Random Conley-Markov Connection Matrix for Gradient Flows

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joint work with Shui-Nee Chow and Hao-Min Zhou

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- R. Bott : to the critical submanifolds with nondegeneracy by applying 'normal bundles' trick in differential topology. The famous Bott perodicity.
- Conley's extension : invariant isolated set and index pair and pointed space are vital to many applications for Dynamical sysytems without differential topology advantages of compact metric space X.

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- Even it is scarce in the sense of the perfect Morse functions. It is called Arnold Conjecture for symplectic closed manifolds with Hamiltonian functions.

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- Results : The Conley-Markov connection matrix we study, would provide a solution.

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- Attractor–Repeller pairs (A_i, R_i) :

$$\emptyset = A_0 \subset A_1 \subset \cdots A_n = X = R_0 \supset R_1 \cdots \supset R_n = \emptyset.$$

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- Morse set : $M_i = A_i \cap R_{i-1}, 1 \le i \le n$.
- $(M_i)_{1\le i\le n}$ is called a Morse decomposition of X. (Morse theory from the dyanmical system point of view).

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- Stratification structure :
  for any $x \in X$, $\omega(x), \alpha(x) \in \cup_{j=1}^{n} M_j$: if $x_1, \cdots, x_p, \alpha(x_k) \subset M_{j_{k-1}}, \omega(x_k) \subset M_{j_k}, 1 \leq k \leq p$, then $j_0 \leq j_p$.

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- $\phi(t)N_0 \subset N_1$ for $t \in \mathbb{R}^+$
- $x \in N_1, x \cdot \mathbb{R}^+$ does not lie in N_1, then there exists $T \geq 0$ such that $\phi[0, T] \cdot x \subset N_1$ and $\phi(T) \cdot x \in N_0$.

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- $\overline{0}$ be the pointed one-point space. If $X \coprod Y/\{x_0, y_0\} = [X, x_0] \vee [Y_0, y_0] = \overline{0}$, then $[X, x_0] = [Y, y_0] = \overline{0}$.

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- $[X/A] \wedge [Y/B] = [X \times Y/(X \times B \cup A \times Y)]$.

Generalized Morse inequality

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- Non-degenerate critical point $z \in Z$,

$$h(\{z\}) = [\dot{S}^m], \quad m = \frac{1}{2} \dim Z - j$$

for the Hamiltonian dynamical system in symplectic manifold Z with the Morse index j. Here m is the Maslov index.

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- $$\sum_{j=1}^{n} p(t, h(M_j)) = p(t, h(S)) + (1+t)Q(t)$$

  See Conley Isolated Invariant sets and the Morse index, AMS (1978), vol.38 ;
  and Conley, Zehnder, Communication in Pure and Applied Math., 37 (1984), 207–253.

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- Cohomology of the homotopy index can be adjusted to different cohomologies.

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- $CH_*(S;\mathbf{R}) = H_*(h(S);\mathbf{R})$ is the homological version of Conley index as a more computable object than the homotopy index.

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- $\Delta : \oplus_{i\in P} CH(M(i);\mathbf{R}) \to \oplus_{i\in P} CH(M(i);\mathbf{R})$ be a linear map such that $\Delta = (\Delta(i,j))_{i,j\in P}$ as a matrix maps, where $\Delta(i,j) : CH_*(M(i);\mathbf{R}) \to CH(M(j);\mathbf{R})$. Similarly, we have $\Delta(I) = (\Delta(i,j))_{i,j\in I}$.

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- Franzosa proved : There exists a Conley connection matrix for a Morse decomposition.

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- Reineck showed : Suppose a Morse-Smale flow has no periodic orbits. Each nonzero map in the connection matrix is flow defined, and the Conley connection matrix is unique.
- Franzosa later relaxed Reineck's result for periodic orbits with doubly connected rest points, and constructed a non-unique Conley connection matrix at the bifurcation point.

Motivations and results

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- A Morse function is called excellent if differential critical points correspond to different critical values. Any Morse function can be arbitrarily approximated in the C^2-topology by excellent Morse functions.
- But the construction from this theoretical result cannot provide any useful way to find the global minimum or maximum.

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- What the small white noise can do to the Dynamic Systems

DRS

- A measurable $(C^k, k = 0, \cdots, \infty, C^\omega)$ random dynamical system on the measurable (C^k) space (C^k-manifold) $(X, \mathcal{B})$ over a metric dynamic system $(\Omega, \mathcal{F}, \mathbf{P}, (\theta(t))_{t\in\mathbb{T}})$ with time $\mathbb{T}$ is a mapping

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- (iii) Cocycle Condition : $\phi(0, \omega) = id_X$ for all $\omega \in \Omega$ and $\phi(t + s, \omega) = \phi(t, \theta(s)\omega) \circ \phi(s, \omega)$ for all $s, t \in \mathbb{T}, \omega \in \Omega$.

White Noise on Gradient-like flow

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- The random perturbation of the gradient-like flow is

$$\dot{x} = -X(x) + \varepsilon Id_{n\times n}\xi(t,\omega),$$

where $x = \pi(t,\omega)$. Its stochastic differential integral equation is given by

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- There exists a unique solution of the stochastic differential equation with an initial condition $\pi(0,\omega) = x_0(\omega)$, where $x_0(\omega)$ is an M-valued measurable function of $(W_s(\omega), s \leq t)$.

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- a simple Markov process on M : for every bounded measurable function h and all $s, t \in [0, T]$ with $s \leq t$

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for P-a.e. $\omega \in \Omega(M)$,

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- and the transition probability law $p(0, x_0, t, x_t)$ (the conditional probability law of $\pi(t, \omega)$ with $\pi(0, \omega) = x_0$) is same with the probability law of the solution $x(t) = \pi(t, \omega)$ of the Fokker-Planck equation with initial condition $x(0) = x_0$.

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- Let $S^1 = \mathbb{R}/(2\pi\mathbb{Z})$ be a compact 1-dimensional circle with a Morse function $f(x) = \cos(2x)$. Hence the random perturbation of the gradient flow is given by

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where ξ is the 1-dimensional Brownian motion and $\omega \in \Omega(S^1)$.

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- For every path of $\omega \in \Omega(S^1)$, the rest point of the random dynamic system can be any point in S^1 since the Brownian motion take any finite value at finite time with probability 1. Hence almost surely $D : \omega \mapsto D(\omega) = S^1$.

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- The random invariant set moves along the path almost surely to cover the whole manifold S^1, Similarly for T^n.

Transition matrix

- For each t and $\omega \in \Omega(M)$, we define the Conley connection matrix as

  $$\Delta_A(t,\omega) : \oplus_{\alpha\in A} CH_*(S_\alpha(t,\omega);\mathbb{R}) \to \oplus_{\alpha\in A} CH_*(S_\alpha(t,\omega);\mathbb{R})$$

  a linear map such that $\Delta_A(t,\omega) = (\Delta(\alpha,\beta)(t,\omega))_{\alpha,\beta\in A}$, where
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  $\Delta(\alpha,\beta)(t,\omega) : CH_*(S_\alpha(t,\omega);\mathbb{R}) \to CH_*(S_\beta(t,\omega);\mathbb{R})$.
- For each isolated invariant compact subset $S_\alpha(t,\omega)$, there is a well-defined $\Delta(\alpha,\alpha)(t,\omega)$ of degree zero.

Markov Property

- For the gradient-like flow with partial ordering $(P, <)$, Let Δ be the connection matrix with respect to the ordering. If $\Delta(\alpha, \beta)$ is of degree -1 for $\alpha \neq \beta \in P$, then $(\Delta - D) \circ (\Delta - D) = 0$, i.e., $\Delta \circ \Delta + D^2 = \Delta \circ D + D \circ \Delta$, where $D = (\Delta(\alpha, \alpha))$ is the diagonal matrix of Δ.

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- For each $\omega \in \Omega(M)$ with $S_\alpha(t, \omega)$ an isolated invariant point for all $\alpha \in A$, then the probability $p_{\alpha\beta}(t, \omega) = p(-\infty, S_\alpha(t, \omega); +\infty, S_\beta(t, \omega))$ form a transition matrix $P(t, \omega) = (p_{\alpha\beta}(t, \omega))_{\alpha, \beta \in A}$ which is Markov of the finite state A.

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  where $\Delta_A(t,\omega) = (\Delta(\alpha,\beta)(t,\omega))_{\alpha,\beta\in A}$ and $p_A(t,\omega) = (p_{\alpha\beta}(t,\omega))_{\alpha,\beta\in A}$.
- For any compact closed manifold M and a gradient-like vector field X, there exists a well-defined Conley-Markov connection matrix for invariant subsets.

Conley-Markov connection matrix

- For $\varepsilon \neq 0$ and $0 \leq s \leq t \leq T$, there exists a transition matrix $T_{st}(\omega)$ such that $T_{st}(\omega)\Delta(s,\omega) = \Delta(t,\omega)$.

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- The limit of $CM(t,\omega)$ for $\varepsilon \to 0$, exists, but not necessary equal to the classical Conley Connection matrix.

Examples

- $dx = -\nabla f(x)dt + \varepsilon dW_t$ with $f(x) = -\alpha(x-1)^2 + x^4/10$, critial points are $x_1 = -5.4, x_2 = 1, x_3 = 4.4.$

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$$CM^{\varepsilon=6}(t,\omega) = \begin{bmatrix} 0.8139 & 0.0000 & 0.0000 \\ 0.7452 & 0.0000 & 0.0556 \\ 0.0012 & 0.0001 & 0.6587 \end{bmatrix}$$

$$CM^{\varepsilon=0.75}(t,\omega) = \begin{bmatrix} 0.9999 & 0.0000 & 0.0000 \\ 0.9996 & 0.0000 & 0.0003 \\ 0.0000 & 0.0000 & 0.9998 \end{bmatrix}$$

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- Conley Connection matrix for finding the global information ; the transition-probability for determining the possible local information.
- If $\Delta(\alpha, \beta)(t, \omega) = 0$, there are still possible trajectory flow linesdue to $p_{\alpha\beta}(t, \omega) > 0$. Detecting possible gradient flows which cannot be detected from the Conley homological connection matrix.
- Even with $p_{\alpha\beta}(t, \omega) = 0$, T_{st} show the strength for detecting the bifurcation with strong possibility.