Indifference Pricing of American Option Underlying Illiquid Stock under Exponential Forward Performance

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Abstract

This work focuses on the indifference pricing of American call option underlying a non-traded stock, which may be partially hedgeable by another traded stock. Under the exponential forward measure, the indifference price is formulated as a stochastic singular control problem. The value function is characterized as the unique solution of a partial differential equation in a Sobolev space. Together with some regularities and estimates of the value function, the existence of the optimal strategy is also obtained. The applications of the characterization result includes a derivation of a dual representation and the indifference pricing on employee stock option. As a byproduct, a generalized Itô's formula is obtained for functions in a Sobolev space.

Keyword: Stochastic control, generalized verification theorem, portfolio optimization, generalized Itô's formula, indifference pricing, exponential forward performance.

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1 Introduction

Recent developments of indifference pricing, particularly the valuation of American options under forward performance measures, have attracted much attention; see [10] and the references therein. One of many applications of the indifference pricing is the valuation of Employee Stock Option (ESO); see [12, 11]. An ESO is an American type of call option on the common stock of a company, which is not available for trade in the market. However, there may be some other partially correlated stocks (ex. Index) available in the market, which may be used for the purpose of partial hedge of ESO. Such a problem can be formulated as a stochastic singular control problem [14] with optimal stopping. Related study also appears in empirical research [2, 7] for the study of early exercise behavior.

In this paper, we study the indifference pricing of the American call options underlying a non-traded common stock by assuming that there exists another perfectly liquid stock available in the market, which may be used for the purpose of partial hedge. The indifference pricing involved is associated to a portfolio optimization problem, which is in fact a stochastic control problem with optimal stopping involved. Using the classical verification theorem [18], the value function may be characterized as the unique solution of an associated variational inequality (VI) under rather strong assumptions. These assumptions are: (1) The VI has the unique classical solution, i.e., unique solvability in $C^{2,2,1}$; (2) The control problem has an underlying process and associated control process that attains the optimal value, i.e., existence of the optimal pair (π^*, τ^*). We refer to [4, 18] for the verification theorem in general control theory, and [10] for the case of indifference pricing of American options.

In the general setting of control problems, the aforementioned conditions for the verification theorem are difficult to verify due to the full non-linearity. As an alternative, one can use the viscosity solution approach [3] to identify the value function. In particular, [15] characterizes the value function V of an investment problem in the framework of indifference pricing. A drawback of this approach is that due to the lack of information on the regularity for the viscosity solution, one cannot obtain further knowledge of optimal controls. Even the existence of the optimal control is problematic.

In this work, we will adopt an intermediate methodology between the classical verification theorem and the viscosity solution approach. First, we show the verification theorem holds under weaker assumptions than the classical verification theorem needs, but stronger than the viscosity solution needs. For the purpose of complete characterization, we obtain some regularity estimates of the partial differential equation (PDE), which verifies all the necessary assumptions of the existence of the PDE solution and associated optimal pair. Thanks to the above results on control problem, one can further develop more useful results: They are the regularity of optimal exercise boundary, the dual representation, and monotonicity of the value function on the system parameters. To be more precise, the idea is illustrated below.

In the standard arguments of the verification theorem, one uses Itô formula and classical solutions of a variation inequality (VI). However, it is too restrictive, since the VI in our case does not admit a classical solution. Therefore, we replace the assumption on the existence of classical solution by solutions in Sobolev space $W^{2,2,1}_{p,loc}$ for large enough p. Then, the question is

- How large p is enough to enable us to use Itô formula?

We will demonstrate that $p = 3$ is large enough. As a by product, we generalize Itô's formula given in [8] from one-side inequality to two-sided equality; see the proof in the appendix, which may be potentially useful in other related control problems.

Moreover, utilizing special structure with the exponential forward performance, one can derive the verification for the price P associated to a PDE. Note that another assumption needed for the verification theorem, namely the existence of optimal pair, can be reduced to an estimation on the first derivative. As a result, the verification theorem holds under weaker assumptions on solvability (H1) and estimates (H2) given at the end of Section 3. To fully characterize the control problem, we answer the questions:

- Does solvability (H1) and estimation (H2) hold for the VI associated to the price P?

To proceed, we start with a simple transformation on the backward equation (in time), which leads to an equivalent counterpart of forward equation on standard domain $\mathcal{Q} = \mathbb{R} \times (0, T]$. Since the original domain is unbounded, we shall use penalized method on the truncated version. As a result, the truncated PDE after penalization is quasi-linear, and Leray-Schauder fixed point theorem provides the solvability in $W_p^{2,1}$, and the comparison principle is standard for the strong solution, see more details in Section 4. By forcing the limit of the parameter of the penalized function $\varepsilon \to 0$, and the parameter of the truncated domain $N \to \infty$, we use local estimate to obtain the $C^{\alpha,\alpha/2}$ estimate (De Giorgi-Nash-Moser estimate) and $W_p^{2,1}$ estimate on compact domain by removing out singularity point $(\ln K, 0)$. Consequently, we obtain the existence of $W_{p,loc}^{2,1}(\mathcal{Q}) \cap C(\overline{\mathcal{Q}})$ solution with some regularity estimates on the first order derivatives, which eventually resolves (H1)-(H2). To this end, we complete the characterization, which is summarized in Theorem 11.

One application of the main result is that, a dual representation Proposition 12 is proved in the framework of stochastic control theory. A simple derivation from this representation implies several economically interesting facts: sensitivity properties with respect to risk aversion γ and some other parameters, and sublinear property with respect to the payoff, etc. Another application is the indifference pricing of ESOs, where vesting periods and job termination risk is involved. Thanks to the characterization theorem, the price can be characterized as the classical solution of its associated PDE.

The rest of this paper is outlined as follows. The precise formulation of the problem is given in Section 2. Section 3 proceeds with a generalized verification theorem. Section 4 presents the main results that gives a complete characterization. Section 5 is concerned with the dual representation and other properties. Section 6 demonstrates application to ESO costs. Finally, an appendix containing a few technical results is provided.

2 Problem Formulation

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a filtered probability space with a filtration $(\mathcal{F}_t)_{t\geq 0}$ that satisfies the usual conditions, and the processes B and $\widetilde{B}$ be two independent standard Brownian motions. Let $\mathcal{T}_{t,T}$ be the set of all stopping times in the interval $[t, T]$.

We consider a financial market with a risk-free interest rate, which consists of a non-traded stock issued by a firm with its price $Y_\nu^{t,y}$ at time $\nu \in [t,T]$ given by

$$dY_\nu = bY_\nu d\nu + cY_\nu(\rho dB_\nu + \sqrt{1-\rho^2}d\widetilde{B}_\nu),\ Y_t = y, \tag{1}$$

and one liquidly traded stock with its discounted price $S_\nu^{t,s}$ at time $\nu \in [t,T]$

$$dS_\nu = S_\nu \sigma(\lambda d\nu + dB_\nu),\ S_t = s. \tag{2}$$

The payoff of American call option with strike $K > 0$ and maturity T underlying non-traded asset $Y_\nu^{t,y}$ is $g(Y_\tau)$, where τ is the exercise time $\tau \in \mathcal{T}_{t,T}$ and g is given by a function given by

$$g(y) = (y-K)^+.$$

If an employee, with initial capital w, dynamically trades in the stock S, then her wealth process $W^{t,w,\pi}$ under self-financing rule satisfying

$$dW_\nu^\pi = \pi_\nu \sigma(\lambda d\nu + dB_\nu),\ W(t) = w, \tag{3}$$

where π_ν represents the cash amount invested in the liquid stock $S_\nu^{t,s}$. We assume the strategy π belongs to $\mathcal{Z}_t$, which is the set of all progressively measurable processes $\pi : [t,T] \times \Omega \to \mathbb{R}$ satisfying integrability condition

$$\mathbb{E}\Big[\int_t^T \pi_\nu^2 d\nu\Big] < \infty. \tag{4}$$

We introduce an exponential forward performance measure given by

$$U_t(w) = -e^{-\gamma w + \frac{1}{2}\lambda^2 t}; \tag{5}$$

see [10, Theorem 3 and (18)]. Under the above performance measure, we compare the following two scenarios for an employee holding the initial capital w and a unit of American call at the initial time t:

1. Find a stopping time $\tau \in \mathcal{T}_{t,T}$ to exercise the call option, while dynamically trading the capital w in liquid stock S_ν, to maximize its performance, and find the corresponding value
$$V(w,y,t) = \sup_{\tau \in \mathcal{T}_{t,T}, \pi \in \mathcal{Z}_{t,\tau}} \mathbb{E}[U_\tau(W_\tau^{t,w,\pi} + g(Y_\tau^{t,y}))|\mathcal{F}_t]. \tag{6}$$

2. Receive a cash payment of amount $P(w,y,t)$ by selling one unit of call at time t, the performance corresponding to the total capital $w + P(w,y,t)$ is $U_t(w + P(w,y,t))$.

Now, we are ready to define the indifference price of the American call option. The indifference price of this call is the cash payment $P(w,y,t)$, which makes the above two scenarios indifferent with respect to the given forward performance of (5). Therefore, the price $P(w,y,t)$ is the value satisfying

$$V(w,y,t) = U_t(w + P(w,y,t)). \tag{7}$$

To determine the above indifference price of the American call, we consider the following two problems:

(Q1) Find the indifference price using the identity $P(w, y, t) = U_t^{-1}(V(w, y, t)) - w$, after solving the stochastic control optimization (6), i.e.,

  (a) characterize the value function V of (6);

  (b) characterize the pair of optimal strategy τ^* and π^* if it exists.

(Q2) Characterize directly the indifference price $P(w, y, t)$ as the unique solution of certain PDE.

To proceed, the following assumptions are imposed throughout this paper:

(A1) Assume σ, $\gamma > 0$, and $\rho \in [0, 1)$.

We exclude the case $\rho = 1$, where the market is complete and the problem of indifference price can be reduced to a Black-Scholes model.

3 Generalized Verification Theorem

We present a version of the verification theorem for the value function V of (6) and P of (7) in this subsection, which generalize the verification theorem from the classical PDE solution to the strong solution in Sobolev function space with certain regularity.

Define a three dimensional domain by $\mathcal{Q}^1 = \mathbb{R} \times \mathbb{R}^+ \times [0, T)$, and denote by $W^{2,2,1}_{p,loc}(\mathcal{Q}^1)$, the collection of all functions $\mathcal{Q}^1 \ni (w, y, t) \mapsto \varphi(w, y, t)$ having $(\partial_{ww}\varphi, \partial_{yy}\varphi, \partial_t\varphi)$ in distribution sense and integrable in any compact subset of $\mathcal{Q}^1$. For a scalar $\pi \in \mathbb{R}$, define a parameterized differential operator $\mathcal{L}_1^\pi : W^{2,2,1}_{p,loc} \mapsto \mathbb{R}$ as

$$\mathcal{L}_1^\pi \varphi(w, y, t) = \frac{1}{2}c^2 y^2 \partial_{yy}\varphi + by\partial_y\varphi + \frac{1}{2}\sigma^2\pi^2\partial_{ww}\varphi + \pi(\rho\sigma c y \partial_{wy}\varphi + \lambda\sigma\partial_w\varphi).$$

We also need to introduce the *continuation region* by

$$\mathcal{C}[v] = \{(w, y, t) \in \mathcal{Q}^1 : v(w, y, t) > U_t(w + g(y))\}. \tag{8}$$

Lemma 1 (Verification Theorem for Function V) *Suppose there exists $v \in W^{2,2,1}_{p,loc}(\mathcal{Q}^1)$ for some $p \geq 3$, satisfying*

$$\min\Big\{ -\partial_t v + \inf_{\pi\in\mathbb{R}}\{-\mathcal{L}_1^\pi v(w, y, t)\}, v(w, y, t) - U_t(w + g(y))\Big\} = 0, \tag{9}$$

with the terminal condition

$$v(w, y, T) = U_T(w + g(y)). \tag{10}$$

Then, $v(w, y, t) \geq V(w, y, t)$. If, in addition, there exists a pair (W^*, π^*) satisfying (3), (4), and

$$(\partial_t v + \mathcal{L}_1^{\pi^*_\nu} v)(W^*_\nu, Y_\nu, \nu) = 0, \ \forall\ t < \nu < \tau^*, \tag{11}$$

where

$$\tau^* := \inf\{\nu > t : (W^*_\nu, Y_\nu, \nu) \notin \mathcal{C}[v]\} \wedge T. \tag{12}$$

Then the variational inequality (9)-(10) admits a unique solution in $W^{2,2,1}_{p,loc}(\mathcal{Q}^1)$, and $v(w, y, t) = V(w, y, t)$.

PROOF: Fix an arbitrary stopping time $\tau \in \mathcal{T}_{t,T}$ and an admissible control $\pi \in \mathcal{Z}_{t,\tau}$, and denote $(W, Y) := (W^{t,w,\pi}, Y^{t,y})$ and $\mathbb{E}_t[\ \cdot\ ] := \mathbb{E}[\ \cdot\ |\mathcal{F}_t]$ for simplicity. By virtue of the generalized Itô's formula (see Proposition 15), we have

$$\mathbb{E}_t[v(W_\tau, Y_\tau, \tau)] = v(w, y, t) + \mathbb{E}_t\Big[\int_t^\tau (\partial_t v + \mathcal{L}_1^\pi v)(W_\nu, Y_\nu, \nu)d\nu\Big].$$

Applying (9), we have $v(w, y, t) \geq U_t(w + g(y))$ for all (w, y, t), hence

$$\mathbb{E}_t[v(W_\tau, Y_\tau, \tau)] \geq \mathbb{E}_t[U_\tau(W_\tau + g(Y_\tau))].$$

Note that (9) also implies $(\partial_t v + \mathcal{L}_1^\pi v)(w, y, t) \leq 0$ for all (w, y, t), which yields

$$\mathbb{E}_t\Big[\int_t^\tau (\partial_t v + \mathcal{L}_1^\pi v)(W_\nu, Y_\nu, \nu)d\nu\Big] \leq 0.$$

Therefore, we conclude

$$\mathbb{E}_t[U_\tau(W_\tau + g(Y_\tau))] \leq v(w, y, t),$$

and arbitrariness of τ and π further implies one-sided inequality

$$v(w, y, t) \geq V(w, y, t).$$

Furthermore, if (11) holds for some (W^*, π^*), Itô's formula together with the definition of $\mathcal{C}[v]$ gives the inequality of the opposite direction:

$$v(w, y, t) = \mathbb{E}_t[v(W_{\tau^*}, Y_{\tau^*}, \tau^*)] = \mathbb{E}_t[U_{\tau^*}(W_{\tau^*} + g(Y_{\tau^*}))] \leq V(w, y, t).$$

This implies $v = V$. $\square$

Next, we discuss the price P of (7). To proceed, we introduce a domain $\mathcal{Q}^2 = \mathbb{R}^+ \times [0, T)$ and its Sobolev space $W^{2,1}_{p,loc}(\mathcal{Q}^2)$. Also define a non-linear differential operator $\mathcal{L}_2 : W^{2,1}_{p,loc}(\mathcal{Q}^2) \to \mathbb{R}$ by

$$\mathcal{L}_2\varphi(y, t) = \frac{1}{2}c^2 y^2 \partial_{yy}\varphi + (by - \rho c\lambda y)\partial_y\varphi - \frac{1}{2}\gamma(1 - \rho^2)c^2 y^2 (\partial_y\varphi)^2.$$

Lemma 2 (Verification Theorem for Function P) *Suppose there exists a function $f \in W^{2,1}_{p,loc}(\mathcal{Q}^2)$ for some $p \geq 3$, satisfying*

$$\min\{-\partial_t f(y, t) - \mathcal{L}_2 f(y, t), f(y, t) - g(y)\} = 0, \quad (y, t) \in \mathcal{Q}^2, \tag{13}$$

with the terminal condition

$$f(y, T) = g(y), \quad y \in \mathbb{R}^+, \tag{14}$$

and $|\partial_y f(y, t)|$ is uniformly bounded. Then, PDE (9) together with the terminal condition (10) is uniquely solvable in $W^{2,2,1}_{p,loc}(\mathcal{Q}^1)$, and there exists a pair (W^*, π^*) satisfying (3), (4), and (11). As a result, the price function $P(w, y, t)$ of (7) is independent to the initial wealth w, and $P(w, y, t) = f(y, t)$.

PROOF: Let $v(w, y, t) = U_t(w + f(y, t))$. Then, we have $v \in W^{2,2,1}_{p,loc}(\mathcal{Q}^1)$. One can directly check $\partial_{ww} v = \gamma^2 v < 0$ in $\mathcal{Q}^1$, and hence, the function $\pi^*[v] : \mathcal{Q}^1 \mapsto \mathbb{R}$ of the form

$$\pi^*[v](w, y, t) = -\frac{\rho c y \partial_{wy} v(w, y, t) + \lambda \partial_w v(w, y, t)}{\sigma \partial_{ww} v(w, y, t)}$$

is well defined. Note that, $\pi^*[v]$ is independent to the variable w, i.e., one can rewrite

$$\pi^*[v](y, t) = -\frac{\rho c}{\sigma} \cdot y \partial_y f(y, t) + \frac{\lambda}{\sigma \gamma}.$$

Now, we can define a process π^* by

$$\pi^*_\nu = \pi^*[v](Y_\nu, \nu) = -\frac{\rho c}{\sigma} \cdot Y_\nu \partial_y f(Y_\nu, \nu) + \frac{\lambda}{\sigma \gamma}, \quad \nu \in (t, T). \tag{15}$$

Note that, Y is the unique strong solution of (1). Therefore, π^*_ν is an admissible strategy, since it satisfies the integrability condition (4) due to boundedness of $\partial_y f$, i.e.,

$$\mathbb{E}\Big[\int_t^T (\pi^*_\nu)^2 d\nu\Big] \leq C + C\mathbb{E}\Big[\int_t^T Y_\nu^2 d\nu\Big] < \infty.$$

Since the optimal strategy π^*_ν is independent to W^*, the solution to (3) can be simply given by its integral form of (3) . Calculations by change of variables leads to (9)-(11). Thus, Lemma 1 together with definition (7) implies $V(w, y, t) = U_t(w + f(y, t))$, and $f(y, t) = P(w, y, t)$. $\square$

Remark 3 (The boundary condition of $f(y, t)$ at $y = 0$) The verification theorem can be thought of the probability counterpart of uniqueness result of PDE solution (13)-(14). From the above verification theorem, the uniqueness holds without the specification of the boundary condition on $y = 0$. Similar observations are addressed in [16] (Fichera condition) for linear equation, and in [1] for the fully non-linear parabolic equation without obstacle.

To this end, we summarize the implication of Lemma 1 and Lemma 2. The above verification theorems give conditional characterization of the control problem since they give the answers to (Q1)-(Q2) based on the assumptions on the solvability of PDE. More precisely,

1. For (Q1-a), V is the unique $W^{2,2,1}_{p,loc}$ solution of PDE (9)-(10);
2. For (Q1-b), the pair of optimal control exists, and they may have representation of (15) and (12), respectively;
3. For (Q2), P is invariant in variable w, and the unique $W^{2,1}_{p,loc}$ solution of PDE (13)-(14) as a function of two variables of (y, t),

provided that the following hypothesis are valid,

(H1) (13)-(14) is solvable in $W^{2,1}_{p,loc}(\mathcal{Q}^2)$;

(H2) $|\partial_y f(y, t)|$ is uniformly bounded.

4 Main Result: Complete Characterization

To obtain the complete characterization, it is crucial to study the solability and its related estimations of PDE (13)-(14).

For the convenience in the analysis, we analyze the backward equation (in time) (13)-(14), by studying its associated forward equation of the following form.

Let $x = \ln y,\ \theta = T - t,\ u(x,\theta) = f(y,t)$ in (13)-(14), then $u(x,\theta)$ satisfies

$$\begin{cases} \min\{(\partial_\theta u - \mathcal{L}u)(x,\theta),\ u - (e^x - K)^+\} = 0, & (x,\theta) \in \mathcal{Q}, \\ u(x,0) = (e^x - K)^+, & x \in \mathbb{R}, \end{cases} \tag{16}$$

where $\mathcal{Q} = \mathbb{R} \times (0,T]$ and $\mathcal{L}$ is the differential operator given by

$$\mathcal{L}\varphi(x,t) = \frac{1}{2}c^2\partial_{xx}\varphi + \Big(b - \rho c\lambda - \frac{1}{2}c^2\Big)\partial_x\varphi - \frac{1}{2}\gamma(1-\rho^2)c^2(\partial_x\varphi)^2.$$

4.1 Solvability of Problem (16)

Since $(-\infty,+\infty) \times (0,T]$ is unbounded, we first confine our attention to the truncated version of (16) in a finite domain $\mathcal{Q}_N = (-N,N) \times (0,T]$. Let $u^N(x,\theta)$ be the solution (if it exists) to the following problem

$$\begin{cases} \min\left\{\partial_\theta u^N - \mathcal{L}u^N,\ u^N - (e^x - K)^+\right\} = 0, & (x,\theta) \in \mathcal{Q}_N, \\ \partial_x u^N(-N,\theta) = 0,\ \partial_x u^N(N,\theta) = e^N, & \theta \in (0,T], \\ u^N(x,0) = (e^x - K)^+, & x \in (-N,N). \end{cases} \tag{17}$$

In order to prove the existence of solution to problem (17), we construct a penalty approximation of problem (17). Suppose $u^N_\varepsilon(x,\theta)$ satisfies

$$\begin{cases} \partial_\theta u^N_\varepsilon - \mathcal{L}u^N_\varepsilon + \beta_\varepsilon(u^N_\varepsilon - \pi_\varepsilon(e^x - K)) = 0, & (x,\theta) \in \mathcal{Q}_N, \\ \partial_x u^N_\varepsilon(-N,\theta) = 0,\ \partial_x u^N_\varepsilon(N,\theta) = e^N, & \theta \in (0,T], \\ u^N_\varepsilon(x,0) = \pi_\varepsilon(e^x - K), & x \in (-N,N), \end{cases} \tag{18}$$

where $\beta_\varepsilon(t)$ (see Fig. 1.) satisfies

$$\begin{aligned} &\beta_\varepsilon(t) \in C^2(-\infty,+\infty),\ \ \beta_\varepsilon(t) \le 0, \\ &\beta'_\varepsilon(t) \ge 0,\ \ \beta''_\varepsilon(t) \le 0,\ \ \beta_\varepsilon(0) = -C_0, \end{aligned}$$

where $C_0 > 0$ is to be determined. Note that

$$\lim_{\varepsilon\to 0}\beta_\varepsilon(t) = \begin{cases} 0, & t > 0, \\ -\infty, & t < 0, \end{cases}$$

and that $\pi_\varepsilon(t)$ (see Fig. 2.) satisfies

$$\pi_\varepsilon(t) = \begin{cases} t, & t \ge \varepsilon, \\ \nearrow, & |t| \le \varepsilon, \\ 0, & t \le -\varepsilon, \end{cases}$$

and $\pi_\varepsilon(t) \in C^\infty$, $\; 0 \le \pi'_\varepsilon(t) \le 1$, $\; \pi''_\varepsilon(t) \ge 0$, $\; \lim\limits_{\varepsilon \to 0^+} \pi_\varepsilon(t) = t^+$.

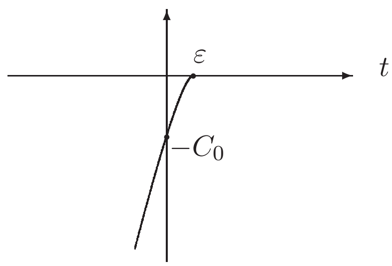

Fig. 1. $\beta_\varepsilon(t)$

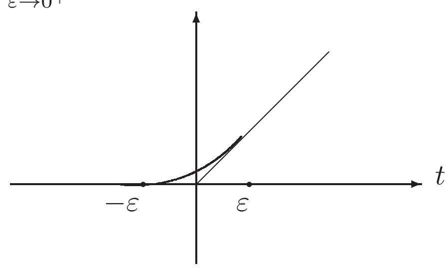

Fig. 2. $\pi_\varepsilon(t)$

Lemma 4 *There exists a unique solution* $u_\varepsilon^N(x,\theta) \in W_p^{2,1}(\mathcal{Q}_N)$ *to problem* (18) *for any* $p \ge 1$. *Moreover, the following estimates hold,*

$$\pi_\varepsilon(e^x - K) \le u_\varepsilon^N(x,\theta) \le k\theta + x^2 + e^{x+(b-\rho c\lambda)^+\theta} + 1, \tag{19}$$

$$0 \le \partial_x u_\varepsilon^N(x,\theta) \le e^{x+(b-\rho c\lambda)^+\theta}, \tag{20}$$

$$\partial_\theta u_\varepsilon^N(x,\theta) \ge 0, \tag{21}$$

where $k \ge \max\left\{ c^2 + \frac{(b-\rho c\lambda - \frac{1}{2}c^2)^2}{2\gamma(1-\rho^2)c^2},\; c^2 + \frac{\left[\gamma(1-\rho^2)c^2 e^{(b-\rho c\lambda)^+ T} - (b-\rho c\lambda - \frac{1}{2}c^2)\right]^2}{2\gamma(1-\rho^2)c^2} \right\}$.

Proof: The Leray-Schauder fixed point theorem implies the existence of $W_p^{2,1}$ solution to problem (18), and the comparison principle holds for the strong solution. The proof of the uniqueness is standard.

First, we prove estimate (19). Note that $u_1(x,\theta) := \pi_\varepsilon(e^x - K)$ satisfies $\partial_\theta u_1 - \mathcal{L}u_1 + \beta_\varepsilon(u_1 - \pi_\varepsilon(\cdot)) \le 0$ in $\mathcal{Q}_N$, if we choose

$$-\beta_\varepsilon(0) = C_0 = \rho c\lambda e^N + \frac{1}{2}\gamma(1-\rho^2)c^2 e^{2N}. \tag{22}$$

Moreover, when N is large enough, we have $\partial_x u_1(-N,\theta) = 0$, $\partial_x u_1(N,\theta) = e^N$, $u_1(x,0) = u_\varepsilon^N(x,0)$. Thus $u_1(x,\theta) = \pi_\varepsilon(e^x - K)$ is a subsolution to problem (18).

On the other hand, one can check $u_2(x,\theta) := k\theta + x^2 + e^{x+(b-\rho c\lambda)^+\theta} + 1$ satisfies supersolution property $\partial_\theta u_2 - \mathcal{L}u_2 + \beta_\varepsilon(u_2 - \pi_\varepsilon(\cdot)) \ge 0$ in $\mathcal{Q}_N$. Moreover, we have

$$\begin{aligned}
&\partial_x u_2(-N,\theta) = -2N + e^{-N+(b-\rho c\lambda)^+\theta} \le 0,\\
&\partial_x u_2(N,\theta) = 2N + e^{N+(b-\rho c\lambda)^+\theta} \ge e^N,\\
&u_2(x,0) = x^2 + e^x + 1 \ge \pi_\varepsilon(e^x - K).
\end{aligned}$$

Applying the comparison principle, we conclude (19).

Next, we verify inequality (20). If we differentiate the equation in (18) w.r.t. x, then $v(x,\theta) = \partial_x u_\varepsilon^N(x,\theta)$ satisfies

$$\begin{cases}
\partial_\theta v - \frac{1}{2}c^2\partial_{xx}v - (b-\rho c\lambda - \frac{1}{2}c^2)\partial_x v & \\
+\gamma(1-\rho^2)c^2 v\partial_x v + \beta'_\varepsilon(\cdot)(v - \pi'_\varepsilon e^x) = 0, & (x,\theta) \in \mathcal{Q}_N,\\
v(-N,\theta) = 0,\; v(N,\theta) = e^N, & \theta \in (0,T],\\
v(x,0) = \pi'_\varepsilon(\cdot)e^x, & x \in (-N,N).
\end{cases} \tag{23}$$

Since $v_1 = 0$ and $v_2(x,\theta) = e^{x+(b-\rho c\lambda)^+\theta}$ are subsolution and supersolution of (23), the estimate (20) follows by comparison principle.

Denote $u^\delta(x,\theta) := u^N_\varepsilon(x,\theta+\delta)$ for $0<\delta<T$, $u^\delta(x,\theta)$ satisfies

$$\begin{cases} \partial_\theta u^\delta - \mathcal{L}u^\delta + \beta_\varepsilon(u^\delta - \pi_\varepsilon(e^x-K)) = 0, & (x,\theta)\in(-N,N)\times(0,T-\delta], \\ \partial_x u^\delta(-N,\theta)=0,\ \partial_x u^\delta(N,\theta)=e^N, & \theta\in(0,T-\delta], \\ u^\delta(x,0) = u^N_\varepsilon(x,\delta) \geq \pi_\varepsilon(e^x-K) = u^N_\varepsilon(x,0), & x\in(-N,N). \end{cases}$$

Combining with (18), applying the comparison principle, we have

$$u^\delta(x,\theta)\geq u^N_\varepsilon(x,\theta),\quad (x,\theta)\in(-N,N)\times(0,T-\delta],$$

which implies (21). $\Box$

Lemma 5 *Problem (17) has a unique solution $u^N(x,\theta)\in W^{2,1}_{p,loc}(\mathcal{Q}_N)\cap C(\overline{\mathcal{Q}}_N)$ satisfying*

$$(e^x-K)^+ \leq u^N(x,\theta) \leq k\theta + x^2 + e^{x+(b-\rho c\lambda)^+\theta} + 1, \tag{24}$$
$$0\leq \partial_x u^N(x,\theta)\leq e^{x+(b-\rho c\lambda)^+\theta}, \tag{25}$$
$$\partial_\theta u^N(x,\theta)\geq 0, \tag{26}$$

where k is defined in Lemma 4.

Proof: Let C_N be a generic constant independent of ε.

Since $u^N_\varepsilon(x,\theta)\geq\pi_\varepsilon(e^x-K)$, then $|\beta_\varepsilon(u^N_\varepsilon-\pi_\varepsilon(e^x-K))|_{L^p(\mathcal{Q}_N)}\leq C_N$. One can treat the nonlinear term in the equation in (18) as a linear term with the coefficient $\frac{1}{2}\gamma(1-\rho^2)c^2\partial_x u^N_\varepsilon$. Thanks to (20), applying $C^{\alpha,\alpha/2}$ estimate (De Giorgi-Nash-Moser estimate, [13]) with $\alpha = 1/2$, we have

$$|u^N_\varepsilon|_{C^{1/2,1/4}(\mathcal{Q}_N)}\leq C_N.$$

Letting $\varepsilon\to0$, we have a continuous limit (of a subsequence if necessary) $u^N(x,\theta)$ up to the boundary, that is,

$$u^N_\varepsilon(x,\theta)\to u^N(x,\theta)\quad \text{in } C(\overline{\mathcal{Q}}_N).$$

In view of (20), applying $W^{2,1}_p$ estimate [9], we have

$$|u^N_\varepsilon|_{W^{2,1}_p(\mathcal{Q}_N\setminus B_\delta)}\leq C_N,$$

where B_δ is a disk with center $(\ln K, 0)$ and radius $\delta>0$. Hence $u^N(x,\theta)\in W^{2,1}_{p,loc}(\mathcal{Q}_N)$ and

$$u^N_\varepsilon(x,\theta)\rightharpoonup u^N(x,\theta)\quad \text{weakly in } W^{2,1}_{p,loc}(\mathcal{Q}_N),$$

which also implies $u^N(x,\theta)$ is a $W^{2,1}_{p,loc}$ solution of (17). Furthermore, (24)-(26) are consequences of (19)- (21).

In this below, we show the uniqueness. If not, we can find two $W_p^{2,1}$ solutions u_1 and u_2 satisfying (24) and (25), and $\mathcal{N} = \{(x,\theta) \in \mathcal{Q}_N : u_1 > u_2\} \neq \emptyset$. Observe that, $\partial_\theta u_1 = \mathcal{L}u_1$ and $\partial_\theta u_2 \geq \mathcal{L}u_2$ in $\mathcal{N}$. Thus, $u_1 - u_2$ satisfies

$$\begin{cases} \partial_\theta(u_1-u_2) - \frac{1}{2}c^2\partial_{xx}(u_1-u_2) - (b-\rho c\lambda - \frac{1}{2}c^2)\partial_x(u_1-u_2) \\ +\frac{1}{2}\gamma(1-\rho^2)c^2(\partial_x u_1 + \partial_x u_2)\partial_x(u_1-u_2) \leq 0, \quad (x,\theta) \in \mathcal{N}, \\ (u_1-u_2)(x,\theta) = 0, & (x,\theta) \in \partial_p\mathcal{N} \setminus \{x = \pm N\}, \\ \partial_x(u_1-u_2)(x,\theta) = 0, & (x,\theta) \in \partial_p\mathcal{N} \cap \{x = \pm N\}. \end{cases}$$

Owing to (25), applying maximum principle (see [17]), we have $u_1 - u_2 \leq 0$ on $\mathcal{N}$, which is a contradiction to the definition of $\mathcal{N}$. $\square$

Lemma 6 *There exists a unique solution $u(x,\theta) \in W_{p,loc}^{2,1}(\mathcal{Q}) \cap C(\overline{\mathcal{Q}})$ to problem (16) satisfying*

$$(e^x - K)^+ \leq u(x,\theta) \leq k\theta + x^2 + e^{x+(b-\rho c\lambda)^+\theta} + 1, \tag{27}$$

$$0 \leq \partial_x u(x,\theta) \leq e^{x+(b-\rho c\lambda)^+\theta}, \tag{28}$$

$$\partial_\theta u(x,\theta) \geq 0, \tag{29}$$

where k is defined in Lemma 4.

Proof: First, we claim problem (17) is equivalent to the following problem

$$\begin{cases} \min\{\partial_\theta u^N - \mathcal{L}u^N,\ u^N - (e^x - K)\} = 0, & (x,\theta) \in \mathcal{Q}_N, \\ \partial_x u^N(-N,\theta) = 0,\ \partial_x u^N(N,\theta) = e^N, & \theta \in (0,T], \\ u^N(x,0) = (e^x-K)^+, & x \in (-N,N). \end{cases} \tag{30}$$

In fact, suppose $w(x,\theta)$ is a $W_{p,loc}^{2,1}$ solution to problem (30), the maximum principle [17] implies $w \geq 0$, combine with $w \geq e^x - K$ to get $w(x,\theta) \geq (e^x-K)^+$. Hence, w is also a solution of (17). Together with the uniqueness of (17), the equivalence follows.

Now we can rewrite problem (17) as

$$\begin{cases} \partial_\theta u^N - \mathcal{L}u^N = f(x,\theta), & (x,\theta) \in \mathcal{Q}_N, \\ \partial_x u^N(-N,\theta) = 0,\ \partial_x u^N(N,\theta) = e^N, & \theta \in (0,T], \\ u^N(x,0) = (e^x-K)^+, & x \in (-N,N), \end{cases} \tag{31}$$

where $f(x,\theta) = \chi_{\{u^N(x,\theta)=(e^x-K)^+\}}\big(-(b-\rho c\lambda)e^x + \frac{1}{2}\gamma(1-\rho^2)c^2e^{2x}\big)$. Thanks to (25), if we apply $W_p^{2,1}$ interior estimate [9] to (31) for arbitrary $M < N$,

$$|u^N|_{W_p^{2,1}(\mathcal{Q}_M \setminus B_\delta)} \leq C_M,$$

where B_δ is a disk with center $(\ln K, 0)$ and radius δ. We emphasize C_M only depends on M, but not on N. Fix $M > 0$, and let $N \to +\infty$, u^N leads to a limit $u^{(M)}$ (possibly a subsequence) in the fixed domain $\mathcal{Q}_M$ in the sense,

$$u^N \rightharpoonup u^{(M)} \quad \text{weakly in } W_{p,loc}^{2,1}(\mathcal{Q}_M) \text{ as } N \to \infty.$$

Moreover the sobolev embedding theorem implies

$$u^N \to u^{(M)} \quad \text{in } C(\mathcal{Q}_M),\ \partial_x u^N \to \partial_x u^{(M)} \quad \text{in } C(\mathcal{Q}_M).$$

It is clear that $u(x,\theta) := u^{(M)}(x,\theta),\ (x,\theta) \in \mathcal{Q}_M$ is well defined and u is the solution to problem (16). Moreover, $\partial_x u \in C(\mathcal{Q})$ and we can deduce $u \in C(\overline{\mathcal{Q}})$ from the C^α estimate. (27)-(29) are consequences of (24)- (26). Lemma 1 and 2 implies the uniqueness of solution to problem (16). □

4.2 The Obstacle of (16)

Now we will characterize the optimal exercising time of V. Problem (16) is an optimal stopping problem, which gives rise to a free boundary that can be expressed as a single-valued function of θ. For later use, we define

$$\mathcal{S} := \{(x,\theta) \in \mathcal{Q} : u(x,\theta) = (e^x - K)^+\} \text{ (Stopping region)},$$
$$\mathcal{C} := \{(x,\theta) \in \mathcal{Q} : u(x,\theta) > (e^x - K)^+\} \text{ (Continuation region)},$$

where u is the solution of (16). Thanks to the continuity of u from Lemma 6, $\mathcal{S}$ is closed and $\mathcal{C}$ is open, respectively.

Lemma 7 *There exists $S(x) : (-\infty, +\infty) \to [0,T]$ such that*

$$\mathcal{S} = \{(x,\theta) \in \mathcal{Q} : 0 < \theta \le S(x)\}. \tag{32}$$

PROOF: If $(x_0,\theta_0) \in \mathcal{S}$, i.e.,

$$u(x_0,\theta_0) = (e^{x_0} - K)^+,$$

according to (27) and (29), we have

$$u(x_0,\theta) = (e^{x_0} - K)^+,\ \theta \in (0,\theta_0].$$

Hence $\{x_0\} \times (0,\theta_0] \subseteq \mathcal{S}$, so we can define $S(x) : (-\infty,+\infty) \to [0,T]$ as

$$S(x) = \begin{cases} 0, & \text{if } u(x,\theta) > (e^x - K)^+ \text{ for any } \theta \in (0,T], \\ \sup\{\theta \in (0,T] : u(x,\theta) = (e^x - K)^+\}. \end{cases}$$

By the definition of $S(x)$, (32) holds. □

Lemma 8 *The free boundary $S(x)$ is strictly increasing w.r.t. x on $\{x : 0 < S(x) < T\}$.*

PROOF: As in the proof of the equivalence between (17) and (30) in Lemma 6, we can show that problem (16) is equivalent to the following problem

$$\begin{cases} \min\left\{\partial_\theta u - \mathcal{L}u,\ u - (e^x - K)\right\} = 0, & (x,\theta) \in \mathcal{Q}, \\ u(x,0) = (e^x - K)^+, & x \in \mathbb{R}. \end{cases} \tag{33}$$

When $u(x,\theta) = (e^x - K)^+ = (e^x - K)$, we have

$$\begin{cases} e^x - K \geq 0, \\ \partial_\theta u(x,\theta) - \mathcal{L}u(x,\theta) \geq 0, \end{cases}$$

which implies

$$e^x \geq \max\left\{K,\ \frac{2(b-\rho c\lambda)}{\gamma(1-\rho^2)c^2}\right\}. \tag{34}$$

For any $x_0 \in \{x : 0 < S(x) < T\}$, denote $S(x_0) = \theta_0 \in (0,T)$, in view of (32), we know

$$u(x_0,\theta) = e^{x_0} - K \geq 0,\ \theta \in (0,\theta_0].$$

Denote $\mathcal{Q}_0 = (-\infty,+\infty)\times(0,\theta_0]$ and define a new function $\overline{u}(x,\theta)$ on $\mathcal{Q}_0$ by

$$\overline{u}(x,\theta) = \begin{cases} u(x,\theta), & (x,\theta) \in (-\infty,x_0]\times[0,\theta_0], \\ e^x - K, & (x,\theta) \in [x_0,+\infty)\times[0,\theta_0]. \end{cases}$$

Since $\{x_0\}\times(0,\theta_0] \subseteq \mathcal{S}$, then $\overline{u}(x,\theta)$, $\partial_x\overline{u}(x,\theta)$ are continuous in $\mathcal{Q}_0$. Now we want to prove $\overline{u}(x,\theta)$ is the solution to problem (16) in the domain $\mathcal{Q}_0$.

By the definition of $\overline{u}(x,\theta)$ we know

$$\overline{u}(x,0) = (e^x - K)^+,\ x \in \mathbb{R}.$$

According to (34), we can check $\overline{u}(x,\theta)$ satisfies

$$\min\left\{\partial_\theta\overline{u} - \mathcal{L}\overline{u},\ \overline{u} - (e^x - K)^+\right\} = 0,\ (x,t) \in \mathcal{Q}_0.$$

Hence, we know $\overline{u}(x,\theta)$ is the solution of (16) in domain $\mathcal{Q}_0$. By the uniqueness of $W^{2,1}_{p,loc}(\mathcal{Q})\cap C(\overline{\mathcal{Q}})$ solution which satisfies (27)-(28) to problem (16) we know

$$u(x,\theta) = \overline{u}(x,\theta), \quad (x,\theta) \in \mathcal{Q}_0.$$

In particular,

$$u(x,\theta) = \overline{u}(x,\theta) = (e^x - K)^+, \quad (x,\theta) \in [x_0,+\infty)\times(0,\theta_0].$$

By the definition of $S(x)$ we know

$$S(x) \geq \theta_0 = S(x_0), \quad x \geq x_0,$$

therefore the monotonicity of $S(x)$ is proved.

Next we will prove the strict monotonicity of $S(x)$ on the set $\{x : 0 < S(x) < T\}$. Suppose not (see Fig. 3). There exists $x_1 < x_2$ such that

$$S(x) = \theta_0 \in (0,T), \quad x \in (x_1,x_2).$$

Thus

$$\begin{cases} \partial_\theta u(x,\theta) - \mathcal{L}u(x,\theta) = 0, & (x,\theta) \in (x_1,x_2)\times(\theta_0,T], \\ u(x,\theta_0) = (e^x - K), & x \in (x_1,x_2). \end{cases}$$

Hence

$$\begin{aligned}\partial_\theta u|_{\theta=\theta_0} &= -\frac{1}{2}\gamma(1-\rho^2)c^2e^{2x}+(b-\rho c\lambda)e^x \\ &< \Big[(b-\rho c\lambda)-\frac{1}{2}\gamma(1-\rho^2)c^2e^x\Big]e^{x_1} \\ &< -\frac{1}{2}\gamma(1-\rho^2)c^2e^{2x_1}+(b-\rho c\lambda)e^{x_1}\le 0, \end{aligned}\tag{35}$$

which contradicts (29), the two inequalities in (35) is due to (34). □

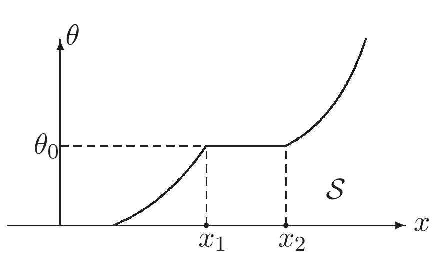

Fig. 3. No strict monotonicity of $S(x)$

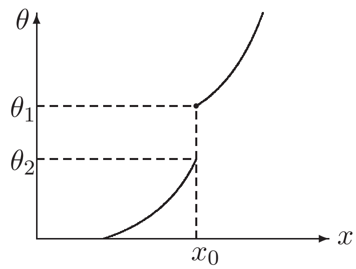

Fig. 4. Discontinuity of $S(x)$

Lemma 9 *The free boundary $S(x)$ is continuous on the set $\{x : 0 < S(x) < T\}$.*

PROOF: Suppose not (see Fig. 4). There would exist an x_0 such that

$$S(x_0) := \theta_1 > \lim_{x\to x_0^-} S(x) := \theta_2.$$

Then we would have

$$\partial_\theta u(x_0,\theta) = \partial_{x\theta}u(x_0,\theta) = 0, \quad \theta\in(\theta_2,\theta_1).$$

Since $\partial_\theta u \ge 0$ and in the domain $(x_0-\varepsilon, x_0)\times(\theta_2,\theta_1)$

$$\partial_\theta(\partial_\theta u) - \frac{1}{2}c^2\partial_{xx}(\partial_\theta u) - \Big[b-\rho c\lambda - \frac{1}{2}c^2 - \gamma(1-\rho^2)c^2\partial_x u\Big]\partial_x(\partial_\theta u) = 0.$$

Applying Hopf's Lemma [5] we obtain

$$\partial_{x\theta}u(x_0,\theta) < 0 \ \text{ or } \ \partial_\theta u \equiv 0, \ \ (x,\theta)\in(x_0-\varepsilon,x_0)\times(\theta_2,\theta_1),$$

which results in a contradiction. □

Since $S(x)$ is strictly increasing with respect to x on the set $\{x : 0 < S(x) < T\}$, then there exists an inverse function of $S(x)$, we denote it as $s(\theta) = S^{-1}(x),\ 0<\theta<T$. Note that the strictly monotonicity of $S(x)$ is equivalent to the continuity of $s(\theta)$, and the continuity of $S(x)$ is equivalent to the strictly monotonicity of $s(\theta)$. Owing to Lemma 8 and Lemma 9, we give the following theorem.

Lemma 10 *There exists an optimal exercising boundary $s(\theta) : (0,T] \to \mathbb{R}$ such that*

$$\mathcal{S} = \{(x,\theta)\in\mathcal{Q} : x \ge s(\theta)\}. \tag{36}$$

Moreover, $s(\theta)$ is strictly increasing with respect to θ and

$$s(\theta) \geq x_0, \tag{37}$$

$$s(0) := \lim_{\theta \to 0^+} s(\theta) = x_0, \tag{38}$$

where $x_0 = \begin{cases} \ln K, & b - \rho c\lambda \leq 0, \\ \max\{\ln K,\ \ln \frac{2(b-\rho c\lambda)}{\gamma(1-\rho^2)c^2}\}, & b - \rho c\lambda > 0. \end{cases}$ *In particular, $\partial_\theta u$ is continuous across $s(\theta)$ and $s(\theta) \in C[0,T] \cap C^\infty(0,T]$.*

Proof: We can define

$$s(\theta) = \begin{cases} S^{-1}(x), & 0 < \theta < T, \\ \inf\{x : S(x) = T\}, & \theta = T. \end{cases}$$

Equation (36) is the consequence of the definition and monotonicity of $S(x)$. According to (34) in the proof of Lemma 8, we know (37) is true. Lemma 8 implies $s(\theta)$ is monotonic increasing with respect to θ, then we can define $s(0) := \lim_{\theta \to 0^+} s(\theta)$, and the proof of (38) is similar to the proof of strictly monotonicity of $S(x)$ in Lemma 8.

Moreover, since $\partial_\theta u \geq 0$ and $(e^x - K)^+$ is a lower obstacle, from [6] we know $\partial_\theta u$ is continuous across $s(\theta)$ and $s(\theta) \in C^\infty(0,T]$. $\square$

4.3 Main Result: Characterization

Now, we are ready to present the complete characterization of the value function V and the indifference price P.

Theorem 11 *1. The indifference price of (7) is independent of the initial capital w, and $P(w,y,t) := P(y,t)$ is the unique $W^{2,1}_{p,loc}(\mathcal{Q}^2) \cap C(\overline{\mathcal{Q}^2})$ solution of the variational inequality (13)-(14) satisfying $|\partial_y P(y,t)| < C$ for some constant C.*

2. The value function of (6) has the form of $V(w,y,t) = U_t(w + P(y,t))$, and is the unique solution of (9)-(10) in $W^{2,2,1}_{p,loc}(\mathcal{Q}^1) \cap C(\overline{\mathcal{Q}^1})$ satisfying

$$|\partial_y V(w,y,t)| < C|V(w,y,t)|. \tag{39}$$

3. There exists C^∞ function $y^* : [0,T) \to \mathbb{R}$, such that the strategy defined by

$$\begin{cases} \pi^*_\nu = -\frac{\rho c}{\sigma} \cdot Y_\nu \partial_y P(Y_\nu, \nu) + \frac{\lambda}{\sigma\gamma}, & \nu \in (t, \tau^*), \\ \tau^* = \inf\{\nu > t : Y_\nu \geq y^*(\nu)\} \wedge T, \end{cases}$$

is optimal of the control problem (6), in the sense that

$$V(w,y,t) = \mathbb{E}[U_{\tau^*}(W^*_{\tau^*} + g(Y^{t,y}_{\tau^*}))|\mathcal{F}_t],$$

where $W^*_{t_1} = w + \int_t^{t_1} \pi^*_\nu \sigma(\lambda d\nu + dB_\nu)$.

Proof: By the fact $y\partial_y f(y,t) = \partial_x u(x,\tau)$ and estimates of (28), we conclude $|\partial_y f(y,t)| \leq C$ for some constant C. Applying Lemma 6, Lemma 2, together with Lemma 1 in order, we conclude (1)-(2) of Theorem 11. Thanks to the representations of (12) and (15), the optimal control can be written as

$$\begin{cases} \pi^*_\nu = -\frac{\rho c}{\sigma} \cdot Y_\nu \partial_y P(Y_\nu, \nu) + \frac{\lambda}{\sigma\gamma}, \quad \nu \in (t, \tau^*), \\ \tau^* = \inf\{\nu > t : (W^*_\nu, Y_\nu, \nu) \notin \mathcal{C}[V]\} \wedge T. \end{cases}$$

Note that the continuation region of (8) satisfies

$$\begin{aligned} \mathcal{C}[V] &= \{(w,y,t) \in \mathcal{Q}^1 : V(w,y,t) > U_t(w + g(y))\} \\ &= \{(w,y,t) \in \mathcal{Q}^1 : U_t(w + P(y,t)) > U_t(w + g(y))\} \\ &= \{(w,y,t) \in \mathcal{Q}^1 : P(y,t)) > g(y)\}. \end{aligned}$$

Therefore, the obstacles of P and V are the same, and the optimal stopping time τ^* can be written invariant to W^*:

$$\tau^* = \inf\{\nu > t : P(Y_\nu, \nu) \leq g(Y_\nu)\} \wedge T.$$

Take $y^*(t) = e^{s(T-t)}$, where $s(\cdot)$ is the function in Lemma 10. Thanks to the the result of Lemma 10, together with the transformation between $P(y,t)$ and $u(x,\tau)$, we conclude the representation of τ^*. $\square$

5 Dual Representation and Ramifications

In this part, we will give an alternative proof of the result on dual representation of the indifference price given in Proposition 7 of [10]. Thanks to the characterization Theorem 11, the proof based on the stochastic control approach is rather straightforward.

Now we recall the market with stock prices given by (1) and (2). Let $\mathcal{Z}$ be the collection of all the processes of the form

$$Z^\varphi_t = \exp\Big\{-\frac{1}{2}\lambda^2 t - \lambda B_t\Big\} \exp\Big\{-\frac{1}{2}\int_0^t \varphi_\nu^2 d\nu - \int_0^t \varphi_\nu d\widetilde{B}_\nu\Big\},$$

where φ is some $\mathcal{F}_\nu$ progressively measurable process satisfying $\int_0^T \varphi_\nu^2 d\nu < \infty$. Then, the collection of equivalent local martingale measures Λ can be written by

$$\Lambda = \{\mathbb{Q}^\varphi : d\mathbb{Q}^\varphi = Z^\varphi_T d\mathbb{P},\ Z^\varphi \in \mathcal{Z}\}.$$

Among of them, we refer $\mathbb{Q}^0$ to the minimal martingale measure (MMM). Under $\mathbb{Q}^\varphi$, two processes

$$B^\lambda_t := B_t + \lambda t \quad \text{and} \quad \widetilde{B}^\varphi_t := \widetilde{B}_t + \int_0^t \varphi_\nu d\nu \tag{40}$$

are both standard Brownian motions. Let $\mathbb{Q}^\varphi_t$ and $\mathbb{P}_t$ be the probability measures of $\mathbb{Q}^\varphi$ and $\mathbb{P}$ restricted to $\mathcal{F}_t$. Then, we have

$$d\mathbb{Q}^\varphi_t = Z^\varphi_t d\mathbb{P}_t.$$

To proceed, we introduce the following concept. The relative entropy of a probability measure $\mathbb{P}_1$ with respect to $\mathbb{P}_2$ is defined by

$$H(\mathbb{P}_1|\mathbb{P}_2) = \begin{cases} \mathbb{E}^{\mathbb{P}_1}\Big[\log \dfrac{d\mathbb{P}_1}{d\mathbb{P}_2}\Big], & \mathbb{P}_1 << \mathbb{P}_2, \\ \infty, & \text{otherwise.} \end{cases}$$

Example 1 The relative entropy $\mathbb{Q}^\varphi$ with respect to $\mathbb{P}$ is

$$\begin{aligned} H(\mathbb{Q}^\varphi|\mathbb{P}) &= \mathbb{E}^{\mathbb{Q}^\varphi}\Big[\log \frac{d\mathbb{Q}^\varphi}{d\mathbb{P}}\Big] = \mathbb{E}^{\mathbb{Q}^\varphi}[\log Z_T^\varphi] \\ &= \mathbb{E}^{\mathbb{Q}^\varphi}\Big[-\frac{1}{2}\lambda^2 T - \lambda B_T - \frac{1}{2}\int_0^T \varphi_\nu^2 d\nu - \int_0^T \varphi_\nu d\widetilde{B}_\nu\Big] \\ &= \frac{1}{2}\lambda^2 T + \mathbb{E}^{\mathbb{Q}^\varphi}\Big[\frac{1}{2}\int_0^T \varphi_\nu^2 d\nu\Big]. \end{aligned}$$

Example 2 Given a stopping time $\tau \in \mathcal{T}_{0,T}$, the relative entropy $\mathbb{Q}_\tau^\varphi$ with respect to $\mathbb{Q}_\tau^0$ is

$$\begin{aligned} H(\mathbb{Q}_\tau^\varphi|\mathbb{Q}_\tau^0) &= \mathbb{E}^{\mathbb{Q}^\varphi}\Big[\log \frac{d\mathbb{Q}_\tau^\varphi}{d\mathbb{Q}_\tau^0}\Big] = \mathbb{E}^{\mathbb{Q}^\varphi}\Big[\log \frac{Z_\tau^\varphi}{Z_\tau^0}\Big] \\ &= \mathbb{E}^{\mathbb{Q}^\varphi}\Big[-\frac{1}{2}\int_0^\tau \varphi_\nu^2 d\nu - \int_0^\tau \varphi_\nu d\widetilde{B}_\nu\Big] \\ &= \mathbb{E}^{\mathbb{Q}^\varphi}\Big[\frac{1}{2}\int_0^\tau \varphi_\nu^2 d\nu\Big]. \end{aligned}$$

Proposition 12 *The indifference price $P(w, y, t)$ of (7) admits following dual representation for all $(w, y) \in \mathbb{R} \times \mathbb{R}^+$,*

$$P(w, y, 0) = \text{esssup}_{\tau \in \mathcal{T}_{0,T}} \text{essinf}_{\mathbb{Q}^\varphi \in \Lambda} \Big\{ \mathbb{E}^{\mathbb{Q}^\varphi}[g(Y_\tau^{y,0})] + \frac{1}{\gamma} H(\mathbb{Q}_\tau^\varphi|\mathbb{Q}_\tau^0) \Big\}. \tag{41}$$

Proof: Thanks to Example 2, it is enough to show that, for all $t \in [0, T]$

$$P(w, y, t) = J(y, t) := \text{esssup}_{\tau \in \mathcal{T}_{t,T}} \text{essinf}_{\varphi \in \Phi} \Big\{ \mathbb{E}_t^{\mathbb{Q}^\varphi}\Big[g(Y_\tau^{y,t}) + \frac{1}{2\gamma}\int_t^\tau \varphi_\nu^2 d\nu\Big] \Big\},$$

where Φ is the collection of all $\mathcal{F}_\nu$ progressively measurable process satisfying $\int_0^T \varphi_\nu^2 d\nu < \infty$, and $\mathbb{E}_t^{\mathbb{Q}^\varphi}[\,\cdot\,] = \mathbb{E}^{\mathbb{Q}^\varphi}[\,\cdot\,|\mathcal{F}_t]$.

Recall that in (40), both B^λ and $\widetilde{B}^\varphi$ are Brownian motions under $\mathbb{Q}^\varphi$. We can rewrite the process Y of (1) in terms of B^λ and $\widetilde{B}^\varphi$,

$$\begin{aligned} dY_\nu &= (b - c\rho\lambda - c\sqrt{1-\rho^2}\varphi_\nu)Y_\nu d\nu + c\rho Y_\nu dB_\nu^\lambda + c\sqrt{1-\rho^2} Y_\nu d\widetilde{B}_\nu^\varphi \\ &= (b - c\rho\lambda - c\sqrt{1-\rho^2}\varphi_\nu)Y_\nu d\nu + cY_\nu dB_\nu^{\lambda,\varphi}, \end{aligned} \tag{42}$$

where $B_\nu^{\lambda,\varphi}$ is another $\mathbb{Q}^\varphi$-Brownian motion.

Since $\mathbb{Q}^\varphi \sim \mathbb{P}$ for arbitrary $\varphi \in \Phi$,

$$\int_0^T \varphi_\nu^2 d\nu < \infty \ \mathbb{P}\text{-almost surely}$$

is equivalent to

$$\int_0^T \varphi_\nu^2 d\nu < \infty \ \mathbb{Q}^\varphi\text{-almost surely}$$

and vice versa. In other words, there exists an 1-1 map $\Gamma : \Phi \mapsto \Phi$, such that the distribution of $\Gamma(\varphi)$ under $\mathbb{P}$ is equal to the distribution of φ under $\mathbb{Q}^\varphi$. Define

$$dY_\nu^\varphi = (b - c\rho\lambda - c\sqrt{1-\rho^2}\varphi_\nu)Y_\nu^\varphi d\nu + cY_\nu^\varphi dB_\nu. \tag{43}$$

Comparing (43) with (42), we conclude Y under $\mathbb{Q}^\varphi$ is equal to $Y^{\Gamma(\varphi)}$ under $\mathbb{P}$ in distribution. Therefore, we write $J(y,t)$ as a standard control problem of the following form,

$$\begin{aligned} J(y,t) &= \operatorname{esssup}_{\tau\in\mathcal{T}_{t,T}} \operatorname{essinf}_{\varphi\in\Phi} \left\{ \mathbb{E}_t\Big[g(Y_\tau^{\Gamma(\varphi),y,t}) + \frac{1}{2\gamma}\int_t^\tau \Gamma(\varphi)_\nu^2 d\nu \Big] \right\} \\ &= \operatorname{esssup}_{\tau\in\mathcal{T}_{t,T}} \operatorname{essinf}_{\widetilde{\varphi}\in\Phi} \left\{ \mathbb{E}_t\Big[g(Y_\tau^{\widetilde{\varphi},y,t}) + \frac{1}{2\gamma}\int_t^\tau \widetilde{\varphi}_\nu^2 d\nu \Big]. \right\} \end{aligned} \tag{44}$$

The second equality of (44) follows from the fact $\Phi = \Gamma(\Phi)$.

Applying exactly the same procedure of the verification theorem Lemma 1, we can conclude $J(y,t) = v(y,t)$, provided that there exists $v \in W_{3,loc}^{2,1}(\mathcal{Q}^2)$ solves

$$\begin{cases} \min\Big\{ -\partial_t v - \frac{1}{2}c^2y^2\partial_{yy}v - (b-c\rho\lambda)y\partial_y v + \sup_{\varphi\in\mathbb{R}}\Big\{ c\sqrt{1-\rho^2}y\varphi\partial_y v - \frac{1}{2\gamma}\varphi^2 \Big\}, \\ \qquad\qquad\qquad\qquad\qquad\qquad v(y,t) - g(y) \Big\} = 0, \ (y,t)\in\mathcal{Q}^2, \\ v(y,T) = g(y), \ y \in \mathbb{R}^+. \end{cases} \tag{45}$$

Thanks to Theorem 11(1), $P(w,y,t) = P(y,t) \in W_{3,loc}^{2,1}(\mathcal{Q}^2)$ solves (13)-(14). By utilizing the quadratic structure of $\sup_\varphi\{\,\cdot\,\}$ in (45), one can check $P(y,t)$ also solves (45) . Therefore, $P(y,t) = v(y,t) = J(y,t)$. $\square$

How does the price $P(y,t)$ change, if we scale its payoff g by n times, or if we change the risk aversion parameter γ? As a result of the dual representation Proposition 12, we have the following properties on the price $P(y,t)$.

Proposition 13 *$P(y,t)$ decreases with respect to γ and λ, and increases with respect to b, satisfying*

$$nP[g](y,t) \geq P[ng](y,t) \ \ (n \geq 1), \tag{46}$$

where $P[\varphi]$ stands for the indifference price with the payoff function φ.

Proof: Note that, Proposition 12 remains valid for the $P[ng]$, if we change g into ng in (41). Since $H(\mathbb{Q}_\tau^\varphi|\mathbb{Q}_\tau^0)$ is non-negative, the monotonicity in γ and the non-linearity of (46) follow directly from Proposition 12.

Suppose Y_i $(i = 1,2)$ are two processes of (43) related to (b_i, λ_i). If $b_1 \geq b_2$ and $\lambda_1 \leq \lambda_2$, then $Y_1 \geq Y_2$ almost surely by comparison result of stochastic differential equation, and (44) implies $P_1 \geq P_2$, where P_i are the associated prices. $\square$

An alternative proof of Proposition 13 using a PDE approach is given in the Appendix, and it is interesting as its own right. In fact, Proposition 13 reveals natural economic facts. For instance, since γ can be interpreted as the absolute risk aversion coefficient, proposition 13 claims that the employee's risk preference directly affect his exercise behavior. It implies a more prudent agent (with a larger coefficient of risk aversion γ) would be likely to exercise the option earlier to realize a cash benefit, and then invest it in other asset to earn the time value of the money obtained from the exercise of the option. So when he exercises, he gets less value than the risky agent, hence the value function decreases w.r.t. γ. In addition, (46) implies that if the agent owns n ($n > 1$) pieces of options, compare with owning one piece of the option, the agent care less about the value of every piece of the option. So the average value of the indifference price about these pieces of options is less than the indifference price about one piece of this option.

As a straightforward consequence of (27) and (28), we present the following estimates on $P(y,t)$ and $V(w,y,t)$.

Proposition 14 *$P(y,t)$ and $V(w,y,t)$ satisfy*

$$(y-K)^+ \le P(y,t) \le k(T-t) + (\ln y)^2 + ye^{(b-\rho c\lambda)^+(T-t)} + 1, \tag{47}$$

$$0 \le \partial_y P(y,t) \le e^{(b-\rho c\lambda)^+(T-t)}, \tag{48}$$

$$-e^{-\gamma(w+(y-K)^+)+\frac{1}{2}\lambda^2 t} \le V \le -e^{-\gamma(w+k(T-t)+(\ln y)^2+ye^{(b-\rho c\lambda)^+(T-t)}+1)+\frac{1}{2}\lambda^2 t}, \tag{49}$$

$$0 \le \partial_y V(w,y,t) \le -\gamma e^{(b-\rho c\lambda)^+(T-t)} V(w,y,t), \tag{50}$$

where k is defined in Lemma 4.

6 Application to ESO Costs

In this part, we briefly describe the application of the above result to Employee stock options (ESO). An ESO is a call option on the common stock of a company, issued as a form of non-cash compensations. Compared to the American call, the main differences of ESO are the vesting period and job termination risk.

Suppose the company's stock price follows Y of (1), which is non-tradable in the market. Given a unit of ESO with maturity T, strike K, and vesting period $t_v \in (0,T)$, its payoff is equivalent to the conventional American call only if the exercise time occurs between t_v and T, otherwise zero. Therefore, we can write its payoff as

$$(Y_\tau - K)^+ I_{\{\tau \ge t_v\}}$$

for the exercise time $\tau \in \mathcal{T}_{0,T}$. In addition, if the job termination is taken into account, we have the following revised payoff,

$$(Y_{\tau\wedge\tau^\alpha} - K)^+ I_{\{\tau\wedge\tau^\alpha \ge t_v\}},$$

where τ^α is the time of employee's job termination. We assume τ^α is a random variable having exponential distribution with parameter $\alpha > 0$, which is independent of Brownian motions B, and $\widetilde{B}$.

As suggested by FASB rules, the price of ESO is evaluated by risk-neutral measure, under which stock price Y is a martingale. For simplicity, we assume $b = 0$ in (1), and $\mathbb{P}$ is the risk-neutral measure specified above. Following the arguments of [11, Section 5], the indifference price has a representation given as follows.

1. When ESO survives throughout the vesting period, that is, if $t \geq t_v$, then the ESO cost $C(\cdot)$ satisfies

$$
\begin{aligned}
C(y,t) &= \mathbb{E}^{\bar{Q}}[(Y_{\tau^* \wedge \tau^\alpha} - K)^+] \\
&= \mathbb{E}^{\bar{Q}}[e^{-\alpha(\tau^*-t)}(Y_{\tau^*} - K)^+ + \int_t^{\tau^*} \alpha (Y_\nu - K)^+ e^{-\alpha(\nu - t)} d\nu],
\end{aligned}
$$

where $\tau^* = \inf\{\nu > t : Y_\nu \geq y^*(\nu)\} \wedge T$ is optimal exercise time in Theorem 11. This corresponds to the following PDE characterization: $C(y,t)$ is the unique $C^{2,1}$ solution of

$$
\begin{cases}
\partial_t C + \frac{1}{2} c^2 y^2 \partial_{yy} C - \alpha C + \alpha (y-K)^+ = 0, \quad (y,t) \in \mathcal{Q}^2 \cap \{y < y^*(t)\} \cap \{t_v \leq t < T\}, \\
C(y,t) = (y-K)^+, \quad (y,t) \in \mathcal{Q}^2 \cap \{y \geq y^*(t)\} \cap \{t_v \leq t < T\}, \\
C(y,T) = (y-K)^+, \quad y \in \mathbb{R}^+,
\end{cases}
\tag{51}
$$

since the boundary curve $y^*(t)$ is smooth due to Theorem 11 and PDE is non-degenerate locally.

2. If $t < t_v$ and ESO is alive at the moment, then the ESO cost is

$$
C(y,t) = \mathbb{E}^{\bar{Q}}[C(Y_{t_v}, t_v) I_{\{\tau^\alpha > t_v\}}] = \mathbb{E}^{\bar{Q}}[e^{-\alpha(t_v - t)} C(Y_{t_v}, t_v)].
$$

Thus, $C(y,t)$ follows,

$$
\partial_t C + \frac{1}{2} c^2 y^2 \partial_{yy} C - \alpha C = 0, \quad (y,t) \in \mathbb{R}^+ \times [0, t_v),
$$

with terminal condition $C(y, t_v)$ given by the solution of (51).

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A Appendix

A.1 A Generalized Itô Formula

We present a generalized Itô formula to the functions in Sobolev spaces which is an extension of the result in [8]. We thank Prof. Krylov for confirming this result by email communication.

Proposition 15 *Let X be a diffusion on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F} = \{\mathcal{F}_t\})$, with generator L, initial time s, and initial state $x \in \mathbb{R}^d$. Suppose $v \in W^{2,1}_{d+1,loc}(Q)$ for some open set $Q \subset \mathbb{R}^{d+1}$. Define*

$$
\tau_Q^{s,x} = \inf\{r > s : X_r \notin Q\}.
$$

Then, for any $\mathbb{F}$-stopping time $\tau \le \tau_Q^{x,s}$, we have

$$\mathbb{E}[v(X_\tau,\tau)] = v(x,s) + \mathbb{E}\Big[\int_s^\tau Lv(X_r,r)dr\Big]. \tag{52}$$

Proof: For any $\mathbb{F}$-stopping time $\tau \le \tau_Q^{s,x}$, by Theorem 2.10.2 of [8], we have

$$\mathbb{E}[v(X_\tau,\tau)] \le v(x,s) + \mathbb{E}\Big[\int_s^\tau Lv(X_r,r)dr\Big]. \tag{53}$$

If we apply (53) to a function $v_1 = -v$, then

$$\mathbb{E}[v_1(X_\tau,\tau)] \le v_1(x,s) + \mathbb{E}\Big[\int_s^\tau Lv_1(X_r,r)dr\Big],$$

which yields

$$\mathbb{E}[v(X_\tau,\tau)] \ge v(x,s) + \mathbb{E}\Big[\int_s^\tau Lv(X_r,r)dr\Big]. \tag{54}$$

From (53) and (54), we conclude equality holds for (53). $\Box$

A.2 Proof of Proposition 13

One can conclude the result of Proposition 13 by the following two lemmas.

Lemma 16 *The solution $u(x,\theta)$ to the problem (16) decreases with respect to γ and λ, and increases with respect to b.*

Proof: We may prove all three monotonicities in the same way by comparison principle. Hence we only present the proof of the monotonicity of $u(x,\theta)$ with respect to γ. Suppose $\gamma_1 > \gamma_2$, and $u^N_{\varepsilon(i)}(x,\theta)(i=1,2)$ is the solution to the following problem

$$\begin{cases} \partial_\theta u^N_{\varepsilon(i)} - \frac{1}{2}c^2\partial_{xx}u^N_{\varepsilon(i)} - (b-\rho c\lambda - \frac{1}{2}c^2)\partial_x u^N_{\varepsilon(i)} & \\ +\frac{1}{2}\gamma_i(1-\rho^2)c^2(\partial_x u^N_{\varepsilon(i)})^2 + \beta_\varepsilon(u^N_{\varepsilon(i)} - \pi_\varepsilon(\cdot)) = 0, & (x,\theta)\in\mathcal{Q}_N, \\ \partial_x u^N_{\varepsilon(i)}(-N,\theta) = 0,\ \partial_x u^N_{\varepsilon(i)}(N,\theta) = e^N, & \theta\in(0,T], \\ u^N_{\varepsilon(i)}(x,0) = \pi_\varepsilon(e^x - K), & x\in(-N,N), \end{cases}$$

Set $w(x,\theta) := u^N_{\varepsilon(1)}(x,\theta) - u^N_{\varepsilon(2)}(x,\theta)$. Then $w(x,\theta)$ satisfies

$$\begin{aligned} \partial_\theta w - \frac{1}{2}c^2\partial_{xx}w - (b-\rho c\lambda - \frac{1}{2}c^2)\partial_x w + \frac{1}{2}\gamma_1(1-\rho^2)c^2(\partial_x u^N_{\varepsilon(1)} + \partial_x u^N_{\varepsilon(2)})\partial_x w \\ +\beta'_\varepsilon(\cdot)w = \frac{1}{2}(\gamma_2-\gamma_1)(1-\rho^2)c^2(\partial_x u^N_{\varepsilon(2)})^2 \le 0. \end{aligned}$$

Combining with the initial and boundary conditions, we have

$$u^N_{\varepsilon(1)}(x,\theta) - u^N_{\varepsilon(2)}(x,\theta) = w(x,\theta) \le 0.$$

Letting $\varepsilon \to 0,\ N \to +\infty$, we know $u(x,\theta)$ is decreasing w.r.t. γ. $\Box$

Lemma 17 *The solution to problem (16) satisfies*

$$nu[\widetilde{g}(x)] \geq u[n\widetilde{g}(x)] \;\; (n \geq 1), \tag{55}$$

where $\widetilde{g}(x) = (e^x - K)^+$, and $u[\widetilde{g}]$ represents the solution to problem (16) with the obstacle and initial condition $\widetilde{g}$.

PROOF: Set $\widetilde{u}(x,\theta) := nu[\widetilde{g}(x)]$, then $\widetilde{u}(x,\theta)$ satisfies

$$\begin{cases} \min\{\partial_\theta \widetilde{u} - \mathcal{L}\widetilde{u} - \frac{1}{2}\gamma n(n-1)(1-\rho^2)c^2(\partial_x u[\widetilde{g}])^2, \; \widetilde{u} - n\widetilde{g}\} = 0, \\ \widetilde{u}(x,0) = n\widetilde{g}(x). \end{cases}$$

Note that $u[n\widetilde{g}(x)]$ satisfies

$$\begin{cases} \min\{(\partial_\theta u - \mathcal{L}u)(x,\theta), \; u - n\widetilde{g}\} = 0, \\ u(x,0) = n\widetilde{g}(x), \end{cases}$$

We can confine the above problems in the bounded domain $\mathcal{Q}_N$. Suppose $\widetilde{u}^N[\widetilde{g}]$ and $u^N[n\widetilde{g}]$ are the solutions of the following problems

$$\begin{cases} \min\big\{\partial_\theta \widetilde{u}^N[\widetilde{g}] - \mathcal{L}\widetilde{u}^N[\widetilde{g}] - \frac{1}{2}\gamma n(n-1)(1-\rho^2)c^2(\partial_x u^N[\widetilde{g}])^2, \\ \qquad\qquad\qquad\qquad \widetilde{u}^N[\widetilde{g}] - n\widetilde{g}(x)\big\} = 0, \; (x,\theta) \in \mathcal{Q}_N, \\ \partial_x \widetilde{u}^N[\widetilde{g}](-N,\theta) = 0, \; \partial_x \widetilde{u}^N[\widetilde{g}](N,\theta) = ne^N, \; \theta \in (0,T], \\ \widetilde{u}^N[\widetilde{g}](x,0) = n\widetilde{g}(x), \; x \in (-N,N). \end{cases} \tag{56}$$

$$\begin{cases} \min\big\{\partial_\theta u^N[n\widetilde{g}] - \mathcal{L}u^N[n\widetilde{g}], \; u^N[n\widetilde{g}] - n\widetilde{g}(x)\big\} = 0, \; (x,\theta) \in \mathcal{Q}_N \\ \partial_x u^N[n\widetilde{g}](-N,\theta) = 0, \; \partial_x u^N[n\widetilde{g}](N,\theta) = ne^N, \; \theta \in (0,T], \\ u^N[n\widetilde{g}](x,0) = n\widetilde{g}(x), \; x \in (-N,N). \end{cases} \tag{57}$$

Comparing (56) with (57), letting $N \to \infty$, we obtain (55). □

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