak{T}_{\mathcal{S}'}$, $\mathfrak{N}_\varphi : \mathfrak{N}_\mathcal{S} \to \mathfrak{N}_{\mathcal{S}'}$ and isomorphisms $\mathfrak{U}^\pm{}_\mathcal{S} \to \mathfrak{U}^\pm{}_{\mathcal{S}'}$. If $\mathbb{A}$ is a suitable apartment for $(\mathfrak{G}_{\mathcal{S}'}, \mathfrak{T}_{\mathcal{S}'})$ (3.5.3a) it is clearly suitable for $(\mathfrak{G}_\mathcal{S}, \mathfrak{T}_\mathcal{S})$ and, for $x \in \mathbb{A}_{F^v}$, $U_{F^v x}^{pm+}$ or $U_{F^v x}^{nm-}$ is the same for $\mathfrak{G}_\mathcal{S}$ or $\mathfrak{G}_{\mathcal{S}'}$. Hence $\widehat{P}_{\mathcal{S}'}(x) = G_\varphi(\widehat{P}_\mathcal{S}(x)).N_{\mathcal{S}'}(x)$. But $G_\varphi^{-1}(N_{\mathcal{S}'}) = N_\mathcal{S}$ [*l.c.* 1.10] hence $\mathrm{Ker}G_\varphi \subset T_\mathcal{S}$, so the lemma 5.5 tells that $G_\varphi^{-1}(\widehat{P}_{\mathcal{S}'}(x)) = \widehat{P}_\mathcal{S}(x)$. It is now clear that $G_\varphi \times Id_\mathbb{A}$ induces a $G_\varphi-$equivariant embedding $\mathscr{I}(\mathfrak{G}_\varphi, K, \mathbb{A}) : \mathscr{I}(\mathfrak{G}_\mathcal{S}, K, \mathbb{A}) \hookrightarrow \mathscr{I}(\mathfrak{G}_{\mathcal{S}'}, K, \mathbb{A})$ which is an isomorphism (bijection between the sets of apartments, isomorphism of the apartments). Hence the affine Weyl groups W^a are the same, but $\nu(N_\mathcal{S}) \subset \nu(N_{\mathcal{S}'})$ are in general different.

The same things are true for the bordered hovels and the embeddings are functorial (note however that $\mathbb{A}$ or $\overline{\mathbb{A}}$ depends on $\mathfrak{G}'$).

2) Changing the group, Levi factors: For a vectorial facet F^v, we may consider the homomorphism $M(F^v) \hookrightarrow G$. More precisely if $F^v = F^v_\varepsilon(J)$ then $\mathfrak{G}_{\mathcal{S}(J)}$ embeds into $\mathfrak{G}_{\mathcal{S}}$ (1.6.5 and 1.8.1). If $\mathbb{A}$ is suitable for $\mathfrak{G}_{\mathcal{S}}$ then it is also suitable for $\mathfrak{G}_{\mathcal{S}(J)}$, but we have only to consider the walls of direction Kerα with $\alpha \in Q(J)$. By construction $\mathscr{I}(\mathfrak{G}_{\mathcal{S}(J)}, K, \mathbb{A})$ is $G_{\mathcal{S}(J)}-$equivariantly isomorphic to the façade $\mathscr{I}(\mathfrak{G}_{\mathcal{S}}, K, \overline{\overline{\mathbb{A}}})_{F^v}$ for $F^v = F^v_\varepsilon(J)$ or $F^v_{-\varepsilon}(J)$ or any other maximal vectorial facet in $\cap_{i\in J}$ Kerα_j. Clearly for $x \in \mathbb{A}$, $\widehat{P}_{\mathcal{S}(J)}(x) \subset \widehat{P}(x)$ and $N_{\mathcal{S}(J)} \subset N$, so $\mathscr{I}(\mathfrak{G}_{\mathcal{S}(J)}, K, \mathbb{A})$ maps onto $\mathscr{I}(\mathfrak{G}_{\mathcal{S}(J)}, \mathfrak{G}_{\mathcal{S}}, K, \mathbb{A}) := G_{\mathcal{S}(J)}.\mathbb{A} \subset \mathscr{I}(\mathfrak{G}_{\mathcal{S}}, K, \mathbb{A})$ and the projection pr_{F^v} maps $\mathscr{I}(\mathfrak{G}_{\mathcal{S}(J)}, \mathfrak{G}_{\mathcal{S}}, K, \mathbb{A})$ onto $\mathscr{I}(\mathfrak{G}_{\mathcal{S}}, K, \overline{\overline{\mathbb{A}}})_{F^v}$. So the three sets $\mathscr{I}(\mathfrak{G}_{\mathcal{S}(J)}, K, \mathbb{A})$, $\mathscr{I}(\mathfrak{G}_{\mathcal{S}(J)}, \mathfrak{G}_{\mathcal{S}}, K, \mathbb{A})$ and $\mathscr{I}(\mathfrak{G}_{\mathcal{S}}, K, \overline{\overline{\mathbb{A}}})_{F^v}$ are $G_{\mathcal{S}(J)}-$equivariantly isomorphic.

For $\mathfrak{G}_{\mathcal{S}(J)}$, the bordered apartment associated to $\mathbb{A}$ is a union of façades with direction facets for $\Phi(J)$. These facets are in one to one correspondence with the facets in F^{v*}, for F^v as above. Let $\overline{\mathbb{A}}^i_J$, $\overline{\mathbb{A}}^e_J$ and $\overline{\overline{\mathbb{A}}}_J$ be the three possible apartments as in 3.7. Then $\overline{\mathscr{I}}(\mathfrak{G}_{\mathcal{S}(J)}, K, \overline{\mathbb{A}}^e_J)$ (resp. $\overline{\mathscr{I}}(\mathfrak{G}_{\mathcal{S}(J)}, K, \overline{\overline{\mathbb{A}}}_J)$) is isomorphic to $\overline{\mathscr{I}}(\mathfrak{G}_{\mathcal{S}}, K, \overline{\mathbb{A}}^e)_{F^v}$ (resp. $\overline{\mathscr{I}}(\mathfrak{G}_{\mathcal{S}}, K, \overline{\overline{\mathbb{A}}})_{F^v}$) as defined in 4.2.3c. And $\overline{\mathscr{I}}(\mathfrak{G}_{\mathcal{S}(J)}, K, \overline{\mathbb{A}}^i_J)$ is isomorphic to $\overline{\mathscr{I}}(\mathfrak{G}_{\mathcal{S}}, K, \overline{\mathbb{A}}^i)_{F^v}$ where we remove $\mathscr{I}(\mathfrak{G}_{\mathcal{S}}, K, \overline{\mathbb{A}}^i)_{F^v}$ and add $\mathscr{I}(\mathfrak{G}_{\mathcal{S}(J)}, \mathfrak{G}_{\mathcal{S}}, K, \mathbb{A})$.

3) Changing the field: Let's consider a field extension $i : K \hookrightarrow L$ and suppose that the valuation ω may be extended to L. Then $\mathfrak{G}_{\mathcal{S}}(K)$ embeds via $\mathfrak{G}_{\mathcal{S}}(i)$ into $\mathfrak{G}_{\mathcal{S}}(L)$. If $\mathbb{A}$ is suitable for $\mathfrak{G}_{\mathcal{S}}(L)$, it is also suitable for $\mathfrak{G}_{\mathcal{S}}(K)$; the three examples of 3.5.1 on K and L are corresponding this way each to the other. There are also embeddings $\mathfrak{G}^{pma}_{\mathcal{S}}(K) \hookrightarrow \mathfrak{G}^{pma}_{\mathcal{S}}(L)$, $\mathfrak{G}^{nma}_{\mathcal{S}}(K) \hookrightarrow \mathfrak{G}^{nma}_{\mathcal{S}}(L)$ and it is clear that, for $x \in \mathbb{A}$, $U^{pm+}_{Kx} = U^{pm+}_{Lx} \cap \mathfrak{G}_{\mathcal{S}}(K)$, $U^{nm-}_{Kx} = U^{nm-}_{Lx} \cap \mathfrak{G}_{\mathcal{S}}(K)$ and $\mathfrak{N}(K)(x) = \mathfrak{N}(L)(x) \cap \mathfrak{G}_{\mathcal{S}}(K)$. So, using Iwasawa decomposition for $\mathfrak{G}_{\mathcal{S}}(K)$, 5.5 and uniqueness in Birkhoff decomposition for $\mathfrak{G}_{\mathcal{S}}(L)$, we have:

$$\widehat{P}_L(x) \cap \mathfrak{G}_{\mathcal{S}}(K) = \widehat{P}_K(x).(\widehat{P}_L(x) \cap (\mathfrak{N}(K).\mathfrak{U}^+(K))) =$$
$$\widehat{P}_K(x).(\widehat{P}_L(x) \cap (\mathfrak{N}(L).\mathfrak{U}^+(L)) \cap (\mathfrak{N}(K).\mathfrak{U}^+(K))) = \widehat{P}_K(x).((\mathfrak{N}(L)(x).U^{pm+}_{Lx}) \cap (\mathfrak{N}(K).\mathfrak{U}^+(K)))$$
$$= \widehat{P}_K(x).(\mathfrak{N}(L)(x) \cap \mathfrak{N}(K)).(U^{pm+}_{Lx} \cap \mathfrak{U}^+(K)) = \widehat{P}_K(x).\mathfrak{N}(K)(x).U^{pm+}_{Kx} = \widehat{P}_K(x).$$

The same calculus gives $\mathfrak{N}(L).\widehat{P}_L(x) \cap \mathfrak{G}_{\mathcal{S}}(K) = \mathfrak{N}(K).\widehat{P}_K(x)$.

Hence there is a $\mathfrak{G}_{\mathcal{S}}(K)-$equivariant embedding $\mathscr{I}(\mathfrak{G}_{\mathcal{S}}, i, \mathbb{A}) : \mathscr{I}(\mathfrak{G}_{\mathcal{S}}, K, \mathbb{A}) \hookrightarrow \mathscr{I}(\mathfrak{G}_{\mathcal{S}}, L, \mathbb{A})$; it sends each apartment onto an apartment. But this embedding is not onto and the bijection between an apartment A_K and its image A_L is in general not an isomorphism: if the extension i is ramified, $\Lambda_L = \omega(L^*)$ is greater than $\Lambda = \omega(K^*)$, so there are more walls in A_L than in A_K and the enclosures or facets are smaller in A_L than in A_K.

This embedding extends clearly to the bordered hovels. Hence $\mathscr{I}(\mathfrak{G}_{\mathcal{S}}, K, \mathbb{A})$ and $\overline{\mathscr{I}}(\mathfrak{G}_{\mathcal{S}}, K, \overline{\mathbb{A}})$ are functorial in (K, ω). In particular a group Γ of automorphisms of K fixing ω acts on $\mathscr{I}(\mathfrak{G}_{\mathcal{S}}, K, \mathbb{A})$ and $\overline{\mathscr{I}}(\mathfrak{G}_{\mathcal{S}}, K, \overline{\mathbb{A}})$.

Actually this possibility of embedding $\mathscr{I}(\mathfrak{G}_{\mathcal{S}}, K, \mathbb{A})$ or $\overline{\mathscr{I}}(\mathfrak{G}_{\mathcal{S}}, K, \overline{\mathbb{A}})$ in a (bordered) hovel where there are more walls or even where all points are special (if $\Lambda_L = \mathbb{R}$) is technically very interesting. It was axiomatized for abstract (bordered) hovels and used by Cyril Charignon: [Ch10], [Ch11].

4) Changing the model apartment: Let's consider an affine map $\psi : \mathbb{A} \to \mathbb{A}'$ between two affine apartments suitable for $G = \mathfrak{G}_{\mathcal{S}}(K)$. We ask that ψ is $N-$equivariant and $({}^t\overrightarrow{\psi})(\Delta) = \Delta$, this makes sense as $\Delta \subset Q$ is in $\overrightarrow{\mathbb{A}}$ and $\overrightarrow{\mathbb{A}'}$. So $\psi^{-1}(V'_0) = V_0$ and the quotients $\mathbb{A}/V_0$, $\mathbb{A}'/V'_0$ are naturally equal to $\mathbb{A}^q$ (with the same walls). In particular there is a one to one correspondence between the enclosed filters in $\mathbb{A}$, $\mathbb{A}'$ or $\mathbb{A}^q$.

For $y \in \mathbb{A}$, $N(y) \subset N(\psi(y))$, $\widehat{P}(y) = U_{\overline{y}}^{pm+}.U_{\overline{y}}^{nm-}.N(y) \subset \widehat{P}'(\psi(y)) = U_{\overline{y}}^{pm+}.U_{\overline{y}}^{nm-}.N(\psi(y))$. We get a $G-$equivariant map $\mathscr{I}(\mathfrak{G}_{\mathcal{S}}, K, \psi) : \mathscr{I}(\mathfrak{G}_{\mathcal{S}}, K, \mathbb{A}) \to \mathscr{I}(\mathfrak{G}_{\mathcal{S}}, K, \mathbb{A}')$. It induces a one to one correspondence between the apartments or facets, chimneys,... of both hovels but it is in general neither into nor onto. The most interesting example is the essentialization map $\mathscr{I}(\mathfrak{G}_{\mathcal{S}}, K, \mathbb{A}) \to \mathscr{I}(\mathfrak{G}_{\mathcal{S}}, K, \mathbb{A}^q)$.

Clearly these maps extend to the bordered hovels.

5.9 Uniqueness of the very good family of parahorics

1) Actually, by 5.3 the family $\widehat{\mathcal{P}}$ satisfies the following strengthening of axiom (P8):

(P8+) For all $x \in \overline{\mathbb{A}}$, for all chamber $C^v \in F^v(x)^*$,

$$Q(x) = (Q(x) \cap U(C^v)).(P(x) \cap U(-C^v)).N(x)$$

By the following lemma, we know that $\widehat{\mathcal{P}}$ is the only very good family of parahorics over $\overline{\mathbb{A}}$.

2) At least for $\mathbb{A} = \mathbb{A}^q$, Charignon defines a maximal good family of parahorics $\overline{\mathcal{P}}$:

$$\text{for } x \in \mathbb{A}_{F^v},\ \overline{P}(x) = \{g \in P(F^v) \mid g.pr_{F_1^v}(x) = pr_{g.F_1^v}(x)\ \forall F_1^v \in \mathscr{I}^v_{sph}, F^v \subset \overline{F_1^v}\}$$

where $pr_{F_1^v}$ is the projection associated to the minimal family $\mathcal{P}$ (supposed good) or to any good family $\mathcal{Q}$ *e.g.* $\widehat{\mathcal{P}}$.

We have $\mathcal{P} \subset \widehat{\mathcal{P}} \subset \overline{\mathcal{P}}$ in the sense that $\forall x \in \overline{\mathbb{A}}$, $P(x) \subset \widehat{P}(x) \subset \overline{P}(x)$ [Ch11, sec. 11.8]. It is likely that $\widehat{\mathcal{P}} = \overline{\mathcal{P}}$, but it seems not to be a clear consequence of the preceding results.

Lemma 5.10. *Let $\mathcal{Q}$ and $\mathcal{Q}'$ be two very good families of parahoric subgroups of G (in the general setting of section 4). Suppose that $\mathcal{Q} \leq \mathcal{Q}'$ (i.e. $Q(x) \subset Q'(x)\ \forall x \in \overline{\mathbb{A}}$) or that $\mathcal{Q}'$ satisfies (P8+), then $\mathcal{Q} = \mathcal{Q}'$.*

Proof. If $\mathcal{Q} \leq \mathcal{Q}'$ there is clearly a $G-$equivariant map $j : \overline{\mathcal{I}} \to \overline{\mathcal{I}}'$ between the bordered hovels associated to G, $\overline{\mathbb{A}}$ and $\mathcal{Q}$ or $\mathcal{Q}'$. This map sends each bordered apartment isomorphically to its image. Let F^v be a vectorial facet in $\mathbb{A}^v$, then 4.14 applies to the map j between the ordered affine hovels $\mathcal{I}_{F^v}$ and $\mathcal{I}'_{F^v}$. So j is one to one. Let $x \in \mathbb{A}_{F^v}$ and $g \in Q'(x)$, then $j(gx) = gj(x) = j(x)$, so $gx = x$ and $g \in Q(x)$.

If $\mathcal{Q}'$ satisfies (P8+) we may apply the first case to $\mathcal{Q}$ or $\mathcal{Q}'$ and $\mathcal{Q}'' = \mathcal{Q} \cap \mathcal{Q}'$ (*i.e.* $Q''(x) = Q(x) \cap Q'(x)\ \forall x$), as this family $\mathcal{Q}''$ is very good. Actually for $\mathcal{Q}''$ (P1) to (P6) and (P9), (P10) are clear. For (P7) we have to prove $Q''(x) \cap NP(F^v) \subset NQ''(pr_{F^v}(x)) = NP(pr_{F^v}(x))$ (as F^v is spherical); it is clear. For (P8) $Q''(x) = Q(x) \cap [(Q'(x) \cap U(C^v)).(P(x) \cap U(-C^v)).N(x)] = (Q(x) \cap Q'(x) \cap U(C^v)).(P(x) \cap U(-C^v)).N(x)$ as $P(x).N(x) \subset Q(x)$. □

5.11 Residue buildings

1) Let x be a point in the apartment $\mathbb{A}$. We defined in [GR08, 4.5] or [Ro11, § 5] the twinned buildings $\mathscr{I}_x^+$ and $\mathscr{I}_x^-$, where $\mathscr{I}_x^+$ (resp. $\mathscr{I}_x^-$) is the set of segment germs $[x, y)$ for $y \in \mathscr{I}$, $y \neq x$ and $x \leq y$ (resp. $y \leq x$). Any apartment A containing x induces a twin apartment $A_x = A_x^+ \cup A_x^-$ where $A_x^\pm = \{[x, y) \mid y \in A\} \cap \mathscr{I}_x^\pm$. As we want to consider thick buildings, we endow the apartments of $\mathscr{I}_x^\pm$ with their unrestricted structure of Coxeter complexes; on $\mathbb{A}_x$ it is associated with the subroot system $\Phi_x = \{\alpha \in \Phi \mid -\alpha(x) \in \Lambda_\alpha\}$ of Φ (*cf.* [Ba96, 5.1]) and the Coxeter subgroup $W_x^{min} \simeq W_x^v$ of W^v. One should note that Φ_x is reduced but could perhaps have an infinite non free basis, corresponding to an infinite generating set of W_x^v.

The group $G_x = \widehat{P}(x)$ contains three interesting subgroups: $P(x) = N(x).G(x) \supset P_x^{min} = Z_0.G(x)$ (see [GR08, § 3.2], they are equal when x is special); the group $G_{\mathscr{I}_x}$ is the pointwise fixator of all $[x,y)\in\mathscr{I}^{\pm}{}_x$ (*i.e.* $g\in G_{\mathscr{I}_x} \iff \forall[x,y)\in\mathscr{I}^{\pm}{}_x, \exists z\in]x,y]$ such that g fixes pointwise $[x,z]$), it is clearly normal in G_x.

We write $\overline{G}_x = G_x/G_{\mathscr{I}_x}$ and $\overline{U}_\alpha$ or $\overline{R}$ the images in $\overline{G}_x$ of $U_{\alpha,-\alpha(x)}$ ($\alpha\in\Phi$) or R any subgroup of G_x.

2) Lemma. *A $g\in G_x$ fixing an element in $\mathscr{I}_x^-$ and fixing pointwise $\mathscr{I}_x^+$ (e.g. $g\in G_{\mathscr{I}_x}$) fixes pointwise each $[x,y)$ for $y\neq x$ in a same apartment as x.*

Proof. So g fixes $[x,z]$ for some $z<x$. By [Ro12, 5.12.4], $[x,z)$ and $[x,y)$ are in a same apartment A. By hypothesis g fixes points $z_1,\cdots,z_n$ in A such that each z_i-x is in the open Tits cone $\mathcal{T}^\circ\subset\overrightarrow{A}$, these vectors generate the vector space $\overrightarrow{A}$ and the interior of the convex hull of $\{x,z_1,\cdots,z_n\}$ contains an opposite of $[x,z)$. By moving each z_i in $]x,z_i]$ one may suppose $x\leq z_1\leq\cdots\leq z_n$. Now as $z<x$, g fixes (pointwise) the convex hull of $\{z,z_1,\cdots,z_n\}$ which is a neighbourhood of x in A, hence contains $[x,y)$. □

3) Lemma. *Any $u\in U^+$ fixing (pointwise) a neighbourhood of x in $\mathbb{A}$, fixes pointwise $\mathscr{I}_x$. This applies in particular to a $u\in U_{\alpha,-\alpha(x)+}$ for $\alpha\in\Phi_x$ or a $u\in U_{\alpha,-\alpha(x)}$ for $\alpha\in\Phi\setminus\Phi_x$.*

Proof. By 5.8.3 we may suppose x special, hence $x=0$. By the above lemma it is sufficient to prove that u fixes $\mathscr{I}_0^+$. An element $[0,y)$ of $\mathscr{I}_0^+$ is in an apartment A containing the chamber $F=F(0,C_f^v)$ and even the sector $\mathfrak{q}=0+C_f^v$ [Ro12, 5.12.4]; this apartment may be written $A=g^{-1}\mathbb{A}$ with $g\in\widehat{P}(\mathfrak{q})$ and even $g\in U_0^{pm+}$. Now $g([0,y))$ is in a sector wC_f^v for some $w\in W^v$ and we have to prove that gug^{-1} fixes a neighbourhood of 0 in this sector.

We argue in U_0^{ma+} (as defined in 5.1 or [Ro12, 4.5.2]). This group may be written as a direct product: $U_0^{ma+}=U_0^{ma}(\Delta^+)=(\prod_{\beta\in\Delta'}U_{\beta,0})\times U_0^{ma}(\Delta^+\setminus\Delta')$ where Δ' is the finite set of positive roots of height $\leq N$ (with N such that $\Delta^+\cap w\Delta^-\subset\Delta'\cap\Phi$) and $U_0^{ma}(\Delta^+\setminus\Delta')$ is a normal subgroup. Moreover each $U_{\beta,0}$ is a finite product of sets in bijection with $\mathcal{O}$, the neutral element corresponding to $(0,\cdots,0)$ (actually for β real, $U_{\beta,0}$ is isomorphic to the additive group of $\mathcal{O}$). For $g_1\in U_0^{ma+}$ the map sending $v\in U_0^{ma+}$ to the component of $g_1vg_1^{-1}$ in $\prod_{\beta\in\Delta'}U_{\beta,0}$, factors through $U_0^{ma+}/U_0^{ma}(\Delta^+\setminus\Delta')=\prod_{\beta\in\Delta'}U_{\beta,0}$ and induces a polynomial map with coefficients in $\mathcal{O}$ and without any constant term.

Now $u\in U^+\cap G_0=U_0^{pm+}$ and u fixes (pointwise) a neighbourhood of x in $\mathbb{A}$, hence some $x'\in -C_f^v$. So $u\in U_{x'}^{ma+}$ and the component of u in $U_{\beta,0}$ is in the maximal ideal $\mathfrak{m}$ of $\mathcal{O}$ if β is real or in $\mathfrak{m}\times\cdots\times\mathfrak{m}$ if β is imaginary. By the above property this is also true for gug^{-1} and gug^{-1} fixes a neighbourhood of 0 in wC_f^v (as wC_f^v is fixed by $U_0^{ma}(\Delta^+\setminus\Delta')$). □

4) Proposition. *$(\overline{P}_x^{min},(\overline{U}_\alpha)_{\alpha\in\Phi_x},\overline{Z}_0)$ is a generating root datum whose associated twin buildings have the same chamber sets or twin-apartment sets as $\mathscr{I}^{\pm}{}_x$.*

Moreover $G_x=G(x).N(x).G_{\mathscr{I}_x}$ and $U_x^{pm+}\subset U_x^{++}.G_{\mathscr{I}_x}$.

Remarks. a) As the basis of Φ_x could be infinite the above generating root datum must be understood in a more general sense than in 1.4: we should consider the free covering $\widetilde{\Phi}_x$ of Φ_x (whose basis is free) which is in one to one correspondence with Φ_x (*cf.* [Ba96, 4.2.8]) and a root datum as in [Re02, 6.2.5]. Another (less precise) possibility is to index the $\overline{U}_\alpha$ by subsets of the Weyl group W_x^v, see [Re02, 1.5.1] or [AB08, 8.6.1]. Actually there is no trouble in defining the combinatorial twin buildings associated to this generalized root datum; but, except for chambers, their facets may not be in one to one correspondence with those of $\mathscr{I}^{\pm}{}_x$, *cf.* [Ro11, 5.3.2].

b) We may define the subgroup $P_x = P_x^{min}.G_{\mathscr{I}_x}$; this generalizes the definition given in [Ro12, 5.14.2], as clearly $P_x \cap N = N_x^{min}$. It is the subgroup of G_x which preserves the "restricted types in x" of the facets $F(x, F^v)$ (*i.e.* their types as defined in the twin buildings $\mathscr{I}^{\pm}{}_x$ endowed with their restricted structures). The greater group $\widehat{P}_x^{sc}$ of [Ro12, 5.14.1] preserves the "unrestricted types" of the local facets $F^l(x, F^v)$ (*i.e.* the (vectorial) type of F^v).

c) These results and [Ro12, 4.13.5] suggest that $G_x = \widehat{P}(x)$ could perhaps be always equal to $P(x) = G(x).N(x) = P_x^{min}.N(x)$ for any $x \in \mathbb{A}$ (*i.e.* $U_x^{pm+} = U_x^+$ and $U_x^{nm-} = U_x^-$). On the contrary we already said in 4.2.1 that U_x^{pm+} or U_x^+ is in general different from U_x^{++}.

Proof. By definition for $\alpha \in \Phi_x$, $-\alpha(x) \in \Lambda$ hence there is a $r \in K$ with $\omega(r) + \alpha(x) = 0$, so $\varphi_\alpha(x_\alpha(r)) + \alpha(x) = 0$ and the fixed point set of $u = x_\alpha(r) \in U_{\alpha,-\alpha(x)}$ is $D(\alpha, -\alpha(x))$. Therefore the image of u in $\overline{U}_\alpha \subset \overline{G}_x$ is non trivial; (RD1) follows.

(RD2) is a consequence of (RD2) and (V3) in G as $\overline{U}_\alpha$ is trivial when $\alpha \notin \Phi_x$ by lemma 3. (RD3) is useless as Φ is reduced. For (RD4) $\overline{u} \in \overline{U}_\alpha \setminus \{1\}$ is the class of an element $u \in U_\alpha$ with $\varphi_\alpha(u) = -\alpha(x)$ (by lemma 3); hence the result follows from 3.1.2.

An element $\overline{g} \in \overline{Z}_0.\overline{U}^+ \cap \overline{U}^-$ is the class of an element $g \in U^-$ and, up to $G_{\mathscr{I}_x}$, g fixes $x + C_f^v$ and $x - C_f^v$, hence g fixes a neighbourhood of x in $\mathbb{A}$ (by convexity) and, by lemma 3, $g \in G_{\mathscr{I}_x}$. So $\overline{g} = 1$ and (RD5) is proved.

The group P_x^{min} is generated by Z_0 and the $U_{\alpha,k}$ for $\alpha \in \Phi$ and $\alpha(x) + k \geq 0$. So the lemma 3 tells that its image $\overline{P}_x^{min}$ is generated by $\overline{Z}_0$ and the $\overline{U}_\alpha$ for $\alpha \in \Phi_x$. Hence $(\overline{P}_x^{min}, (\overline{U}_\alpha)_{\alpha \in \Phi_x}, \overline{Z}_0)$ is a generating root datum. We define $\overline{U}^{\pm}$ as the group generated by the $\overline{U}_{\pm\alpha}$ for $\alpha \in \Phi_x^+$ (and not the image $\overline{U}^{\pm}{}_x$ of $U^{\pm}{}_x$).

Let $\mathcal{I}^{\pm}{}_c$ be its associated (combinatorial) twin buildings and $C^{\pm}{}_c$ its fundamental chambers [AB08, 8.81]. The twin apartments of $\mathcal{I}^{\pm}{}_c$ or $\mathscr{I}^{\pm}{}_x$ are both the twin Coxeter complexes associated to W_x^v and N acts transitively on their four chamber sets. Moreover the chambers in $\mathcal{I}_c^+$ (resp. $\mathscr{I}_x^+$) sharing with C_c^+ (resp. $C = F(x, C_f^v)$) a panel of type $r_\alpha \in W_x^v$ (for α simple in Φ_x) are in one to one correspondence with $\overline{U}_\alpha$ by [AB08, 8.56] (resp. $U_{\alpha,-\alpha(x)}/U_{\alpha,-\alpha(x)+}$ by 4.11e). By lemma 3 these two groups are isomorphic. So the chamber sets of $\mathcal{I}_c^+$ and $\mathscr{I}_x^+$ are in one to one correspondence. The same thing is true for the negative buildings.

The twin apartments in $\mathcal{I}^{\pm}{}_c$ are permuted transitively by P_x^{min} and the stabilizer of the fundamental one is $N_x^{min} = P_x^{min} \cap N$. So the twin apartments in $\mathcal{I}^{\pm}{}_c$ correspond bijectively to some apartments of $\mathscr{I}^{\pm}{}_x$. But two chambers in $\mathscr{I}^{\pm}{}_x$ correspond to chambers in $\mathcal{I}^{\pm}{}_c$, hence are in a twin apartment of $\mathcal{I}^{\pm}{}_c$ and their distance or codistance is the same in $\mathcal{I}^{\pm}{}_c$ or $\mathscr{I}^{\pm}{}_x$. As a twin apartment is uniquely determined by a pair of opposite chambers, every twin apartment of $\mathscr{I}^{\pm}{}_x$ comes from a twin apartment in $\mathcal{I}^{\pm}{}_c$.

The chambers in $\mathcal{I}_c^-$ opposite C_c^+ are transitively permuted by $\overline{Z}_0.\overline{U}^+$ [AB08, 6.87] hence in one to one correspondence with $\overline{U}^+$, as $\overline{Z}_0.\overline{U}^- \cap \overline{U}^+ = \{1\}$ [*l.c.* 8.76]. In $\mathscr{I}_x^-$ the chambers opposite C are in a same apartment as the sector $x + C_f^v$ [Ro12, 5.12.4] hence transitively permuted by U_x^{pm+} (4.6.4). Now the fixator in U_x^{pm+} of the chamber $F(x, -C_f^v)$ is actually in $G_{\mathscr{I}_x}$ by lemma 3. So the chambers opposite C in $\mathscr{I}_x^-$ are in one to one correspondence with the image $\overline{U}_x^{pm+}$ of U_x^{pm+} in $\overline{G}_x$. Thus $\overline{U}_x^{pm+} = \overline{U}^+$. But $\overline{U}^+$ is the image of U_x^{++}, hence $U_x^{pm+} \subset U_x^{++}.G_{\mathscr{I}_x}$. As $G_x = \widehat{P}(x) = U_x^{pm+}.U_x^-.N(x) = U_x^{pm+}.G(x).N(x)$ we have $\overline{G}_x = \overline{G}(x).\overline{N}(x)$, hence $G_x = G(x).N(x).G_{\mathscr{I}_x}$. □

6 Hovels and bordered hovels for almost split Kac-Moody groups

6.1 Situation and goal

We consider an almost split Kac-Moody group $\mathfrak{G}$ over a field K, as in section 2. We suppose the field K endowed with a non trivial real valuation $\omega = \omega_K$ which may be extended functorially to all extensions in $\mathcal{S}ep(K)$. This condition is satisfied (with uniqueness of the extension) if K is complete for ω_K (or more generally if (K, ω_K) is henselian).

We built in 5.4 a bordered hovel $\overline{\mathscr{I}}(\mathfrak{G}, L, \overline{\mathbb{A}})$ for any $L \in \mathcal{S}ep(K)$ splitting $\mathfrak{G}$. We want a bordered hovel $\overline{\mathscr{I}}(\mathfrak{G}, K, {}^K\overline{\mathbb{A}})$ on which $G = \mathfrak{G}(K)$ would act strongly transitively and $G-$equivariant embeddings $\overline{\mathscr{I}}(\mathfrak{G}, K, {}^K\overline{\mathbb{A}}) \hookrightarrow \overline{\mathscr{I}}(\mathfrak{G}, L, \overline{\mathbb{A}})$ for $L \in \mathcal{S}ep(K)$ splitting $\mathfrak{G}$.

An idea (already used in the classical case [BrT84]) is to suppose L/K finite Galois, to build an action of the Galois group $\Gamma = Gal(L/K)$ over $\overline{\mathscr{I}}(\mathfrak{G}, L, \overline{\mathbb{A}})$ and to find $\overline{\mathscr{I}}(\mathfrak{G}, K, {}^K\overline{\mathbb{A}})$ in the fixed point set $\overline{\mathscr{I}}(\mathfrak{G}, L, \overline{\mathbb{A}})^\Gamma$. As already known in the classical case [Ro77], the equality of these last two objects is in general impossible.

6.2 Action of the Galois group on the bordered hovel

1) We consider a finite Galois extension L of K which splits $\mathfrak{G}$. The Galois group $\Gamma = Gal(L/K)$ acts on $\mathscr{I}^v(L) = \mathscr{I}^v(\mathfrak{G}, L, \mathbb{A}^v)$ and the action of $\mathfrak{G}(L)$ on $\mathscr{I}^v(L)$ is $\Gamma-$equivariant. More precisely we suppose L such that there exists a maximal $K-$split torus $\mathfrak{S}$ in $\mathfrak{G}$ contained in a maximal torus $\mathfrak{T}$ defined over K and split over L (*cf.* 2.4.4 and 2.5). We described in section 2 the fixed point set $\mathscr{I}^v(L)^\Gamma$, its $K-$apartments and $K-$facets. In particular the apartment $A^v = A^v(\mathfrak{T})$ corresponding to $\mathfrak{T}$ is stable under Γ and $(A^v)^\Gamma$ is the $K-$apartment ${}_KA^v(\mathfrak{S})$ corresponding to $\mathfrak{S}$.

We want an action of Γ on the bordered hovel $\overline{\mathscr{I}}(L) = \overline{\mathscr{I}}(\mathfrak{G}, L, \overline{\mathbb{A}})$ compatible with its action on $\mathscr{I}^v(L)$ and the action of $\mathfrak{G}(L)$. Hence Γ must permute the apartments and façades of $\overline{\mathscr{I}}(L)$ as the apartments and facets of $\mathscr{I}^v(L)$. In particular Γ has to stabilize the bordered apartment $\overline{A} = \overline{A}(\mathfrak{T})$ corresponding to $\mathbb{A}^v$ *i.e.* to $\mathfrak{T}$.

2) Action on $\overline{A}$: $\gamma \in \Gamma$ must act affinely on $\overline{A}$ with associated linear action the action of γ on $V = \overrightarrow{A^b} = \overrightarrow{A}$. Moreover this action has to be compatible with the action on the root groups ($\forall \alpha \in \Phi\ \gamma(U_{\alpha,\lambda}) = U_{\gamma\alpha,\lambda'} \Rightarrow \gamma(D(\alpha, \lambda)) = D(\gamma\alpha, \lambda')$) and we know that the action of Γ on $\mathfrak{G}(L)$ is compatible with its action on its Lie algebra ($\gamma(exp(ke_\alpha)) = exp(\gamma(k)\gamma(e_\alpha))$). Using these results and conditions, C. Charignon succeeds in finding a (unique) good action of Γ on the essentialization $A^e = A/V_0$ of A; in particular the action of N is $\Gamma-$equivariant [Ch11, 13.2]. As Γ is finite and acts affinely, it has a fixed point $x_0 + V_0$ in A^e.

Now Γ has to fix a point in $x_0 + V_0$. But all points in $x_0 + V_0$ play the same rôle with respect to the conditions; so we may choose a point in $x_0 + V_0$, *e.g.* x_0, and say that Γ fixes x_0 *i.e.* that Γ acts on A as on $\overrightarrow{A^b}$ (after choosing x_0 as origin). This action is compatible with the above action on A^e. It permutes the walls, facets, ... and extends clearly to $\overline{A}$ ($= \overline{A}^e$, $\overline{A}^i$ or $\overline{\overline{A}}$).

N.B. We had to make a choice to define this action. This is not a surprise: as in the classical case, V_0 is a group of $G-$equivariant automorphisms of $\overline{\mathscr{I}}(L)$.

3) Lemma. *This action of* Γ *on* $\overline{A}$ *stabilizes* $\widehat{\mathcal{P}}$*:* $\quad \gamma(\widehat{P}(x)) = \widehat{\mathcal{P}}(\gamma x)$, $\forall x \in \overline{A}$, $\forall \gamma \in \Gamma$.

Proof. By Charignon's work (2) above) we know that Γ stabilizes $\mathcal{P}$. Hence $\gamma \in \Gamma$ sends $\widehat{\mathcal{P}}$ to a family $\mathcal{Q}$ which is still a very good family of parahorics. So 5.9.1 tells that $\mathcal{Q} = \widehat{\mathcal{P}}$. Note that a longer proof may also be given using the star actions instead of 5.9.1. □

4) We have got compatible actions of Γ on $G = \mathfrak{G}(L)$ and $\overline{A}$ satisfying the above lemma. As $\overline{\mathscr{I}}(L) = \overline{\mathscr{I}}(\mathfrak{G}, L, \overline{\mathbb{A}})$ is defined by the formula in 4.1, we obtain an action of Γ on this bordered hovel, for which the $G-$action is $\Gamma-$equivariant. Each $\gamma \in \Gamma$ acts as an automorphism: it induces a permutation of the apartments, facets, walls, façades, chimneys, ... and the bijection between an apartment and its image is an affine automorphism.

This action of Γ on $\overline{\mathscr{I}}(L)$ is compatible with its action on $\mathscr{I}^v(L)$ ($\gamma(\mathscr{I}_{F^v}) = \mathscr{I}_{\gamma(F^v)}$) and on the sector faces ($\gamma(x + F^v) = \gamma(x) + \gamma(F^v)$) or the chimneys. Moreover the projections on the façades are $\Gamma-$equivariant ($\gamma \circ pr_{F^v} = pr_{\gamma(F^v)} \circ \gamma$). These results are first proved (easily) for the actions on A and A^v, and then extended (easily) to $\overline{\mathscr{I}}$.

As Γ has fixed points in $\mathscr{I}$, any $\Gamma-$fixed point in a façade $\mathscr{I}_{F^v} \subset \overline{\mathscr{I}}$ is associated to a $\Gamma-$stable sector face $x + F^v$ in $\mathscr{I}$.

6.3 The descent problem

In $\overline{\mathscr{I}}$ we have got an apartment $\overline{A}$ stable under Γ. But Γ is finite and acts affinely so it has a fixed point in A and A^Γ is an affine space directed by $(\overrightarrow{A^v})^\Gamma$. It seems interesting to choose $\overline{A}^\Gamma$ as affine bordered $K-$apartment and define ${}_K\overline{\mathscr{I}} = \mathfrak{G}(K).\overline{A}^\Gamma$. Unfortunately we are not sure then that $\overline{A}^\Gamma$ is stable under ${}_KN$ or fixed by ${}_KZ$; so this ${}_K\overline{\mathscr{I}}$ is not a good candidate for a bordered hovel associated to the root datum $(\mathfrak{G}(K), (V_{K\alpha})_{K\alpha \in {}_K\Phi}, {}_KZ)$.

It is possible to find in $\overline{\mathscr{I}}^\Gamma$ a subspace of apartment ${}_KA_1$ directed by $(\overrightarrow{A^v})^\Gamma$ and stable under ${}_KN$. But then it is not clear that there exists an apartment $\overline{A}_2$ in $\overline{\mathscr{I}}$ containing ${}_K\overline{A}_1$ and stable under Γ, or even such that $\overline{A}_2 \cap \overline{\mathscr{I}}^\Gamma = {}_K\overline{A}_1$ [Ch11, 13.3].

This problem is the same as in the classical case of reductive groups: [BrT72], [BrT84], [Ro77]. Charignon solves it the same way: under some hypothesis on $\mathfrak{G}$ or K and by a two steps descent.

6.4 The descent theorem of Charignon

This abstract result is largely inspired by the descent theorem of Bruhat and Tits in the classical case [BrT72, 9.2.10]. We explain here the hypotheses and conclusions of [Ch11, § 12], but, to simplify, we consider a more concrete framework. We keep the notations of *loc. cit.* or we indicate them in brackets when they are too far from ours. We keep our idea to replace many Charignon's overrightarrows by an exponent ^{v} and to use often an overrightarrow to indicate the generated vector spaces.

1) Vectorial data: We consider a finite Galois extension L/K which splits $\mathfrak{G}$ as in 6.2.1. So there exists in $\mathfrak{G}$ a maximal $K-$split torus ${}_K\mathfrak{S}$ and a maximal torus $\mathfrak{T}$ split over L containing ${}_K\mathfrak{S}$ (we don't ask $\mathfrak{T}$ to be defined over K).

We consider the fixed point set $\overrightarrow{\mathcal{I}}_\natural = {}_K\mathscr{I}^{vq} = (\mathscr{I}^{vq})^\Gamma$ of $\Gamma = Gal(L/K)$ in $\overrightarrow{\mathcal{I}} = \mathscr{I}^{vq}$. The group $\mathfrak{G}(K)$ ($= G^\natural$) acts on $\overrightarrow{\mathcal{I}}_\natural$. By 2.7, 2.8 $\overrightarrow{\mathcal{I}}_\natural$ is a good geometrical representation of the combinatorial twin building ${}^K\mathscr{I}^{vc} = \mathscr{I}^{vc}(\mathfrak{G}, K)$.

To ${}_K\mathfrak{S}$ and $\mathfrak{T}$ correspond apartments $A^v_\natural = {}_KA^{vq}({}_K\mathfrak{S}) \subset \overrightarrow{\mathcal{I}}_\natural$ included in $A^{vq} = A^{vq}(\mathfrak{T}) \subset \overrightarrow{\mathcal{I}}$; they are cones in the vector spaces $\overrightarrow{A^v_\natural}$ ($= \overrightarrow{V}^\natural$) included in $\overrightarrow{A^{vq}}$ ($= \overrightarrow{V}$). The real root system Φ (resp. the real relative root system ${}_K\Phi = \Phi^\natural$) is included in the dual $(\overrightarrow{A^{vq}})^*$ (resp. $(\overrightarrow{A^v_\natural})^*$) and has a free basis. Its associated vectorial Weyl group is $W^v = N/T$ ($= W(\Phi)$)

(resp. ${}_KW^v = {}_KN/{}_KZ = W(\Phi^\natural)$). Here ${}_KZ = T^\natural$ or ${}_KN = N^\natural$ is the generic centralizer or normalizer in $G^\natural$ of ${}_K\mathfrak{S}$. We write $\overrightarrow{A^v_{\natural 0}} = \cap_{a\in\Phi^\natural} Ker(a)$.

We consider also the Weyl$-K-$apartment $A^{v\natural} = {}^KA^{vq}({}_K\mathfrak{S})$ with $A^v_\natural \subset A^{v\natural} \subset \overrightarrow{A^b_\natural}$ and the corresponding building $\overrightarrow{\mathcal{I}}^\natural = G^\natural.A^{v\natural}$ (*cf.* 2.8). As in [Ch11] we define the facets in $A^v_\natural$ or $\overrightarrow{\mathcal{I}}_\natural$ as the traces $F^v_\natural = F^{v\natural} \cap \overrightarrow{\mathcal{I}}_\natural$ of the Weyl$-K-$facets $F^{v\natural}$ of $A^{v\natural}$ or $\overrightarrow{\mathcal{I}}^\natural$. The same set $A^v_\natural$, endowed as facets with the non empty traces $F^v_\# = F^{vq} \cap A^v_\natural$ for F^{vq} a facet in A^{vq}, will be written $A^v_\#$. There is a one to one correspondence between facets of $A^{v\natural}$ and $A^v_\natural$. But a facet $F^{v\natural}$ or $F^v_\natural = F^{v\natural} \cap A^v_\natural$ contains several facets in $A^v_\#$; among them one $F^{v\natural}_\#$ is maximal, open in $F^{v\natural}$, generates the same vector space and $F^{v\natural}_\# + \overrightarrow{A^v_{\natural 0}} = F^v_\natural + \overrightarrow{A^v_{\natural 0}} = F^{v\natural}$ (*cf.* 2.8).

The combinatorial twin building ${}^K\mathscr{I}_\pm{}^{vc}$ is associated to the root datum $(\mathfrak{G}(K), (V_{K\alpha})_{K\alpha\in_K\Phi}, {}_KZ)$ $(= (G^\natural, (U^\natural_a)_{a\in\Phi^\natural}, {}_KZ))$. Everything associated as in § 1 to this root datum will be written with an exponent $^\natural$ or a subscript $_K$. The reader will check easily the conditions (DSR), (DDR1),...,(DDR3.2) and (DIV) of [*l.c.* 12.1], *cf.* [*l.c.* 13.4.1]. In particular for $a = {}_K\alpha \in \Phi^\natural$, $U^\natural_a = V_{K\alpha}$ is included in the group U_a generated by the groups U_β for the roots β in the finite set $\Phi_a = \{\beta \in \Phi \mid \beta_{|\overrightarrow{V}^\natural} \in \mathbb{R}^{+*}a\} = \{\beta \in \Phi \mid {}_K\beta = {}_K\alpha \text{ or } (\frac{1}{2}).{}_K\alpha \text{ or } 2.{}_K\alpha\}$. Actually $U_a = \prod_{\beta\in\Phi_a} U_\beta$ for any order [Re02, 6.2.5].

2) Affine data: We consider the essential bordered hovel $\overline{\mathscr{I}} = \overline{\mathscr{I}}(\mathfrak{G}, L, \overline{\mathbb{A}}^e)$ $(= \mathcal{I})$ and $\overline{A}$ $(= A(T))$ the bordered apartment associated to A^{vq} whose main façade A $(= A^\circ(T))$ is an affine space under $\overrightarrow{A^{vq}} = \overrightarrow{V}$. The façades of $\overline{\mathscr{I}}$ are indexed by the facets $F^v \in \mathscr{I}^{vc}$.

We consider moreover a subset $\overline{\mathcal{I}}_\#$ in $\overline{\mathscr{I}}$, we write $A^\# = A \cap \overline{\mathcal{I}}_\#$ and suppose:

(DM1) $\overline{\mathcal{I}}_\#$ is $G^\natural-$stable and, $\forall F^v \in \mathscr{I}^{vc}_{sph}$, $\overline{\mathcal{I}}_\# \cap \mathscr{I}_{F^v}$ is convex in $\mathscr{I}_{F^v}$.

(DM2) $A^\#$ is affine in A, directed by $\overrightarrow{V}^\natural$ and $\overline{A} \cap \overline{\mathcal{I}}_\#$ is the closure $\overline{A}^\#$ of $A^\#$ in $\overline{A}$.

(DM3) $\forall F^v \in \mathscr{I}^{vc}_{sph}$, if $F^v \cap A^{v\natural} \neq \emptyset$, there exists a facet F in the (classical) apartment A_{F^v} with $F \cap \overline{\mathcal{I}}_\# \neq \emptyset$ and F is equal to any facet F' in $\mathscr{I}_{F^v}$ with $F' \cap \overline{\mathcal{I}}_\# \neq \emptyset$ and $F \subset \overline{F'}$.

(DM4) $\overline{A}^\#$ is stable under ${}_KN = N^\natural$.

Axiom (DM3) means essentially that, in appropriate spherical façades, $\overline{A} \cap \overline{\mathcal{I}}_\#$ cannot be enlarged by modifying the apartment $\overline{A}$.

For $a \in \Phi^\natural$ and $u \in U^\natural_a$, one defines $\varphi^\natural_a(u)$ as the supremum in $\mathbb{R} \cup \{+\infty\}$ of the k such that u is in the group $U_{a,k}$ generated by the $U_{\alpha,r_\alpha k} = \varphi_\alpha^{-1}([r_\alpha k, +\infty])$ for $\alpha \in \Phi_a$, $r_\alpha \in \mathbb{R}^{+*}$ and $\alpha_{|\overrightarrow{V}^\natural} = r_\alpha a$. Actually $U_{a,k} = \prod_{\alpha\in\Phi_a} U_{\alpha,r_\alpha k}$ and $U^\natural_{a,k} := (\varphi^\natural_a)^{-1}([k, +\infty]) = U^\natural_a \cap U_{a,k}$.

There are two more axioms, one normalizing φ (among equipollent valuations, in such a way that the associated origin 0_φ of A is in $A_\#$) and one avoiding triviality for each $\varphi^\natural_a$. They are easily verified in our situation [*l.c.* 13.4.1].

As we have three types of vectorial facets in $\overrightarrow{V}^\natural = \overrightarrow{A^b_\natural}$, we may define three bordered apartments with $A^\#$: $\overline{A}_\natural$ (resp. $\overline{A}^\natural$, $\overline{A}^\#$) is the disjoint union of the façades $A^\#_{F^v_\natural} = A^\#/\overrightarrow{F^b_\natural}$ (resp. $A^\#_{F^{v\natural}} = A^\#/\overrightarrow{F^{v\natural}}$, $A^\#_{F^v_\#} = A^\#/\overrightarrow{F^v_\#}$), for $F^v_\natural$ (resp. $F^{v\natural}$, $F^v_\#$) a facet in $A^v_\natural$ (resp. $A^{v\natural}$, $A^v_\#$). Actually $\overline{A}^\#$ is the closure of $A^\#$ in $\overline{A}$ as in (DM2) above. Moreover the sets $\overline{A}_\natural$ and $\overline{A}^\natural$ are equal (as $\overrightarrow{F^b_\natural} = \overrightarrow{F^{v\natural}}$ when $F^v_\natural = F^{v\natural} \cap A^v_\natural$) but they differ by their facets, sectors, ...

When $F^{v\natural} \supset F^{v\natural}_\# = F^v \cap A^v_\natural$ for F^v a (maximal) facet in A^v, we have $\overrightarrow{F^{v\natural}} = \overrightarrow{F^b}$, so $A^\#_{F^{v\natural}} \subset A_{F^v} \subset \overline{A}$. Hence for $x \in A^\#_{F^{v\natural}} \subset \overline{A}^\natural$, we may define: $\quad Q^\natural(x) = \widehat{P}(x) \cap G^\natural$.

This is the same definition as in [*l.c.* 12.4] as, for us, $F^{v\natural}_{\#}$ is uniquely determined by $F^{v\natural}$.

3) Theorem. *We suppose satisfied all conditions or axioms in 1) or 2) above, then:*

a) ${}_KN = N^\natural \subset N.\widehat{P}(\overline{A^{\#}})$.

b) $\forall a \in \Phi^\natural$, $\forall u \in U^\natural_a \setminus \{1\}$ *the fixed point set of* u *in* $A^{\#}$ *is* $D_{\#}(a, \varphi^\natural_a(u)) := \{x \in A^{\#} \mid a(x - 0_\varphi) + \varphi^\natural_a(u) \geq 0\}$ *and* $m^\natural(u) \in N^\natural$ *induces on* $A^{\#}$ *the reflection with respect to the wall* $M_{\#}(a, \varphi^\natural_a(u)) = \partial D_{\#}(a, \varphi^\natural_a(u))$.

c) The family $\varphi^\natural$ *is a valuation for the root datum* $(G^\natural, (U^\natural_a)_{a\in\Phi^\natural}, {}_KZ)$.

d) The family $\mathcal{Q}^\natural = (Q^\natural(x))_{x\in\overline{A}^\natural}$ *is a very good family of parahorics.*

e) There is an injection of the essential bordered hovel $\overline{\mathcal{I}}^\natural$ *associated to* $\mathcal{Q}^\natural$ *into* $\overline{\mathscr{I}}$ *which may be described on the façades as follows:*

For $F^{v\natural}_{\#} = F^v \cap A^v_\natural$ *open in* $F^{v\natural}$ *as above in 2), the* ${}_KN-$*equivariant embedding* $A^{\#}_{F^{v\natural}} \hookrightarrow A_{F^v}$ *between apartment-façades may be extended uniquely in a* $P_K(F^{v\natural})-$*equivariant embedding* $\mathcal{I}^\natural_{F^{v\natural}} \hookrightarrow \mathscr{I}_{F^v}$, *where* $\mathcal{I}^\natural_{F^{v\natural}}$ *is the façade of* $\overline{\mathcal{I}}^\natural$ *associated to* $F^{v\natural}$.

Proof. a), b) c) and a great part of d), e) are among the main results of Charignon [*l.c.* 12.3, 12.4]. For $\mathcal{Q}^\natural$ he proves (P1) to (P7), but then (P8) is got for free in this framework (*cf.* 4.3.6) and (P10) is clearly satisfied.

He proves (P9) actually for $\overline{A}_\natural$ *i.e.* for (spherical) vectorial facets in $A^v_\natural$: if $F^{v\natural}$ is spherical, $F^{v\natural}_1 \subset \overline{F^{v\natural}}$ and $x \in A^{\#}_{F^{v\natural}_1} = A^{\#}_{F^v_{1\natural}}$ (with $F^v_{1\natural} = F^{v\natural}_1 \cap A^v_\natural$ and $F^v_\natural = F^{v\natural} \cap A^v_\natural$) he proves only $Q^\natural(x) \cap P_K(F^v_\natural) = Q^\natural(\overline{x + F^v_\natural})$. But $P_K(F^v_\natural) = P_K(F^{v\natural})$, $F^v_\natural + \overrightarrow{A^v_{\natural 0}} = F^{v\natural}$ (2.8.3) and the "torus" S_Z in the center of $G^\natural$ (2.9.2) acts on $A^{\#}$ as a group (of translations) $\overrightarrow{T}$ generating $\overrightarrow{A^v_{\natural 0}}$. So $Q^\natural(x) \cap P_K(F^{v\natural}) = Q^\natural(\cup_{\tau\in\overrightarrow{T}} \overline{x + \tau + F^v_\natural}) = Q^\natural(\overline{x + F^{v\natural}})$.

The maps in e) between façades are described in [*l.c.* 12.5] and proved to be injective in the spherical case; but 4.14 gives the general injectivity. □

6.5 Tamely quasi-splittable descent

1) Let $\mathfrak{S}$ be a maximal $K-$split torus in the almost split Kac-Moody group $\mathfrak{G}$ over K. The generic centralizer $\mathfrak{Z}_g(\mathfrak{S})$ of $\mathfrak{S}$ in $\mathfrak{G}$ (1.10 and 2.5.3) is actually a reductive group defined over K [Re02, 12.5.2]. We suppose satisfied the following condition (independent of the choice of $\mathfrak{S}$, as different choices are conjugated by 2.5.1).

(TRQS) $\mathfrak{Z}_g(\mathfrak{S})$ becomes quasi-split over a finite tamely ramified Galois extension M of K.

(Actually $\mathfrak{Z}_g(\mathfrak{S})$ is quasi-split over M if and only if $\mathfrak{G}$ itself is quasi-split. It is an easy consequence of 2.7 NB 2) applied to M and a maximal $M-$split torus containing $\mathfrak{S}$.)

There are two important cases where this condition is satisfied for any $\mathfrak{G}$: when the field K is complete (or henselian) for a discrete valuation with perfect residue field (we then may replace tamely ramified by unramified, *cf.* [BrT84, 5.1.1] or [Ro77, 5.1.3]) or when the residue field of K has characteristic 0 (we then may replace quasi-split by split).

A consequence of this hypothesis is that there exists a finite Galois extension L of K containing M, a maximal $K-$split torus ${}_K\mathfrak{S}$, a maximal $M-$split torus ${}_M\mathfrak{S}$ and a maximal torus $\mathfrak{T}$ with $\mathfrak{T}$ $L-$split, $M-$defined and ${}_K\mathfrak{S} \subset {}_M\mathfrak{S} \subset \mathfrak{T}$ [Ch11, 13.4.2]. We shall now apply the abstract descent theorem successively to L/M and L/K to build a bordered hovel for $\mathfrak{G}$ over K.

2) Quasi-split descent: We consider the extension L/M, so we apply 6.4 with $K = M$: $\mathfrak{G}$ is quasi-split over K and split over L. Then $\mathfrak{T} = \mathfrak{Z}_g({}_K\mathfrak{S})$ is the only maximal torus containing ${}_K\mathfrak{S}$.

We choose the essential bordered hovel $\overline{\mathscr{I}} = \overline{\mathscr{I}}(\mathfrak{G}, L, \overline{\mathbb{A}}^e)$ and set $\overline{\mathcal{I}}_\# = \overline{\mathscr{I}}^\Gamma$. Then the bordered apartment $\overline{A} = \overline{A}^e(\mathfrak{T})$ is $\Gamma-$stable. The Galois group Γ has a fixed point in its main façade $A = A^q(\mathfrak{T})$ and $A^\# = A^\Gamma = A \cap \overline{\mathcal{I}}_\#$ is an affine subspace directed by $\vec{V}^\Gamma = \vec{V}^\natural$. It is easy to verify (DM1), (DM2) and also (DM4) (as ${}_KN$ is the normalizer in $\mathfrak{G}(K)$ of ${}_K\mathfrak{S}$). For (DM3) there exists a chamber F in A_{F^v} meeting $\overline{\mathcal{I}}_\#$, so the condition is clearly satisfied.

Therefore theorem 6.4.3 applies. Actually in the classical case (Φ finite) $\mathfrak{G}(K).A^\#$ is the extended Bruhat-Tits building of $\mathfrak{G}$ over M *cf.* [BrT84] or [Ro77].

3) General descent: We come back to the situation and notations in 1) above. We still choose the essential bordered hovel $\overline{\mathscr{I}} = \overline{\mathscr{I}}(\mathfrak{G}, L, \overline{\mathbb{A}}^e)$ with $\mathbb{A} = \mathbb{A}^q$.

The generic centralizer $\mathfrak{Z}_g(\mathfrak{S})$ of $\mathfrak{S} = {}_K\mathfrak{S}$ is a $K-$defined reductive group generated over L by $T = \mathfrak{T}(L)$ and the groups U_α for $\alpha \in \Phi$, $\alpha_{|\mathfrak{S}}$ trivial. In particular over L, $\mathfrak{Z}_g(\mathfrak{S})$ is isomorphic to some $\mathfrak{G}_{\mathcal{S}(I_0)}$ and by 5.8.2 $\mathscr{I}(\mathfrak{Z}_g(\mathfrak{S}), L, \mathbb{A}^q)$ may be embedded in $\mathscr{I}$. The image is the union $\mathscr{I}_\mathfrak{S} = \mathscr{I}(\mathfrak{Z}_g(\mathfrak{S}), \mathfrak{G}, L, \mathbb{A}^q)$ of the apartments of $\mathscr{I}$ corresponding to $L-$split maximal tori of $\mathfrak{G}$ containing $\mathfrak{S}$. This set is stable by Γ and $Z_L(\mathfrak{S}) = \mathfrak{Z}_g(\mathfrak{S})(L)$ or the normalizer $N_L(\mathfrak{S})$ of $\mathfrak{S}$ in $\mathfrak{G}(L)$. If we choose a vectorial $K-$chamber ${}_KC_0^{vq} \subset A_\natural^v$ and let $F_0^{vq} \in \mathscr{I}^{vc}(L)$ be the spherical vectorial facet containing ${}_KC_0^{vq}$, the projection map π from $\mathscr{I}_\mathfrak{S}$ to $\mathscr{I}_{F_0^{vq}}$ is onto and $\Gamma \ltimes N_L(\mathfrak{S})-$equivariant; it identifies the essentialization of $\mathscr{I}_\mathfrak{S}$ with $\mathscr{I}_{F_0^{vq}}$.

In $\mathscr{I}_\mathfrak{S}$ we consider the union $\mathscr{I}_\mathfrak{S}^{ord}$ of the apartments corresponding to a torus containing a maximal $M-$split torus ${}_M\mathfrak{S}$ (containing $\mathfrak{S}$). It is stable by $\Gamma \ltimes N_L(\mathfrak{S})$ and we saw in 2) above that $\mathscr{J}_\mathfrak{S} = Z_M(\mathfrak{S}).A_M^\# = (\mathscr{I}_\mathfrak{S}^{ord})^{Gal(L/M)}$ is a good candidate for the hovel of $\mathfrak{Z}_g(\mathfrak{S})$ over M. More precisely its image $\mathscr{J}_{F_0^{vq}} = \pi(\mathscr{J}_\mathfrak{S})$ in $\mathscr{I}_{F_0^{vq}}$ is the Bruhat-Tits building of $\mathfrak{Z}_g(\mathfrak{S})$ over M: it is the set of ordinary $Gal(L/M)-$invariant points in the Bruhat-Tits building over L [Ro77, 2.5.8c].

We consider now $A^\# = (\mathscr{J}_\mathfrak{S})^{Gal(M/K)} = (\mathscr{I}_\mathfrak{S}^{ord})^\Gamma$; its image by π is in $(\mathscr{J}_{F_0^{vq}})^{Gal(M/K)}$. But the semi-simple quotient of $\mathfrak{Z}_g(\mathfrak{S})$ is $K-$anisotropic and M/K is tamely ramified, so we know that $(\mathscr{J}_{F_0^{vq}})^{Gal(M/K)}$ contains at most one point [Ro77, 5.2.1]. Moreover Koen Struyve [S11] proved what was missing in [Ro77] (condition (DE) of [BrT84, 5.1.5]): this set is non empty (even if the valuation is not discrete). So $(\mathscr{J}_{F_0^{vq}})^{Gal(M/K)}$ is reduced to one point x_0 and $A^\# = \pi^{-1}(x_0)^\Gamma$. But $\pi^{-1}(x_0)$ is an affine space directed by $\overrightarrow{F_0^{vq}}$, Γ is finite and acts affinely, so $A^\#$ is a (non empty) affine space directed by $(\overrightarrow{F_0^{vq}})^\Gamma = \overrightarrow{{}_KC_0^{vq}} = \overrightarrow{A_\natural^v} = \vec{V}^\natural$. We shall apply 6.4 with $A^\#$, A any apartment of $\mathscr{I}_\mathfrak{S}^{ord}$ containing $A^\#$ and $\overline{A}$ its closure in $\overline{\mathscr{I}} = \overline{\mathscr{I}}(\mathfrak{G}, L, \overline{\mathbb{A}}^e)$.

We define $\mathscr{J} = \mathfrak{G}(M).\mathscr{J}_\mathfrak{S} = \mathfrak{G}(M).A_M^\#$ (resp. its closure $\overline{\mathscr{J}} = \overline{\mathfrak{G}(M).A_M^\#}$); it is the set of $Gal(L/M)-$fixed points in the union of the apartments in $\mathscr{I}$ (resp. $\overline{\mathscr{I}}$) corresponding to a maximal torus containing a maximal $M-$split torus, itself containing a maximal $K-$split torus. We take $\overline{\mathcal{I}}_\# = \overline{\mathscr{J}}^{Gal(M/K)} = \overline{\mathscr{J}}^\Gamma$. The verification of axioms (DM1) to (DM4) is made in [Ch11, 13.4.4]. Actually (DM4) is clear, (DM2) not too difficult and (DM1), (DM3) have to be verified in spherical façades, hence are corollaries of the classical Bruhat-Tits theory.

4) Conclusion: We keep the notations as in 1); let ${}_KA^{vq}$ be the $K-$apartment in ${}_K\mathscr{I}^{vq}(\mathfrak{G})$ and ${}_K\Phi$ the real root system associated to $\mathfrak{S}$. Then theorem 6.4.3 gives us a valuation ${}_K\varphi = \varphi^\natural = ({}_K\varphi_{{}_K\alpha})_{{}_K\alpha \in {}_K\Phi}$ of the root datum $(\mathfrak{G}(K), (V_{{}_K\alpha})_{{}_K\alpha \in {}_K\Phi}, {}_KZ)$ (*cf.* 2.7) and a very good family of parahorics $(\widehat{P}_K(x))_{x \in \overline{A^\natural}}$. The corresponding bordered hovel is written $\overline{\mathscr{I}}(\mathfrak{G}, K, \overline{A^\natural})$.

For ${}^KF^{vq} = F^{v\natural}$ a vectorial Weyl$-K-$facet and F^{vq} a vectorial facet with $F^{vq} \cap {}_KA^{vq}$ open in ${}^KF^{vq}$, we have a $P_K({}^KF^{vq})-$equivariant embedding $\mathscr{I}(\mathfrak{G}, K, \overline{A}^\natural)_{{}^KF^{vq}} \hookrightarrow \mathscr{I}(\mathfrak{G}, L, \overline{\mathbb{A}}^e)_{F^{vq}}$ between the façades. The image $\mathfrak{G}(K).A^{\#}_{F^{v\natural}}$ is pointwise fixed by Γ.

Actually the set $\overline{A}^\natural$ is the essential bordered apartment associated to $A^{\#}$ and ${}_K\Phi$, its façades are the $A^{\#}_{F^{v\natural}}$ for $F^{v\natural}$ as above. Such a façade $A^{\#}_{F^{v\natural}}$ may be identified with the closure of $A^{\#}$ in $A^q(\mathfrak{T})_{F^{vq}}$. Moreover $A^{\#}$ is the set of $Gal(L/K)-$fixed points in the union of the apartments $A^q(\mathfrak{T}) \subset \mathscr{I}(\mathfrak{G}, L, \mathbb{A}^q)$ for $\mathfrak{T}$ a $L-$split maximal torus containing a maximal $M-$split torus, itself containing the maximal $K-$split torus ${}_K\mathfrak{S}$. More precisely for each such apartment $A^q(\mathfrak{T})$, $A^q(\mathfrak{T}) \cap \mathscr{I}^\Gamma$ is empty or equal to $\mathbb{A}^{\#}$ (an affine subspace directed by $\overrightarrow{{}_KA^v({}_K\mathfrak{S})} \subset \overrightarrow{A^q(\mathfrak{T})}$) and, for each F^{vq} as above, the intersection $\overline{A}^q(\mathfrak{T}) \cap \mathscr{I}^\Gamma_{F^{vq}}$ is empty or equal to $\overline{A}^{\#} \cap A^q(\mathfrak{T})_{F^{vq}}$ (as the arguments in 3) give analogous results in the $Gal(L/K)-$stable façades).

So the image of $\mathscr{I}(\mathfrak{G}, K, \overline{A}^\natural)_{{}^KF^{vq}}$ in $\mathscr{I}(\mathfrak{G}, L, \overline{\mathbb{A}}^e)_{F^{vq}}$ is the set of $Gal(L/K)-$fixed points in the union of the apartments $A^q(\mathfrak{T})_{F^{vq}} \subset \mathscr{I}(\mathfrak{G}, L, \mathbb{A}^q)_{F^{vq}}$ for $\mathfrak{T}$ a $L-$split maximal torus containing a maximal $M-$split torus, itself containing a maximal $K-$split torus.

6.6 More general relative apartments

Most of the preceding arguments apply with a more general choice of apartments. We keep the hypotheses as in 6.5.1, but we choose for $\mathbb{A}$ one of the model apartments associated to $\mathfrak{G}$ and $\mathfrak{T}$ as in 3.5.1 (*i.e.* via a commutative extension $\varphi : \mathcal{S} \to \mathcal{S}'$ of RGS) or eventually a quotient by a subspace V_{00} of $V_0 \subset V = Y' \otimes \mathbb{R}$. We suppose moreover $\mathcal{S}'$ endowed with a star action of Γ for which φ is Γ^*-equivariant and V_{00} Γ^*-stable; *cf.* remark 2.2 and the choice made in 2.4.1. We write $\mathbb{A}^v$ the corresponding vectorial apartment in $\overrightarrow{\mathbb{A}} = V/V_{00}$ and $\overline{\mathbb{A}}$ one of the three associated bordered apartments.

The Galois group Γ acts on $\overline{\mathscr{I}}(\mathfrak{G}, L, \overline{\mathbb{A}})$ and $\mathscr{I}^v(\mathfrak{G}, L, \mathbb{A}^v)$, *cf.* 6.2.4. These actions are compatible with each other, with the $\mathfrak{G}(L)-$actions and the essentialization maps η : $\overline{\mathscr{I}}(\mathfrak{G}, L, \overline{\mathbb{A}}) \to \overline{\mathscr{I}}(\mathfrak{G}, L, \overline{\mathbb{A}}^e) = \overline{\mathscr{I}}$, $\eta^v : \mathscr{I}^v(\mathfrak{G}, L, \mathbb{A}^v) \to \mathscr{I}^v(\mathfrak{G}, L, \mathbb{A}^{vq}) = \mathscr{I}^{vq}$. We define ${}_K\mathbb{A} = \eta^{-1}(A^{\#})^\Gamma$; it is an affine space directed by $(\overrightarrow{F_0^v})^\Gamma = \overrightarrow{{}_K\mathbb{A}^v}$ (where $F_0^v = (\eta^v)^{-1}(F_0^{vq})$ and $A^{\#}$, F_0^{vq} are as in 6.5.3). The group ${}_KN$ acts on ${}_K\mathbb{A}$, we write ν_K this action.

We choose ${}_K\mathbb{A}$ as model relative apartment. We may suppose ${}_KA \subset \mathbb{A}$, but then $\mathbb{A}$, as apartment in $\mathscr{I}(\mathfrak{G}, L, \mathbb{A})$, is non necessarily $\Gamma-$stable. We choose in ${}_K\mathbb{A}$ a special origin x_0 *i.e.* its image by η is the special point in $A^{\#}$ chosen as origin in 6.4.2 to define the valuation ${}_K\varphi = \varphi^\natural$ of $(\mathfrak{G}(K), (V_{{}_K\alpha})_{{}_K\alpha \in {}_K\Phi^{re}}, {}_KZ)$. For $x \in {}_K\mathbb{A}$ we define $\widehat{P}_K(x) = \widehat{P}(x) \cap \mathfrak{G}(K)$.

The (real) walls in ${}_K\mathbb{A}$ are the inverse images by η of the walls in $A^{\#}$ defined in 6.4.3b, *i.e.* they are described as $M_K({}_K\alpha, {}_K\varphi_\alpha(u)) = \{x \in {}_K\mathbb{A} \mid {}_K\alpha(x - x_0) + {}_K\varphi_\alpha(u) = 0\}$ for ${}_K\alpha \in {}_K\Phi$ and $u \in V_{{}_K\alpha} \setminus \{1\}$; their set is written $\mathcal{M}_K$. Note that, even if $\mathbb{A} = \mathbb{A}^q$ is essential, ${}_K\mathbb{A}$ may be inessential (as essentiality does not involve the imaginary walls defined below).

For $\alpha \in \Phi$ (resp. $\alpha \in \Delta$), $\lambda \in \Lambda_L = \omega_L(L^*)$, if ${}_K\alpha = \alpha_{|\mathfrak{S}}$ is in ${}_K\Phi$ (resp. if ${}_K\alpha = \alpha_{|\mathfrak{S}} \neq 0$ is not in ${}_K\Phi$), then $M_K({}_K\alpha, \lambda)$ is a real (ghost) wall (resp. an imaginary wall); we write $\mathcal{M}_{L/K}$ (resp. $\mathcal{M}^i_{L/K}$) the set of these walls. So $\mathcal{M}_{L/K} \cup \mathcal{M}^i_{L/K}$ is the set of the non trivial traces on ${}_K\mathbb{A}$ of the real or imaginary walls of $\mathbb{A}$.

We define the enclosure map $cl^{{}_K\Delta^r}_{L/K}$ as in 3.6.1: it is associated to $\mathcal{P} = {}_K\Delta^r \subset {}_K\Delta$ (defined in 2.9.3b) and the sets $\Lambda'_{{}_K\alpha} = \Lambda_L \subset \mathbb{R}$ *i.e.* to $\mathcal{M}_{L/K}$ and a subset of $\mathcal{M}^i_{L/K}$. A more natural enclosure map would be $cl_K^{K\Delta^r}$ associated to $\mathcal{M}_K$ and the same subset of $\mathcal{M}^i_{L/K}$ (or a smaller one $\mathcal{M}^i_K$?), but we have not succeeded in using it.

Proposition 6.7. *In the above situation, we have:*

a) The action ν_K is affine and ${}_KN \subset N.\widehat{P}({}_K\mathbb{A}) \subset N.{}_KZ$. In particular for $n \in {}_KN$, the linear map associated to $\nu_K(n)$ is $\nu^v_K(n) \in {}_KW^v = {}_KN/{}_KZ$.

b) The group ${}_KZ$ acts on ${}_K\mathbb{A}$ by translations. More precisely for $z \in {}_KZ$, the vector $\nu_K(z)$ of this translation is the class modulo V_{00} of a vector $\tilde{\nu}_K(z) \in V$ which satisfies the formula: $\chi_1(\tilde{\nu}_K(z)) = -\omega_K(\chi_2(z))$, for any $\chi_1 \in X' \subset V^*$ and χ_2 in the group $X(\mathfrak{Z})$ of characters of the reductive group $\mathfrak{Z}_g(\mathfrak{S})$ with the condition that χ_2 and $\varphi^*(\chi_1) \in X(\mathfrak{T})$ coincide on $\mathfrak{S}$.

As $X(\mathfrak{Z})$ is identified by restriction to a finite index subgroup of $X(\mathfrak{S})$, this formula determines completely $\tilde{\nu}_K(z)$ and $\nu_K(z)$.

c) For any real relative root ${}_K\alpha$ and $u \in V_{{}_K\alpha}\setminus\{1\}$ u fixes the half apartment $D_K({}_K\alpha, {}_K\varphi_\alpha(u)) = \{x \in {}_K\mathbb{A} \mid {}_K\alpha(x - x_0) + {}_K\varphi_\alpha(u) \geq 0\}$ and $\nu_K(m_K(u))$ is the reflection $r_{{}_K\alpha, {}_K\varphi_\alpha(u)}$ with respect to the wall $M_K({}_K\alpha, {}_K\varphi_\alpha(u)) = \partial D_K({}_K\alpha, {}_K\varphi_\alpha(u))$.

d) If moreover ${}_K\alpha$ is non multipliable, $m_K(u)^2 = {}_K\alpha^\vee(-1)$ and $m_K(u)^4 = 1$.

e) $\nu_K({}_KN)$ is a semi-direct product of $\nu_K({}_KZ)$ by a subgroup fixing x_0 and isomorphic, via ν^v_K, to ${}_KW^v = {}_KN/{}_KZ$.

f) The action of ${}_KN$ on the closure $\eta^{-1}(\overline{A\#})^\Gamma$ of ${}_K\mathbb{A}$ in $\overline{\mathbb{A}}$ is deduced from its actions on ${}_K\mathbb{A}$ and ${}_K\mathbb{A}^v$: $\nu_K(n).pr_{{}_KF^v}(x) = pr_{\nu^v_K(n)({}_KF^v)}(\nu_K(n).x)$, for $n \in {}_KN$, $x \in {}_K\mathbb{A}$ and ${}_KF^v$ a K-facet in ${}_K\mathbb{A}^v$.

N.B. The equations defining $\overrightarrow{{}_K\mathbb{A}^b}$ in $\overrightarrow{\mathbb{A}^b}$ are in Q and correspond (via *bar*) to the equations defining ${}_KY = Y(\mathfrak{S})$ in $Y = Y(\mathfrak{T})$ *i.e.* $\mathfrak{S}$ in $\mathfrak{T}$, *cf.* 2.4.4 an 2.5.2. So the formula in b) above defines a vector $\nu_K(z) \in \overrightarrow{{}_K\mathbb{A}^b} = \overrightarrow{{}_K\mathbb{A}}$. Moreover $\nu_K(z)$ is in the image of the map $Y(\mathfrak{S}) \otimes \mathbb{R} \hookrightarrow Y(\mathfrak{T}) \otimes \mathbb{R} \xrightarrow{\varphi} V \to V/V_{00}$ (analogous to the map in 1.10).

Proof. With the notations as in 2.8.1, let $j \in {}_KI_{re}$; then $J = \underline{{}_K\{j\}} = I_0 \cup \{i \in I \mid \Gamma^* i = j\}$ is spherical. So $\mathfrak{G}_{\mathcal{S}(J)}$ is a reductive group, containing $\mathfrak{Z}_g(\mathfrak{S}) = \mathfrak{G}_{\mathcal{S}(I_0)}$ and defined over K; we write $\mathfrak{G}_J$ the corresponding $K-$subgroup-scheme of $\mathfrak{G}$. By 5.8.2 the extended Bruhat-Tits building $\mathscr{I}(\mathfrak{G}_J, L, \mathbb{A})$ embeds in the hovel $\mathscr{I}(\mathfrak{G}, L, \mathbb{A})$: the way we have chosen $\mathbb{A}$ ensures us that $\mathbb{A}$ is really endowed with the same action of the normalizer of $\mathfrak{T}$ in $\mathfrak{G}_J(L)$ as in the case of an extended Bruhat-Tits building [Ro77, § 2.1]. Moreover the actions of Γ are compatible.

As the classical construction of $\mathscr{I}(\mathfrak{G}_J, K, \mathbb{A})$ uses the same methods as in 6.5 above, we know that a), b) and c) are satisfied for ${}_KN \cap \mathfrak{G}_J(K)$ and ${}_K\alpha = \pm{}_K\alpha_j, \pm 2{}_K\alpha_j$ [Ro77, 5.1.2]. So b) is completely proved. Now ${}_KW^v = {}_KN/{}_KZ$ is generated by simple reflections in $({}_KN \cap \mathfrak{G}_J(K))/{}_KZ$ for $j \in {}_KI_{re}$ (as ${}_KZ \subset \mathfrak{G}_J(K)$, $\forall j$). So a) is satisfied and also c) as any ${}_K\alpha \in {}_K\Phi_{re}$ is conjugated by ${}_KW^v$ to some $\pm{}_K\alpha_j$ or $\pm 2{}_K\alpha_j$.

Let ${}_K\alpha$ and u be as in d); we choose $s \in \mathfrak{S}(K_s)$ such that ${}_K\alpha(s) = -1$. By [BoT65, 7.2 (2)], $m_K(u)^2 = m_K(u).s.m_K(u)^{-1}.s^{-1} = r_{{}_K\alpha}(s).s^{-1} = {}_K\alpha^\vee(-1)$; so d) follows. As x_0 was chosen special, $\forall i \in {}_KI_{re}$, $\exists u_i \in V_{{}_K\alpha_i} \setminus \{1\}$ with ${}_K\varphi_{\alpha_i}(u_i) = 0$ hence $m_K(u_i)$ fixes x_0. So the subgroup fixing x_0 in e) is the image by ${}_K\nu$ of the subgroup of ${}_KN$ generated by the $m_K(u_i)$ and e) follows from a) and b).

We know that, for the action ν of N on $\overline{\mathbb{A}}$, $\nu(n).pr_{F^v}(x) = pr_{\nu^v(n)(F^v)}(\nu(n).x)$; so f) follows from a). □

6.8 Embeddings of bordered apartments

1) To define the bordered apartment ${}_K\overline{\mathbb{A}}$, we always choose the vectorial Weyl$-K-$facets in ${}^K\mathbb{A}^v$ (as for $\overline{A}^\natural$ in 6.4.2 but differently from 6.7f). We still have three choices for ${}_K\overline{\mathbb{A}}$ (as in the general definition 3.7.2): ${}_K\overline{\overline{\mathbb{A}}}$ (resp. ${}_K\overline{\mathbb{A}}^e$) is the disjoint union of the inessential façades ${}_K\mathbb{A}^{ne}_{{}^KF^v} = {}_K\mathbb{A}$ (resp. the essential façades ${}_K\mathbb{A}^e_{{}^KF^v} = {}_K\mathbb{A}/\overrightarrow{{}^KF^v}$) for ${}^KF^v$ a Weyl$-K-$facet in ${}^K\mathbb{A}^v$, and ${}_K\overline{\mathbb{A}}^i$ differs from ${}_K\overline{\mathbb{A}}^e$ only by its main façade which is inessential. A Weyl$-K-$facet ${}^KF^v$ contains a unique maximal $K-$facet ${}_KF^v_{max}$ which is open in ${}^KF^v$, hence $\overrightarrow{{}_KF^v_{max}} = \overrightarrow{{}^KF^v}$. So ${}_K\mathbb{A}^e_{{}^KF^v}$ is equal to ${}_K\mathbb{A}^e_{{}_KF^v_{max}}$. Now the proposition 6.7f tells us that the action ν_K of ${}_KN$ on ${}_K\mathbb{A}$ extends naturally to ${}_K\overline{\mathbb{A}}$ ($= {}_K\overline{\mathbb{A}}^e$, or ${}_K\overline{\mathbb{A}}^i$ or ${}_K\overline{\overline{\mathbb{A}}}$).

2) For any choice of $\mathbb{A}$ (suitable for $\mathfrak{G}$ and L), we chose a unique ${}_K\mathbb{A}$ (inside $\mathbb{A}$ for some embedding). So it is interesting to define a good choice for ${}_K\overline{\mathbb{A}}$ for each choice of $\overline{\mathbb{A}}$. And it is natural to choose ${}_K\overline{\mathbb{A}}^i$ (resp. ${}_K\overline{\mathbb{A}}^e$, ${}_K\overline{\overline{\mathbb{A}}}$) when $\overline{\mathbb{A}} = \overline{\mathbb{A}}^i$ (resp. ${}_K\overline{\mathbb{A}}^e$, ${}_K\overline{\overline{\mathbb{A}}}$). Then we have a ${}_KN-$equivariant embedding ${}_K\overline{\mathbb{A}} \hookrightarrow \overline{\mathbb{A}}$ defined as follows on each façade: for ${}^KF^v$ a vectorial Weyl$-K-$facet, let F^v be the facet in $\mathbb{A}^v$ containing ${}_KF^v_{max}$, then ${}_K\mathbb{A}^{ne}_{{}^KF^v} = {}_K\mathbb{A} \hookrightarrow \mathbb{A} = \mathbb{A}^{ne}_{F^v}$ and ${}_K\mathbb{A}^e_{{}^KF^v} = {}_K\mathbb{A}/\overrightarrow{{}_KF^v_{max}} \hookrightarrow \mathbb{A}^e_{F^v} = \mathbb{A}/\overrightarrow{F^v}$.

Note that the main façade does not embed in general in the main façade when we choose ${}_K\overline{\mathbb{A}}^e$ (as was the case for Charignon, *cf.* 6.4.3e). Moreover, if ${}_KF^v$ is positive and negative, the definition of ${}_KF^v_{max}$ may include a choice of sign. For example the main façade ${}_K\mathbb{A}^e$ of ${}_K\overline{\mathbb{A}}^e$ may embed in $\mathbb{A}^e_{F^v}$ or $\mathbb{A}^e_{-F^v}$ (they are equal but included separately in $\overline{\mathbb{A}}$).

3) For $x \in {}_K\overline{\mathbb{A}}$, more precisely $x \in {}_K\mathbb{A}^{(n)e}_{{}^KF^v}$, we define $\widehat{P}_K(x) = \widehat{P}(x) \cap \mathfrak{G}(K)$ where x is considered in $\mathbb{A}^{(n)e}_{F^v}$ as above. This coincides with the above definition for $x \in {}_K\mathbb{A}$ and it is compatible with the projections: $\widehat{P}_K(x) \subset \widehat{P}_K(pr_{{}^KF^v_1}(x))$.

Theorem 6.9. *We suppose that the Kac-Moody group $\mathfrak{G}$ satisfies the condition (TRQS) of 6.5 and we keep the notations as in 6.5 to 6.8.*

a) The family ${}_K\varphi$ is a valuation for the root datum $(\mathfrak{G}(K), (V_{K\alpha})_{K\alpha\in{}_K\Phi}, {}_KZ)$.

b) The family $\widehat{\mathcal{P}}_K = (\widehat{P}_K(x))_{x\in{}_K\overline{\mathbb{A}}}$ is a very good family of parahorics.

We write $\mathscr{I}(\mathfrak{G}, K, {}_K\mathbb{A})$ (resp. $\overline{\mathscr{I}}(\mathfrak{G}, K, {}_K\overline{\mathbb{A}})$) the corresponding hovel (resp. bordered hovel).

c) The family $\widehat{\mathcal{P}}_K$ is compatible with the enclosure map $cl^{{}^K\Delta^r}_{L/K}$. In particular $\mathscr{I}(\mathfrak{G}, K, {}_K\mathbb{A})$ is an ordered affine hovel of type $({}_K\mathbb{A}, cl^{{}^K\Delta^r}_{L/K})$ (without generalization) on which $\mathfrak{G}(K)$ acts strongly transitively.

d) The ${}_KN-$equivariant embedding ${}_K\overline{\mathbb{A}} \hookrightarrow \overline{\mathbb{A}}$ may be extended uniquely in a $\mathfrak{G}(K)-$equivariant embedding $\overline{\mathscr{I}}(\mathfrak{G}, K, {}_K\overline{\mathbb{A}}) \hookrightarrow \overline{\mathscr{I}}(\mathfrak{G}, L, \overline{\mathbb{A}})$. Its image is in $\overline{\mathscr{I}}(\mathfrak{G}, L, \overline{\mathbb{A}})^\Gamma$.

e) If the valuation ω_K of K is discrete, then ${}_K\mathbb{A}$ (or $\mathscr{I}(\mathfrak{G}, K, {}_K\mathbb{A})$) is semi-discrete: in $\mathscr{M}_{L/K}$ or $\mathscr{M}_K$ the set of walls of given direction is locally finite.

f) The hovel $\mathscr{I}(\mathfrak{G}, K, {}_K\mathbb{A})$ is thick. More precisely the set of chambers adjacent to a chamber C along a panel in a wall $M_K({}_K\alpha, k)$ with ${}_K\alpha \in {}_K\Phi$ non divisible, is in one to one correspondence with a finite dimensional vector space over the residue field κ of K.

Definition. $\mathscr{I}(\mathfrak{G}, K, {}_K\mathbb{A})$ (resp. $\overline{\mathscr{I}}(\mathfrak{G}, K, {}_K\overline{\mathbb{A}})$) is the *affine hovel* (resp. *affine bordered hovel*) *of* $\mathfrak{G}$ *over* K *with model apartment* $\mathbb{A}$ (resp. $\overline{\mathbb{A}}$).

Remark. $\mathscr{I}(\mathfrak{G}, K, {}_K\mathbb{A})$ is the main façade of $\overline{\mathscr{I}}(\mathfrak{G}, K, {}_K\overline{\mathbb{A}})$ for ${}_K\overline{\mathbb{A}} = {}_K\overline{\mathbb{A}}^i$ or ${}_K\overline{\overline{\mathbb{A}}}$. By the definition of ${}_K\mathbb{A}$ in 6.6 and of $A^\#$ in 6.5.4, the image of $\mathscr{I}(\mathfrak{G}, K, {}_K\mathbb{A})$ in $\mathscr{I}(\mathfrak{G}, L, \mathbb{A})$ is the set of $\Gamma-$fixed points in the union of the apartments $A(\mathfrak{T}) \subset \mathscr{I}(\mathfrak{G}, L, \mathbb{A})$ for $\mathfrak{T}$ a $L-$split maximal torus containing a maximal $M-$split torus, itself containing a maximal $K-$split torus.

Proof. a) The family ${}_K\varphi$ is actually defined by the essentialization of ${}_K\mathbb{A}$. So it is a valuation by 6.4.3c.

b) The family $\widehat{\mathcal{P}}_K$ satisfies clearly to axioms (P1), (P2), (P4), (P5) and (P10). (P3) is proved in 6.7c for the main façade; the result is analogous in the other façades and the link is made by 6.7f.

If ${}^KF^v(x)$ is spherical, then the corresponding facet F^v (as in 6.8.2) is spherical and ${}_K\mathbb{A}_{{}^KF^v(x)}$ embeds in $\mathbb{A}_{F^v}$ which is an apartment in the Bruhat-Tits building $\mathscr{I}(\mathfrak{G}, L, \overline{\mathbb{A}})_{F^v}$ for the reductive group $P(F^v)/U(F^v) \simeq M(F^v)$. Moreover ${}_K\mathbb{A}_{{}^KF^v(x)}$ is chosen in $\mathscr{I}(\mathfrak{G}, L, \overline{\mathbb{A}})_{F^v}$ as in 6.5.3 *i.e.* as in the descent theorems of (extended) Bruhat-Tits buildings. So $\widehat{P}_K(x)$ is generated by ${}_KN(x)$ and the $V_{{}_K\alpha} \cap \widehat{P}_K(x)$ for ${}_K\alpha \in {}_K\Phi$ and (P6) is satisfied.

For $x \in {}_K\mathbb{A}_{{}^KF_1^v}$ and ${}^KF_1^v \subset \overline{{}^KF^v}$ we write F^v_{1max} and F^v_{max} the corresponding maximal facets in $\mathbb{A}^v$. Then $\widehat{P}_K(x) \cap P_K({}^KF^v) \subset \mathfrak{G}(K) \cap \widehat{P}(x) \cap P(F^v_{max}) \subset \mathfrak{G}(K) \cap \widehat{P}(\overline{x + F^v_{max}}) \subset \mathfrak{G}(K) \cap \widehat{P}(\overline{x + F^v_{max} \cap {}_K\mathbb{A}^v}) = \mathfrak{G}(K) \cap \widehat{P}(\overline{x + {}_KF^v})$ with ${}_KF^v = F^v_{max} \cap {}_K\mathbb{A}^v \subset {}_KF^v$. But ${}_KF^v + \overrightarrow{{}_K\mathbb{A}^v}_0 = {}^KF^v$ and the "torus" S_Z in the center of $\mathfrak{G}(K)$ (2.9.2) acts on ${}_K\mathbb{A}$ as a group (of translations) $\overrightarrow{T}$ generating $\overrightarrow{{}_K\mathbb{A}^v}_0$; so $\widehat{P}_K(x) \cap P_K({}^KF^v) = \widehat{P}_K(\cup_{\tau \in \overrightarrow{T}} \overline{x + \tau + {}_KF^v}) = \widehat{P}_K(\overline{x + {}^KF^v})$ and (P9) is satisfied.

For $x \in \overline{\mathbb{A}}$ and ${}^KC^v$ a chamber in ${}^KF^v(x)^*$, we have by 6.4.3d $\widehat{P}_K(x) \subset \widehat{P}_K(\eta x) = (\widehat{P}_K(\eta x) \cap U({}^KC^v)).(\widehat{P}_K(\eta x) \cap U(-{}^KC^v)).{}_KN(\eta x)$. We know by construction that $\widehat{P}(\eta x) \cap U(\pm C^v) = \widehat{P}(x) \cap U(\pm C^v)$ (*cf.* 5.1 to 5.3) for any chamber C^v in F^{v*} (F^v as above) *e.g.* C^v containing ${}_KC^v = {}^KC^v \cap {}_K\mathbb{A}^v$. So $\widehat{P}(\eta x) \cap U(\pm {}^KC^v) = \widehat{P}(x) \cap U(\pm {}^KC^v)$ and $\widehat{P}_K(\eta x) \cap U(\pm {}^KC^v) \subset \widehat{P}_K(x)$. Hence $\widehat{P}_K(x) = (\widehat{P}_K(x) \cap U({}^KC^v)).(\widehat{P}_K(x) \cap U(-{}^KC^v)).(\widehat{P}_K(x) \cap {}_KN(\eta x))$ and $\widehat{P}_K(x) \cap {}_KN = {}_KN(x)$. So (P8) is satisfied.

In the situation of (P7), let $B = \{x, pr_{F^v}(x)\}$. We saw above that $\widehat{P}_K(x).{}_KN = \widehat{P}_K(\eta x).{}_KN$. So, by 6.4.3d, $\widehat{P}_K(x).{}_KN \cap P_K({}^KF^v).{}_KN \subset \widehat{P}_K(\eta B).{}_KN$. Let ${}^KC^v$ be such that ${}^KF^v \subset \overline{{}^KC^v}$, then arguing as in [GR08, 4.3.4] we see that $\widehat{P}_K(\eta B) = [\widehat{P}_K(\eta B) \cap U({}^KC^v)].[\widehat{P}_K(\eta B) \cap U(-{}^KC^v)].{}_KN(\eta B) \subset [\widehat{P}_K(B) \cap U({}^KC^v)].[\widehat{P}_K(B) \cap U(-{}^KC^v)].{}_KN$. So (P7) follows.

c) Let Ω be a non empty filter in ${}_K\mathbb{A}_{{}^KF^v}$ and ${}^KC^v \in {}^KF^v(x)^*$. We consider a chamber C^v in $\mathbb{A}^v$ such that ${}_KC^v = C^v \cap {}_K\mathbb{A}^v$ is open in ${}^KC^v$. Then $\widehat{P}_K(\Omega) \cap U_K(\pm {}^KC^v) \subset \mathfrak{G}(K) \cap \widehat{P}(\Omega) \cap U(\pm C^v) \subset \mathfrak{G}(K) \cap \widehat{P}(cl^{\Delta}(\Omega)) \subset \mathfrak{G}(K) \cap \widehat{P}(cl^{{}_K\Delta}_{L/K}(\Omega))$, where $cl^{\Delta}(\Omega)$ (resp. $cl^{{}_K\Delta}_{L/K}(\Omega)$) is the enclosure of Ω in $\mathbb{A}$ (resp. ${}_K\mathbb{A}$) for the root system Δ (resp. for ${}_K\Delta$ and $\mathcal{M}_{L/K}$, $\mathcal{M}^i_{L/K}$). We use once more the torus S_Z in the center of $\mathfrak{G}(K)$: we have $cl^{{}_K\Delta}_{L/K}(\cup_{\tau \in \overrightarrow{T}} \Omega + \tau) = cl^{{}_K\Delta^r}_{L/K}(\Omega)$. So $\widehat{P}_K(\Omega) \cap U_K(\pm {}_KC^v) \subset \widehat{P}_K(cl^{{}_K\Delta^r}_{L/K}(\Omega))$. Hence $\widehat{\mathcal{P}}_K$ is compatible with $cl^{{}_K\Delta^r}_{L/K}$ and c) is a consequence of 4.12.5.

d) The existence of a unique $P_K({}^KF^v)-$equivariant map $\mathscr{I}(\mathfrak{G}, K, {}_K\overline{\mathbb{A}})_{{}^KF^v} \to \mathscr{I}(\mathfrak{G}, L, \overline{\mathbb{A}})_{F^v}$ extending $\mathbb{A}_{{}^KF^v} \hookrightarrow \mathbb{A}_{F^v}$ is an easy consequence of the definitions of ν_K and $\widehat{P}_K(x)$: ${}_KN \subset N.\widehat{P}({}_K\mathbb{A})$, "$\nu_K = \nu_{|_{KN}}$" and $\widehat{P}_K(x) = \mathfrak{G}(K) \cap \widehat{P}(x)$. As for 6.4.3e we conclude with 4.14.

e) As ω_K is discrete and L/K finite, ω_L is discrete. Suppose ${}_K\alpha \in {}_K\Phi$ non divisible, then the walls in $\mathcal{M}_{L/K}$ of direction $\mathrm{Ker}({}_K\alpha)$ are the traces of walls in $\mathbb{A}$ of direction $\mathrm{Ker}\beta$ for $\beta \in \Phi$ with ${}_K\beta = {}_K\alpha$ or ${}_K\beta = 2.{}_K\alpha$. There is only a finite number of such β and, for each β, $\Lambda_\beta = \omega_L(L^*)$ is discrete. So the set of these walls of direction $\mathrm{Ker}({}_K\alpha)$ is locally finite.

f) We write $a = {}_K\alpha$. By 4.11e this set of chambers is in one to one correspondence with $V_{a,k}/V_{a,k+}$. Suppose $2a \notin {}_K\Phi$, then by 6.4.2, $V_{a,k} = V_a \cap (\prod_{_K\beta=a} U_{\beta,k})$ is an $\mathcal{O}_K-$module and $V_{a,k+} = V_a \cap (\prod_{_K\beta=a} U_{\beta,k+})$ an $\mathcal{O}_K-$submodule such that $\mathfrak{m}_K.V_{a,k} \subset V_{a,k+}$. So $V_{a,k}/V_{a,k+}$ is a $\kappa-$vector space of dimension $\leq dim_K(V_a) = |(a)|$. When $2a \in {}_K\Phi$ we prove, the same way, that $V_{a,k}/V_{a,k+}$ is a group extension of two $\kappa-$vector spaces of dimensions at most $|(2a)|$ and $|(a)|-|(2a)|$ *cf.* 2.6. To see that $V_{a,k}/V_{2a,2k}$ is an $\mathcal{O}_K-$submodule of $\prod_{_K\beta=a} U_{\beta,k} \subset V_{a,L}/V_{2a,L}$, we may use the coroot $(2a)^\vee$ in $Y(\mathfrak{S})$; as $a((2a)^\vee) = 1$ the exterior multiplication by $K \setminus \{0\}$ in $V_{a,L}/V_{2a,L}$ is given by the action of the torus $\mathfrak{S}(K)$. □

6.10 Remarks

1) The condition (TRQS) is certainly non necessary for the existence of an hovel $\mathscr{I}(\mathfrak{G}, K, {}_K\mathbb{A})$; the existence of this hovel for any almost split Kac-Moody group $\mathfrak{G}$ (over a complete field) seems a reasonable conjecture, as in the classical (= reductive) case. On the contrary the existence of a $\mathfrak{G}(K)-$equivariant embedding of $\mathscr{I}(\mathfrak{G}, K, {}_K\mathbb{A})$ in $\mathscr{I}(\mathfrak{G}, L, {}_L\mathbb{A})$ for any extension L/K seems to necessitate (TRQS) or ω_K discrete. And the functoriality of these embeddings seems to necessitate (TRQS). There are counter-examples even in the classical case [Ro77, 3.5.9 and 3.4.12a].

2) As explained in 6.6 the good enclosure map we may expect gives greater enclosures than $cl_{L/K}^{_K\Delta^r}$. We hope to prove that it is compatible with $\widehat{\mathcal{P}}_K$.

3) We chose to define the façades of ${}_K\overline{\mathbb{A}}$ and $\overline{\mathscr{I}}(\mathfrak{G}, K, {}_K\overline{\mathbb{A}})$ using the Weyl$-K-$facets as indexing set. This is more natural for the bordered hovel of the root datum $(\mathfrak{G}(K), (V_{_K\alpha})_{_K\alpha\in{}_K\Phi}, {}_KZ)$ but a definition using $K-$facets seems richer. This is largely an illusion:

Let ${}^KF^v \subset {}^K\mathbb{A}^v$ be a Weyl$-K-$facet and ${}_KF^v$ (resp. ${}_KF^v_{min}$, ${}_KF^v_{max}$) be any (resp. the minimal, maximal) $K-$facet in ${}_K\mathbb{A}^v$ corresponding to ${}^KF^v$ and F^v (resp. F^v_{min}, F^v_{max}) the corresponding facet in $\mathbb{A}^v$: hence $\overline{{}_KF^v_{min}} \subset \overline{{}_KF^v} \subset \overline{{}_KF^v_{max}}$ and $\overline{F^v_{min}} \subset \overline{F^v} \subset \overline{F^v_{max}}$. The $K-$façade ${}_K\mathbb{A}^{ne}_{_KF^v} = {}_K\mathbb{A}$ or ${}_K\mathbb{A}^e_{_KF^v} = {}_K\mathbb{A}/\overrightarrow{{}_KF^v}$ is endowed with a system of relative real roots ${}_K\Phi^m({}_KF^v)$ (and even a system of relative almost real roots ${}_K\Delta^{rm}({}_KF^v)$) which is independent of the choice of ${}_KF^v$. So ${}_K\mathbb{A}^{ne}_{_KF^v}$ and the essentialization of ${}_K\mathbb{A}^e_{_KF^v}$ do not depend of the choice: we have projections maps ${}_K\mathbb{A} = {}_K\mathbb{A}^{ne}_{_KF^v} \to {}_K\mathbb{A}^e_{_KF^v_{min}} \to {}_K\mathbb{A}^e_{_KF^v} \to {}_K\mathbb{A}^e_{_KF^v_{max}} = {}_K\mathbb{A}^e_{^KF^v}$ where the last term is the essentialization of the first three (actually $\overrightarrow{{}_KF^v}$ is in general non enclosed, as ${}_KF^v$ is defined by inequalities involving ${}_K\Delta$, and not only ${}_K\Phi$ or ${}_K\Delta^r$).

We saw in 2.8.3c that the fixator $P_K({}_KF^v)$ of ${}_KF^v$ in $\mathfrak{G}(K)$ is independent of the choice of ${}_KF^v$. In particular the above maps are ${}_KN \cap P_K({}_KF^v)-$equivariant. Moreover the fixator of a point x in an apartment is included in the fixator of the image of x in another apartment. So we have corresponding projection maps between the façades corresponding to these $K-$facets: $\mathscr{I}(\mathfrak{G}, K, {}^K\overline{\overline{\mathbb{A}}})_{^KF^v} \to \mathscr{I}(\mathfrak{G}, K, {}_K\mathbb{A})_{_KF^v_{min}} \to \mathscr{I}(\mathfrak{G}, K, {}_K\mathbb{A})_{_KF^v} \to \mathscr{I}(\mathfrak{G}, K, {}_K\mathbb{A})_{_KF^v_{max}} = \mathscr{I}(\mathfrak{G}, K, {}_K\overline{\mathbb{A}}^e)_{^KF^v}$ and the last hovel is the essentialization of the first three .

Hence these hovels are more or less the same and it is not really interesting to include all of them in a bordered hovel. Perhaps the only interesting thing to do could be to define a fourth bordered apartment ${}_K\overline{\mathbb{A}}^{min}$ associated to ${}_K\mathbb{A}$ with ${}_K\mathbb{A}^{min}_{^KF^v} = {}_K\mathbb{A}_{_KF^v_{min}}$ and $\mathscr{I}(\mathfrak{G}, K, {}_K\mathbb{A}^{min})_{^KF^v} = \mathscr{I}(\mathfrak{G}, K, {}_K\mathbb{A})_{_KF^v_{min}}$. Then $\overline{\mathscr{I}}(\mathfrak{G}, K, {}_K\overline{\mathbb{A}}^{min})$ coincides with $\overline{\mathscr{I}}(\mathfrak{G}, K, {}_K\overline{\mathbb{A}}^e)$ when $\mathfrak{G}$ is split over K (or if ${}_KI_{re} = {}_KI$ *i.e.* ${}_K\Delta = {}_K\Delta^r$).

4) The microaffine building of a split Kac-Moody group $\mathfrak{G}_{\mathcal{S}}$ over a "local" field is defined in [Ro06]. In its Satake realization [*l.c.* 4.2.3] it is the union $\overline{\mathscr{I}}_{sph}(\mathfrak{G}_{\mathcal{S}},K,\overline{\mathbb{A}}^e)$ of the spherical façades in the essential bordered hovel $\overline{\mathscr{I}}(\mathfrak{G}_{\mathcal{S}},K,\overline{\mathbb{A}}^e)$. Hence, as explained in this section, Charignon proved the existence of such a microaffine building for any almost split Kac-Moody group satisfying (TRQS). This building satisfies clearly the functorial properties proved below for bordered hovels.

6.11 Functoriality

1) Changing the group, commutative extensions: We consider a morphism $\psi:\mathfrak{G}\to\mathfrak{G}'$ between two almost split Kac-Moody groups and we suppose that, over K_s, $\psi=\mathfrak{G}_\varphi:\mathfrak{G}_{\mathcal{S}}\to\mathfrak{G}_{\mathcal{S}'}$ for a commutative extension of RGS $\varphi:\mathcal{S}\to\mathcal{S}'$. This extension is then automatically $Gal(K_s/K)^*-$equivariant.

The conditions (TRQS) for $\mathfrak{G}$ and $\mathfrak{G}'$ are equivalent: ψ induces a bijection between the combinatorial vectorial buildings of $\mathfrak{G}$ and $\mathfrak{G}'$ over K_s [Ro12, 1.10] which is clearly $Gal(K_s/K)-$equivariant; so $\mathfrak{G}$ has a Borel subgroup defined over a field $M\in\mathcal{S}ep(K)$ if and only if $\mathfrak{G}'$ has one.

Suppose (TRQS), then $\mathfrak{G}'$ and $\mathfrak{G}$ are quasi-split over a tamely ramified finite Galois extension M/K and split over a finite Galois extension L/K with $L\supset M$. We choose an apartment $\mathbb{A}$ for $\mathfrak{G}'$ as in 6.8, hence associated to a morphism $\varphi':\mathcal{S}'\to\mathcal{S}''$ of RGS and some V_{00} in $V=Y''\otimes\mathbb{R}$ compatible with the star action of $\Gamma=Gal(L/K)$ associated to $\mathfrak{G}'$. Then the same thing is true for $\varphi'\circ\varphi$ and $\mathfrak{G}$. Now the constructions of $\overline{\mathscr{I}}(\mathfrak{G},K,{}_K\overline{\mathbb{A}})$ inside $\overline{\mathscr{I}}(\mathfrak{G},L,\overline{\mathbb{A}})$ or of $\overline{\mathscr{I}}(\mathfrak{G}',K,{}_K\overline{\mathbb{A}})$ inside $\overline{\mathscr{I}}(\mathfrak{G}',L,\overline{\mathbb{A}})$ are completely parallel. So the ψ_L-equivariant morphism $\overline{\mathscr{I}}(\psi_L,L,\overline{\mathbb{A}}):\overline{\mathscr{I}}(\mathfrak{G},L,\overline{\mathbb{A}})\to\overline{\mathscr{I}}(\mathfrak{G}',L,\overline{\mathbb{A}})$ of 5.8.1 induces a ψ_K-equivariant morphism $\overline{\mathscr{I}}(\psi_K,K,{}_K\overline{\mathbb{A}}):\overline{\mathscr{I}}(\mathfrak{G},K,{}_K\overline{\mathbb{A}})\to\overline{\mathscr{I}}(\mathfrak{G}',K,{}_K\overline{\mathbb{A}})$.

This is functorial (up to the problem that $\mathbb{A}$ or $\overline{\mathbb{A}}$ has sometimes to change with $\mathfrak{G}'$).

2) Changing the group, Levi factors: Suppose that $\mathfrak{G}$ satisfies (TRQS) and let $M,L,\Gamma,\cdots$ be as in 6.5.

Let F^v_+ and F^v_- be opposite $\Gamma-$stable vectorial facets in $\mathscr{I}^v(\mathfrak{G},L,\mathbb{A}^v)$. They determine completely a subgroup $M(F^v)$ in $\mathfrak{G}(L)$ which is $\Gamma-$stable. We write $\mathfrak{G}'=\mathfrak{G}_{F^v_\pm}$ the corresponding subgroup-scheme of $\mathfrak{G}$. We know that, over L, $\mathfrak{G}'$ is isomorphic to some $\mathfrak{G}_{\mathcal{S}(J)}$.

The parabolic subgroup-scheme $\mathfrak{P}(F^v)$ of $\mathfrak{G}_L$ associated to F^v is defined over K, hence over M, and contains a minimal $M-$parabolic *i.e.* a Borel subgroup defined over M. The parabolics in $\mathfrak{P}(F^v)$ correspond bijectively to the parabolics of its Levi factor $\mathfrak{G}'$ and this correspondence is $\Gamma-$equivariant as $\mathfrak{G}'$ is $\Gamma-$stable. So $\mathfrak{G}'$ is quasi-split over M: it satisfies (TRQS).

If $\mathbb{A}$ is chosen as in 6.6 for $\mathfrak{G}$, then it satisfies the same conditions for $\mathfrak{G}'\subset\mathfrak{G}$. Here also the constructions of the bordered hovels over K inside the bordered hovels over L for $\mathfrak{G}$ and $\mathfrak{G}'$ are completely parallel. We deduce from 5.8.2 a $\mathfrak{G}'(K)-$equivariant isomorphism of $\mathscr{I}(\mathfrak{G}',K,{}_K\mathbb{A})$ with the façade $\mathscr{I}(\mathfrak{G},K,{}_K\overline{\overline{\mathbb{A}}})_{{}^KF^v}$ (where ${}^KF^v$ is the Weyl$-K-$facet corresponding to F^v) or with $\mathscr{I}(\mathfrak{G}',\mathfrak{G},K,{}_K\mathbb{A})=\mathfrak{G}'(K).({}_K\mathbb{A})\subset\mathscr{I}(\mathfrak{G},K,{}_K\mathbb{A})$.

The reader will write the results for bordered hovels analogous to those in 5.8.2.

3) Changing the field: We asked in 6.1 that the valuation $\omega=\omega_K$ of K may be extended functorially to all extensions in $\mathcal{S}ep(K)$. We ask also that the almost split Kac-Moody group $\mathfrak{G}$ satisfies (TRQS), hence is quasi-split over a tamely ramified finite Galois extension M/K and split over a finite Galois extension L/K with $L\supset M$.

Let's consider now a field extension $i:K\hookrightarrow K'$ in $\mathcal{S}ep(K)$. We define in K_s, $L'=K'L$ and $M'=K'M$; we write $i_L:L\hookrightarrow L'$. The extensions L'/K' and L'/M' are Galois, $Gal(L'/K')\subset$

$Gal(L/K)$, $Gal(M'/K')\subset Gal(M/K)$ and M'/K' is tamely ramified. Moreover $\mathfrak{G}$ is split on L' and quasi-split on M', so $\mathfrak{G}$ satisfies (TRQS) on K'.

We saw in 5.8.3 that $\mathbb{A}$ (with some added walls) is still a suitable apartment for $(\mathfrak{G},\mathfrak{T})$ over L' and there is a $\mathfrak{G}(L)-$equivariant embedding $\mathscr{I}(\mathfrak{G},i_L,\mathbb{A}):\mathscr{I}(\mathfrak{G},L,\mathbb{A})\hookrightarrow\mathscr{I}(\mathfrak{G},L',\mathbb{A})$. This embedding is also $Gal(L'/K')-$equivariant. Now $\mathscr{I}(\mathfrak{G},K,{}_K\mathbb{A})\subset\mathscr{I}(\mathfrak{G},L,\mathbb{A})^{Gal(L/K)}$ and $\mathscr{I}(\mathfrak{G},K',{}_{K'}\mathbb{A})\subset\mathscr{I}(\mathfrak{G},L',\mathbb{A})^{Gal(L'/K')}$. Moreover $\mathscr{I}(\mathfrak{G},K,{}_K\mathbb{A})$ is the union of the apartments $\mathbb{A}_{\mathfrak{S}}=\mathscr{I}(\mathfrak{Z}_g(\mathfrak{S}),\mathfrak{G},K,{}_K\mathbb{A})$ for $\mathfrak{S}$ a maximal $K-$split torus in $\mathfrak{G}$ and $\mathfrak{Z}_g(\mathfrak{S})$ its generic centralizer, which is a reductive group. By 2) above and [Ro77, 5.12], $\mathscr{I}(\mathfrak{G},i_L,\mathbb{A})(\mathbb{A}_{\mathfrak{S}})=\mathscr{I}(\mathfrak{Z}_g(\mathfrak{S}),i_L,\mathbb{A})(\mathscr{I}(\mathfrak{Z}_g(\mathfrak{S}),\mathfrak{G},K,{}_K\mathbb{A})\subset\mathscr{I}(\mathfrak{Z}_g(\mathfrak{S}),\mathfrak{G},K',{}_{K'}\mathbb{A})\subset\mathscr{I}(\mathfrak{G},K',{}_{K'}\mathbb{A})$. We have thus defined a $\mathfrak{G}(K)-$equivariant embedding $\mathscr{I}(\mathfrak{G},i,\mathbb{A}):\mathscr{I}(\mathfrak{G},K,{}_K\mathbb{A})\hookrightarrow\mathscr{I}(\mathfrak{G},K',{}_{K'}\mathbb{A})$.

This is clearly functorial. We leave to the reader the "pleasure" to formulate a result for bordered hovels; there is the problem of the choice of the façade of ${}_{K'}\mathbb{A}$ in which embeds a façade of ${}_K\mathbb{A}$. This is easier for the essential spherical façades *i.e.* for the microaffine buildings.

4) Changing the model apartment: Suppose that $\mathfrak{G}$ satisfies (TRQS) and let $M,L,\Gamma,\cdots$ be as in 6.5.

The apartment $\mathbb{A}$ is associated to a commutative extension $\varphi:\mathcal{S}\to\mathcal{S}'$ of RGS and a subspace V'_{00} of $V'_0\subset V'=Y'\otimes\mathbb{R}$ with the condition that $\mathcal{S}'$ is endowed with a star action of Γ for which φ is Γ^*-equivariant and V'_{00} Γ^*-stable.

Now let $\psi:\mathcal{S}'\to\mathcal{S}''$ be a commutative extension of RGS and V''_{00} a subspace of $V''_0\subset V''=Y''\otimes\mathbb{R}$ containing $\psi(V'_{00})$, with the condition that $\mathcal{S}''$ is endowed with a star action of Γ for which ψ is Γ^*-equivariant and V''_{00} Γ^*-stable. Then $\varphi'=\psi\circ\varphi:\mathcal{S}\to\mathcal{S}''$ satisfies the above condition and can be used to define a new apartment $\mathbb{A}'=V''/V''_{00}$. We have an affine map $\psi:\mathbb{A}\to\mathbb{A}'$ which is $N-$equivariant.

By 5.8.4 we get a $\mathfrak{G}(L)-$equivariant map $\mathscr{I}(\mathfrak{G},L,\psi):\mathscr{I}(\mathfrak{G},L,\mathbb{A})\to\mathscr{I}(\mathfrak{G},L,\mathbb{A}')$. It is $\Gamma-$equivariant and induces a $\mathfrak{G}(K)-$equivariant map $\mathscr{I}(\mathfrak{G},K,\psi):\mathscr{I}(\mathfrak{G},K,{}_K\mathbb{A})\to\mathscr{I}(\mathfrak{G},K,{}_K\mathbb{A}')$ (by the characterization given in remark 6.9).

This construction is functorial and extends clearly to the bordered hovels.

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