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Automated Reasoning in Differential Geometry and Mechanics Using Characteristic Method[1]

IV. Bertrand Curves

Shang-Ching Chou
Department of Computer Science
The Wichita State University
Wichita, KS 67208, USA

Xiao-Shan Gao
Institute of Systems Science
Academia Sinica
Beijing 100080, P.R. China

Abstract This paper reports the study of properties of the curve pairs of the Bertrand types using our automated reasoning program based on Wu's method of mechanical theorem proving for differential geometry. A complete list of results about Bertrand curves in metric and affine spaces is derived mechanically. The list includes most of the known results of various Bertrand curves. We also derive some new results about Bertrand curves.

Keywords Mechanical theorem proving, metric differential geometry, affine differential geometry, Bertrand curves.

Abbreviated title: Automated Reasoning in Differential Geometry

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1. Introduction to the Problem

This paper reports results about various Bertrand curves obtained using a computer program based on an improved version of Ritt-Wu's zero decomposition algorithm [2]. We adopt two approaches to the problems. First, we use Formulation II to prove known results under some explicitly given non-degenerate conditions [3]. Second, we derive "unknown" relations among certain variables using Ritt-Wu's characteristic method [4]. In this way, we have proved or derived most of the known results for various Bertrand curves mechanically. We have also derived some results which we have not found in textbooks of differential geometry or relevant papers.

The Bertrand curves problem is first studied using a computer by Wu in [6]. This paper is a further study of the same problem, but contains more results than those of Wu's: totally 18 types of Bertrand curves in metric and affine differential geometries are studied and a complete list of results are given. Also our study here follows a different approach: we use the complete decomposition algorithm to derive or prove certain results under some explicitly given conditions. Also, the proving procedure for the known or derived results is automatically carried out by our program without any human assistance.

Theorems on various Bertrand curves are among the most eminent results in the local theory of space curves. The success of our method in dealing with these problems shows that our program based on the Ritt-Wu's decomposition algorithm can be used to solve quite difficult problems in elementary differential geometry, or even to discover new results.

A pair of space curves having their principal normals in common are said to be associate Bertrand curves [1]. Here following Wu [6], we shall consider more general problems. Given two space curves C_1 and C_2 in a one to one correspondence, let us attach *moving triads* $(C_1, e_{11}, e_{12}, e_{13})$ and $(C_2, e_{21}, e_{22}, e_{23})$ to C_1 and C_2 at the corresponding points of C_1 and C_2 respectively. We denote the arcs, curvatures and torsions of C_1 and C_2 by s_1, k_1, t_1 and s_2, k_2, t_2 respectively. Then all the quantities introduced above can be looked as functions of s_1. Let $r = \frac{ds_2}{ds_1}$, and let

$$C_2 = C_1 + a_1 E_{11} + a_2 E_{12} + a_3 E_{13} \tag{1.1}$$

$$\begin{aligned} e_{21} &= u_{11}e_{11} + u_{12}e_{12} + u_{13}e_{13} \\ e_{22} &= u_{21}e_{11} + u_{22}e_{12} + u_{23}e_{13} \\ e_{23} &= u_{31}e_{11} + u_{32}e_{12} + u_{33}e_{13} \end{aligned} \tag{1.2}$$

where a_i are variables and (u_{ij}) is a matrix of variables satisfying certain relations which will be given in the following sections.

Roughly speaking, the problem is to find under what conditions for the curve pairs (C_1 and C_2) their moving triads will satisfy some given relations. For example, the original Bertrand curve problem is to ask under what conditions C_1 and C_2 will have identical principal normals at the corresponding points, i.e. $e_{22} = e_{12}$ at the corresponding points.

In this paper, we mainly consider the following three groups of problems.

MI_{ij} $(1 \le i \le j \le 3)$ means that e_{2j} is identical with e_{1i} in metric differential geometry.

MP_{ij} $(1 \le i \le j \le 3)$ means that e_{2j} is parallel to e_{1i} in metric differential geometry.

AI_{ij} $(1 \le i \le j \le 3)$ means that e_{2j} has the same direction with e_{1i} in affine differential geometry.

So totally 18 kinds of Bertrand curves are studied.

In this paper, we assume the reader has already known the Ritt-Wu's decomposition algorithm and Wu's method of mechanical theorem proving in the differential case. A detailed description of the algorithm can be found in [7] or [2, 3].

2. Bertrand Curves In Metric Space

Let (e_{11}, e_{12}, e_{13}) and (e_{21}, e_{22}, e_{23}) be the Frenet triads of C_1 and C_2 at their corresponding points respectively, then we have the following Frenet formulae.

$$e'_{11} = k_1 e_{12},\ e'_{12} = -k_1 e_{11} + t_1 e_{12},\ e'_{13} = -t_1 e_{12} \tag{2.1}$$
$$e'_{21} = rk_2 e_{22},\ e'_{22} = -rk_2 e_{21} + rt_2 e_{22},\ e'_{23} = -rt_2 e_{22} \tag{2.2}$$

where $r = \frac{ds_2}{ds_1}$ and the differentiations here and in what follows are all with respect to (abbreviated to wrpt) s_1.

Differentiating (1.1) and (1.2); eliminating $e'_{11}, e'_{12}, e'_{13}, e'_{21}, e'_{22}$ and e'_{23} using (2.1) and (2.2); eliminating e_{21}, e_{22}, and e_{23} using (1.2); at last, comparing coefficients for the vectors e_{11}, e_{12}, and e_{13}, we have:

$$\begin{array}{l}
a_2 t_1 - r u_{13} + a'_3 = 0 \\
a_3 t_1 - a_1 k_1 + r u_{12} - a'_2 = 0 \\
a_2 k_1 + r u_{11} - a'_1 - 1 = 0 \\
r u_{23} k_2 - u_{12} t_1 - u'_{13} = 0 \\
r u_{22} k_2 + u_{13} t_1 - u_{11} k_1 - u'_{12} = 0 \\
r u_{21} k_2 + u_{12} k_1 - u'_{11} = 0 \\
r u_{33} t_2 - r u_{13} k_2 - u_{22} t_1 - u'_{23} = 0 \\
r u_{32} t_2 - r u_{12} k_2 + u_{23} t_1 - u_{21} k_1 - u'_{22} = 0 \\
r u_{31} t_2 - r u_{11} k_2 + u_{22} k_1 - u'_{21} = 0 \\
r u_{23} t_2 + u_{32} t_1 + u'_{33} = 0 \\
r u_{22} t_2 - u_{33} t_1 + u_{31} k_1 + u'_{32} = 0 \\
r u_{21} t_2 - u_{32} k_1 + u'_{31} = 0
\end{array} \tag{2.3}$$

To transform a right-handed orthogonal system $\{e_{11}, e_{12}, e_{13}\}$ to another right-handed orthogonal system $\{e_{21}, e_{22}, e_{23}\}$, (u_{ij}) must satisfy

$$\begin{array}{l}
u_{13}^2 + u_{12}^2 + u_{11}^2 - 1 = 0 \\
u_{23}^2 + u_{22}^2 + u_{21}^2 - 1 = 0 \\
u_{33}^2 + u_{32}^2 + u_{31}^2 - 1 = 0 \\
u_{13} u_{23} + u_{12} u_{22} + u_{11} u_{21} = 0 \\
u_{13} u_{33} + u_{12} u_{32} + u_{11} u_{31} = 0 \\
u_{23} u_{33} + u_{22} u_{32} + u_{21} u_{31} = 0 \\
(u_{11} u_{22} - u_{12} u_{21}) u_{33} + (-u_{11} u_{23} + u_{13} u_{21}) u_{32} + (u_{12} u_{23} - u_{13} u_{22}) u_{31} - 1 = 0
\end{array} \tag{2.4}$$

(2.3) and (2.4) are first given by Wu in [6] except the last equation in (2.4) which is added by us to preserve the right-handness of the moving triads.

2.1. The Identical Case

At case MI_{ij}, the a_i and $u_{i,j}$ must satisfy

$$a_m = 0 \text{ for } m \neq i; u_{ji} - 1 = 0;\ u_{jn} = 0 \text{ for } n \neq i;\ u_{ki} = 0 \text{ and } k \neq j. \tag{2.5}$$

The following non-degenerate conditions are often used: $k_1 \neq 0$ means curve C_1 is not a straight line. $k_2 \neq 0$ means curve C_2 is not a straight line. $r \neq 0$ means the arc length of C_2 as a function of the arc length of C_1 is not a constant, i.e., C_2 is not a fixed point. At first, we list some known results.

Case MI_{11}**.** Under the non-degenerate condition $rk_1k_2 \neq 0$, C_1 and C_2 must be identical, i.e. $C_1 = C_2$.

Case MI_{12}**.** Under condition $r \neq 0$, C_2 and C_1 are both plane curves satisfying

$$\begin{array}{ll} e_{21} = -e_{12}, e_{22} = e_{11}, e_{23} = e_{13}; & k_1 = -r/a_1, k_2 = -1/a_1, a_1' = -1 \text{ or} \\ e_{21} = e_{12}, e_{22} = e_{11}, e_{23} = -e_{13}; & k_1 = r/a_1, k_2 = -1/a_1, a_1' = -1. \end{array}$$

The geometric meaning of the above results can be stated as follows.

If C_2 is the involute of C_1 in the strong sense that the principal normals of C_2 are identical with the tangent vectors of C_1, then both curves must be plane curves, and

(i) $C_2 = C_1 + (c_0 - s_1)e_{11}$ for a constsnt c_0;

(ii) $C_1 = C_2 + \frac{1}{k_2}e_{22}$, i.e C_1 is the locus of the curvature center of C_2;

(iii) The arc length of C_1 between two points of C_1 equals the difference of the reciprocal of the curvatures of C_2 at the corresponding points.

Case MI_{13}**.** There exist no curves satisfying $e_{11} = e_{23}$ under the condition $r \neq 0$.

Case MI_{22}**.** Under the non-degenerate condition $ra_2 \neq 0$ ($C_2 \neq C_2$), we have

a. The distance from C_1 to C_2 is a constant.

b. The angle formed by the tangent lines at C_1 and C_2 respectively is a constant.

c. (Bertrand) There exists a linear relation between k_1 and t_1 with constant coefficients.

d. (Schell) The production of t_1 and t_2 is a constant.

Case MI_{23}**.** Under the non-degenerate condition $rk_1 \neq 0$, we have

a. The distance from C_1 to C_2 is a constant.

b. (Mannheim) $k_1^2 + t_1^2 = ck_1$ for a constant c.

Case MI_{33}**.** Under the non-degenerate condition $rk_1k_2 \neq 0$, we have either $C_2 = C_1$ or C_2 and C_1 are on two parallel planes respectively and C_2 is the translation of C_1 along the binormal of C_1.

Take MI_{22}, the original case of Bertrand, as an example. Other cases can be proved similarly. Using Ritt-Wu's decomposition algorithm under the following variable order $r < a_1 < a_2 < a_3 < u_{11} < u_{12} < u_{13} < u_{21} < u_{22} < u_{23} < u_{31} < u_{32} < u_{33} < k_1 < t_1 < k_2 < t_2$, we have $Zero((2.3) \cup (2.4) \cup (2.5)/ra_2) = \cup_{i=1}^{3} Zero(PD(ASC_i))$ where

$ASC_1 =$	$ASC_2 =$	$ASC_3 =$
a_1	a_1	a_1
a_2'	a_2'	a_2'
a_3	a_3	a_3
$u_{11} - 1$	$u_{11} + 1$	u_{11}'
u_{12}	u_{12}	u_{12}
u_{13}	u_{13}	$u_{13}^2 + u_{11}^2 - 1$
u_{21}	u_{21}	u_{21}

$u_{22}-1$	$u_{22}-1$	$u_{22}-1$
u_{23}	u_{23}	u_{23}
u_{31}	u_{31}	$u_{31}+u_{13}$
u_{32}	u_{32}	u_{32}
$u_{33}-1$	$u_{33}+1$	$u_{33}-u_{11}$
a_2k_1+r-1	a_2k_1-r-1	$a_2k_1+ru_{11}-1$
t_1	t_1	$a_2t_1-ru_{13}$
ra_2k_2+r-1	ra_2k_2+r+1	$ra_2k_2-u_{11}+r$
t_2	t_2	$ra_2t_2-u_{13}$

The four conclusions of MI_{22} are equivalent to $c_1 = a_2' = 0$, $c_2 = u_{11}' = 0$, $c_3 = k_1''t_1' - t_1''k_1' = 0$, and $c_4 = (t_1t_2)' = 0$ respectively. The pseudo remainders of c_i, $i = 1, ..., 4$, wrpt ASC_i, $i = 1, 2, 3$, are zero which implies the results are correct.

On the other hand, we can obtain the results from ASC_3, the main component of the problem (see [3]) directly. The differential equations representing results a ($a_2' = 0$) and b ($u_{11}' = 0$) are already in ASC_3. Eliminating r from the last four equations of ASC_3, we have:

$$\begin{aligned}&a_2u_{11}t_1 + a_2u_{13}k_1 - u_{13} = 0\\&a_2^2t_1t_2 - u_{13}^2 = 0\\&a_2^2t_1k_2 + a_2t_1 - u_{11}u_{13} = 0\end{aligned} \tag{2.6}$$

As a_2, u_{11}, (and hence $u_{13} = \sqrt{1-u_{11}^2}$) are constants, the first two formulae of (2.6) actually give the concrete expression for Bertrand's theorem and Schell's theorem. From (2.6) we can derive the following formulae.

$$\begin{aligned}&(1-a_2k_1)(1+a_2k_2) - u_{11}^2 = 0\\&a_2^2k_1t_2 - a_2t_2 + u_{11}u_{13} = 0\\&a_2u_{11}t_2 - a_2u_{13}k_2 - u_{13} = 0\end{aligned}$$

The above formulae are proved (mechanically) correct under condition $k_1k_2r \neq 0$.

For MI_{23}, we can find the following formulae among k_1, t_1, and t_2 similarly

$$\begin{aligned}&a_2t_1^2 + a_2k_1^2 - k_1 = 0\\&a_2^2t_1t_2^2 - t_2 + t_1 = 0\\&a_2t_1t_2 - k_1 = 0\\&k_1^2 + t_1^2 - t_1t_2 = 0\\&(a_2^2k_1 - a_2)t_2^2 + k_1 = 0\end{aligned}$$

where a_2 is a constant. For r, we have

$r = u_{11} = \sqrt{t_1/t_2} = \sqrt{t_1^2/(t_1^2+k_1^2)}$.

All the above results are true under the nondegenerate condition $k_1k_2r \neq 0$.

2.2. The Parallel Case

For MP_{ij}, the u_{ij} must satisfy $u_{jk} = 0$ for $k \neq i$; $u_{mi} = 0$ for $m \neq j$. The following results can be derived automatically under condition $k_1k_2r \neq 0$ similar to Section 2.1.

Case MP_{11}**.** C_2 and C_1 must satisfy one of the following conditions.

$$
\begin{array}{ll}
a. & e_{21}=-e_{11}, e_{22}=e_{12}, e_{23}=-e_{13} \\
 & r=-k_1/k_2=-t_1/t_2 \text{ and } a_3a_3'+a_2a_2'+a_1a_1'+(r+1)a_1=0. \\
b. & e_{21}=-e_{11}, e_{22}=-e_{12}, e_{23}=e_{13} \\
 & r=k_1/k_2=-t_1/t_2 \text{ and } a_3a_3'+a_2a_2'+a_1a_1'+(r+1)a_1=0. \\
c. & e_{21}=e_{11}, e_{22}=-e_{12}, e_{23}=-e_{13} \\
 & r=-k_1/k_2=t_1/t_2 \text{ and } a_3a_3'+a_2a_2'+a_1a_1'+(-r+1)a_1=0. \\
d. & e_{21}=e_{11}, e_{22}=e_{12}, e_{23}=e_{13} \\
 & r=k_1/k_2=t_1/t_2 \text{ and } a_3a_3'+a_2a_2'+a_1a_1'+(-r+1)a_1=0.
\end{array}
$$

Case MP_{12}**.** C_2 and C_1 must satisfy one of the following conditions.

$$
\begin{array}{ll}
a. & u_{21}-1=0, r=\sqrt{\frac{k_1^2}{t_2^2+k_2^2}}=-\frac{k_1}{k_2u_{12}}=\frac{k_1}{t_2u_{13}}. \\
b. & u_{21}+1=0, r=\sqrt{\frac{k_1^2}{t_2^2+k_2^2}}=\frac{k_1}{k_2u_{12}}=\frac{k_1}{t_2u_{13}}.
\end{array}
$$

Case MP_{13}**.** C_2 and C_1 must satisfy one of the following conditions.

$$
\begin{array}{ll}
a. & e_{21}=-e_{13}, e_{22}=-e_{12}, e_{23}=-e_{11} \text{ and } r=-t_1/k_2=-k_1/t_2. \\
b. & e_{21}=e_{13}, e_{22}=e_{12}, e_{23}=-e_{11}, \text{ and } r=-t_1/k_2=k_1/t_2. \\
c. & e_{21}=-e_{13}, e_{22}=e_{12}, e_{23}=e_{11}, \text{ and } r=t_1/k_2=-k_1/t_2. \\
d. & e_{21}=e_{13}, e_{22}=-e_{12}, e_{23}=e_{11} \text{ and } r=t_1/k_2=k_1/t_2.
\end{array}
$$

Case MP_{22}**.** C_2 and C_1 must satisfy one of the following conditions.

$$
\begin{array}{ll}
a. & u_{22}-1=0,\ u_{11}'=0,\ u_{13}'=0. \\
 & u_{13}(t_1t_2+k_1k_2)=u_{11}(k_1t_2-t_1k_2). \\
 & r^2=\frac{t_1^2+k_1^2}{t_2^2+k_2^2}. \\
b. & u_{22}+1=0,\ u_{11}'=0,\ u_{13}'=0. \\
 & u_{13}(t_1t_2-k_1k_2)=u_{11}(k_1t_2+t_1k_2). \\
 & r^2=\frac{t_1^2+k_1^2}{t_2^2+k_2^2}.
\end{array}
$$

The second equation implies that t_1, k_1, k_2, and t_2 satisfy a quadratic equation with constant coefficients. This formula cannot be found in textbooks of differential geometry.

Case MP_{23}. We have $r=\frac{t_1}{t_2u_{11}}=\frac{k_1}{t_2u_{13}}=\sqrt{\frac{t_1^2+k_1^2}{t_2^2}}$.

Case MP_{33}**.** We have the same results as MP_{11}.

3. Bertrand Curves in Affine Space

In affine differential geometry, let $e_{11}=\frac{dC_1}{ds_1}, e_{12}=\frac{de_{11}}{ds_1}, e_{13}=\frac{de_{12}}{ds_1}$ and $e_{21}=\frac{dC_2}{ds_2}, e_{22}=\frac{de_{21}}{ds_2}, e_{23}=\frac{de_{22}}{ds_2}$ be the moving triads of C_1 and C_2 at their corresponding points respectively, where s_i are the arc length of curves C_i for $i=1,2$. Then we have the following Frenet formulae.

$$
\begin{array}{l}
e_{11}'=e_{12},\ e_{12}'=e_{13},\ e_{13}'=-k_1e_{12}+t_1e_{11}; \\
e_{21}'=re_{22},\ e_{22}'=re_{23},\ e_{23}'=-rk_2e_{22}+rt_2e_{21}.
\end{array}
$$

where $r = \frac{ds_2}{ds_1}$. Similar to section 2, we can get the following differential equations.

$$
\begin{aligned}
& ru_{13} - a_3' - a_2 = 0 \\
& a_3 k_1 + r u_{12} - a_2' - a_1 = 0 \\
& a_3 t_1 + r u_{11} - a_1' - 1 = 0 \\
& r u_{23} - u_{13}' - u_{12} = 0 \\
& u_{13} k_1 + r u_{22} - u_{12}' - u_{11} = 0 \\
& u_{13} t_1 + r u_{21} - u_{11}' = 0 \\
& r u_{33} - u_{23}' - u_{22} = 0 \\
& u_{23} k_1 + r u_{32} - u_{22}' - u_{21} = 0 \\
& u_{23} t_1 + r u_{31} - u_{21}' = 0 \\
& r u_{13} t_2 + r u_{23} k_2 + u_{33}' + u_{32} = 0 \\
& r u_{12} t_2 + r u_{22} k_2 - u_{33} k_1 + u_{32}' + u_{31} = 0 \\
& r u_{11} t_2 + r u_{21} k_2 - u_{33} t_1 + u_{31}' = 0
\end{aligned}
$$

We also have $(e_{11}, e_{12}, e_{13}) = 1$ and $(e_{21}, e_{22}, e_{23}) = 1$. Then by (1.2), the determinant of the matrix (u_{ij}) is 1, i.e.,

$$(u_{11}u_{22} - u_{12}u_{21})u_{33} + (-u_{11}u_{23} + u_{13}u_{21})u_{32} + (u_{12}u_{23} - u_{13}u_{22})u_{31} - 1 = 0.$$

Let AI_{ij} be the case such that e_{2j} has the same direction[2]*In affine space, there is no identical case as in Section 2.1.* as e_{1i} at the corresponding points. Then for case AI_{ij}, we have:

$$a_k = 0 \text{ for } k \neq i; \quad u_{jm} = 0 \text{ for } m \neq i.$$

Similar to section 2.1, we have derived the following results mechanically.

Case AI_{11}. C_2 and C_1 are identical under the non-degenerate condition $r \neq 0$.

Case AI_{12}. There exist no curves such that e_{11} has the same direction as e_{22}.

Case AI_{13}. Under condition $r \neq 0$, we have $k_2 = 0$ and $t_2 = -r^3/a_1^3 = -u_{31}/a_1$.

Case AI_{22}. Under the non-degenerate condition $ra_2 \neq 0$, we have two conditions:

$$
\begin{array}{ll}
r^2 a_2 k_2 + a_2 k_1 + 2r^3 - 2 = 0 & r^2 a_2 k_2 + a_2 k_1 - 2r^3 - 2 = 0 \\
t_2 = t_1 = -\frac{r'}{ra_2} = -\frac{2a_2'}{a_2^2} & t_2 = -t_1 = \frac{r'}{ra_2} = \frac{2a_2'}{a_2^2} \\
k_1 + k_2 = \frac{2-2r^3}{a_2} & k_1 + k_2 = \frac{2+2r^3}{a_2}
\end{array}
$$

For k_1 and t_1, we obtain an algebraic equation for $k_1, k_1', k_1''.t_1, t_1', t_1''$, and t_1''' of 55 terms. We have $r = ca_2^2$ for a constant c. For the transformation matrix, we have $u_{11} = \frac{1}{r}$, $u_{12} = \frac{a_2'}{r}$, $u_{13} = \frac{a_2}{r}$, $u_{22} = \pm r$, $u_{31} = 0$, $u_{32} = \pm r'$, $u_{33} = \pm 1$. Our mechanically obtained results here are more complete than that in [5].

Case AI_{23}. We obtain the following equations for k_1, t_1, k_2, and t_2 under condition $rk_1k_2 \neq 0$.

$$
\begin{aligned}
& r^2 a_2 k_1 - 3r^2 a_2'' + 6rr' a_2' + (rr'' - 3r'^2) a_2 - r^2 = 0 \\
& a_2^2 u_{32} t_1 + 2a_2' u_{32} + r^3 = 0 \\
& r^2 k_2 - u_{32}^2 = 0 \\
& r^4 a_2 t_2 + (2ra_2' - r' a_2) u_{32}^2 + r^4 u_{32} = 0
\end{aligned}
$$

Case AI_{33}. We have $k_1/k_2 = r^2/u_{33}^2$ under the non-degenerate condition $ra_3 \neq 0$.

[2]*

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