A note on super-hedging for investor-producers

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Abstract

We study the situation of an agent who can trade on a financial market and can also transform some assets into others by means of a production system, in order to price and hedge derivatives on produced goods. This framework is motivated by the case of an electricity producer who wants to hedge a position on the electricity spot price and can trade commodities which are inputs for his system. This extends the essential results of [2] to continuous time markets. We introduce the generic concept of *conditional sure profit* along the idea of the *no sure profit* condition of Rásonyi [17]. The condition allows one to provide a closedness property for the set of super-hedgeable claims in a very general financial setting. Using standard separation arguments, we then deduce a dual characterization of the latter and provide an application to power futures pricing.

Key words : arbitrage pricing theory, markets with proportional transaction costs, non-linear returns, super replication theorem, electricity markets, energy derivatives.

1 Introduction

The recent deregulation of electricity markets in many countries has opened a new range of applications for financial techniques in order to hedge energy risks. However, the non-storability of electricity forbids any trading strategy based on the spot price and the standard mathematical toolbox cannot be exploited to hedge and price derivative products upon this asset. The challenge must still be taken up for electricity producers who are endowed with such claims. It also concerns financial agents possessing a power plant, as an asset for diversification purposes. These economical agents can produce power out of a storable commodity, and sell it to benefit from electricity prices variation. Hence, they perform a sort of financial strategy, as studied in [1] where the production is a structural function of electricity demand. Here, our goal is to study the general situation of an agent who can trade financial assets and inputs for his production system, and who can transform a position into an other by the mean of a production system he controls.

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As in [2], the reasoning is the following. In the framework of purely financial portfolios, Arbitrage Pricing Theory ensures by an economical assumption, the no-arbitrage condition, a closedness property for the set of attainable terminal wealth for self financing portfolios. This key property has direct applications, such as a dual formulation, which provides an equivalent martingale measure for pricing purposes. In our particular framework, if the financial market runs as usual, production is not bound up with any particular economical condition : it is an idiosyncratic action of the agent. We thus propose in this note a general constraint upon the production possibilities of the agent in order to apply arbitrage pricing techniques. In practice, the additional condition is calibrated to market data and the producer's activity. In theory, this condition implies the closedness property of the set of attainable terminal positions, as it is sought in the purely financial case. This property allows to display many financial techniques, such as risk measures or portfolio optimization. The purpose of this note is to demonstrate and apply the undermentioned super-replication theorem for the investor-producer. If we denote $\mathfrak{X}_0^R(T)$ the set of possible portfolio outcomes at time T that the investor-producer can reach starting from 0 at time 0, and $\mathcal{M}$ the set of *pricing measures* for the financial market model, we thus show afterwards the following result:

Theorem 1.1. *Let H be a contingent claim. Then*

$$H \in \mathfrak{X}_0^R(T) \iff \mathbb{E}\left[Z_T' H\right] \leq \alpha_0^R(Z),\ \forall Z \in \mathcal{M}$$

where $\alpha_0^R(Z) := \sup\left\{\mathbb{E}\left[Z_T' V_T\right] \ : \ V_T \in \mathfrak{X}_0^R(T)\right\}$ is the support function of $Z \in \mathcal{M}$ on $\mathfrak{X}_0^R(T)$.

The usual interpretation is that a contingent claim is replicable with a strategy starting from nothing at time 0 if and only if the expectation with respect to a pricing measure $Z \in \mathcal{M}$ always verifies a given bounding condition. The paper is thus structured around that theorem as follows. In Section 2, we introduce properly the entities $\mathfrak{X}_0^R(T)$, $\mathcal{M}$ and H. In Section 3, we propose the economical condition under which Theorem 1.1 holds. In Section 4, we give an application to Theorem 1.1. Section 5 is dedicated to the proof of Theorem 1.1.

This problem has actually been explored in a discrete time framework for markets with proportional transaction costs in [2]. In the latter, the authors propose to extend the no-arbitrage of second kind condition of Rásonyi [17] to portfolios augmented by a linear production system. A condition for general production functions, the *no marginal arbitrage for high production regime* condition, has then been introduced using the extended condition above in order to allow marginal arbitrages for reasonable levels of production. In the present note, we push forward this study by proposing an alternative condition which has a close economical interpretation: the *conditional sure profit* condition. Contrary to the *no marginal arbitrage condition* of [2], it deals directly with general production possibilities and avoids to introduce a linear production system. This is the contribution of Section 3. We also focus on investors-producers with specific means of production: production possibilities are in discrete time as in [2] but we additionally assume concavity and boundedness of the production function. In counterpart, our framework encompasses continuous time financial market models with and without transaction costs. This is the contribution of Section 2. The contribution of Section 4 is to apply Theorem 1.1 in order to put a price on a power futures contract for an electricity producer endowed with a simple mean of production.

General notations: throughout this note, $x \in \mathbb{R}^d$ will be viewed as a column vector with entries x^i, $i \leq d$. The transpose of a vector x will be denoted x', so that $x'y$ stands for the scalar product.

As usual, $\mathbb{R}^d_+$ and $\mathbb{R}^d_-$ stand for the positive and negative orthants of $\mathbb{R}^d$ respectively, i.e., $[0,+\infty)^d$ and $(-\infty,0]^d$. For a given probability space $(\Omega,\mathcal{G},\mathbb{P})$ and a $\mathcal{G}$-measurable random set E, $L^0(E,\mathcal{G})$ will denote the set of $\mathcal{G}$-measurable random variables taking values in E $\mathbb{P}$-almost surely, $L^1(E,\mathcal{G})$ the set of $\mathbb{P}$-integrable random variables takings values in E $\mathbb{P}$-almost surely and $L^\infty(E,\mathcal{G})$ the set of essentially bounded random variables taking values in E. The notation $\mathrm{conv}(E)$ will denote the closed convex hull of E, and $\mathrm{cone}(E)$ the closed convex cone generated by $\mathrm{conv}(E)$. All the inclusions or inequalities are to be understood in the almost sure sense unless otherwise specified.

2 The framework

We first introduce the financial possibilities of the agent. We consider an abstract setting mainly inspired by [8], which allows to deal with a very large class of market models. To illustrate our framework, we provide two examples in Section 2.4. We then introduce production possibilities for the investor.

Preamble. Let $(\Omega,\mathcal{F},\mathbb{F}=(\mathcal{F}_t)_{t\in[0,T]},\mathbb{P})$ be a continuous-time filtered stochastic basis on a finite time interval $[0,T]$ satisfying the usual conditions. We assume without loss of generality that $\mathcal{F}_0$ is trivial and $\mathcal{F}_{T^-}=\mathcal{F}_T$. For any $0\le t\le T$, let $\mathcal{T}$ denote the family of stopping times taking values in $[0,T]$ $\mathbb{P}$-almost surely. From now on, we consider a pair of set-valued $\mathbb{F}$-adapted process $\widehat{K}$ and $\widehat{K}^*$ such that $\widehat{K}_t(\omega)$ is a proper convex closed cone of $\mathbb{R}^d$ including $\mathbb{R}^d_+$, $dt\otimes d\mathbb{P}$-a.e. The process $\widehat{K}^*$ is defined by

$$\widehat{K}^*_t(\omega) := \left\{ y\in\mathbb{R}^d_+ \ :\ xy\ge 0,\ \forall x\in\widehat{K}_t(\omega)\right\} \ . \tag{2.1}$$

Since $\widehat{K}_t(\omega)$ is proper, its dual $\widehat{K}^*_t(\omega)\neq\{0\}$ $dt\otimes d\mathbb{P}$-a.s. In the literature on markets with transaction costs, $\widehat{K}_t$ usually stands for the solvency region at time t, and $-\widehat{K}_t$ for the set of possible trades at time t, see [12] and the reference therein. In practice, $\widehat{K}$ and $\widehat{K}^*$ are given by the market model we consider, see the examples of Sections 2.4 and 4. We use here the process $\widehat{K}$ to introduce a partial order on $\mathbb{R}^d$ at any stopping time in $\mathcal{T}$.

Definition 2.1. *Let $\tau\in\mathcal{T}$. For $(\xi,\kappa)\in L^0(\mathbb{R}^{2d},\mathcal{F}_\tau)$, $\xi\succeq_\tau -\kappa$ if and only if $\xi+\kappa\in L^0(\widehat{K}_\tau,\mathcal{F}_\tau)$.*

Definition 2.2. *A contingent claim is a random variable $H\in L^0(\mathbb{R}^d,\mathcal{F}_T)$ such that $H\succeq_T -\kappa$ for some $\kappa\in\mathbb{R}^d_+$.*

2.1 The set of financial positions

We consider a financial market on $[0,T]$ with d assets. The market also includes the prices of commodity entering in the production process, e.g., fuel or raw materials. The agent we consider has the possibility to trade on this market by starting a portfolio strategy at any time $\rho\in\mathcal{T}$. The financial possibilities of the agent are then represented by a family of sets of wealth processes denoted $(\mathfrak{X}^0_\rho)_{\rho\in\mathcal{T}}$. The superscript 0 stands for *no production*, or *pure financial*.

Definition 2.3. *For any $\rho\in\mathcal{T}$, the set $\mathfrak{X}^0_\rho$ is a set of $\mathbb{F}$-adapted d-dimensional processes ξ defined on $[0,T]$ such that $\xi_t=0$ $\mathbb{P}-a.s.$ for all $t\in[0,\rho)$. We denote by $\mathfrak{X}^0_\rho(T):=\left\{\xi_T\ :\ \xi\in\mathfrak{X}^0_\rho\right\}$ the corresponding set of attainable financial positions at time T.*

We do not give more details on what a financial strategy is. In all the considered examples, it will denote a self financing portfolio value as commonly defined in Arbitrage Pricing Theory. The multidimensional setting is justified by models of financial portfolios in markets with proportional transaction costs, see [12]. In that case, portfolio are expressed in physical units of assets. Just note that we implicitly assume that the initial wealth of the agent does not influence his financial possibilities, so that a portfolio generically starts with a null wealth in our setting.

Assumption 2.1. *For any $\rho \in \mathcal{T}$, the set $\mathfrak{X}^0_\rho(T)$ has the following properties:*

(i) Convexity*: $\mathfrak{X}^0_\rho(T)$ is a convex subset of $L^0(\mathbb{R}^d, \mathcal{F}_T)$ containing* 0.

(ii) Liquidation possibilities*:* $\mathfrak{X}^0_\rho(T) - L^\infty(\widehat{K}_s, \mathcal{F}_s) \subseteq \mathfrak{X}^0_\rho(T), \quad \forall s \in [\rho, T]\ \mathbb{P} - a.s.$

(iii) Concatenation*:* $\mathfrak{X}^0_\rho(T) = \left\{\xi_\sigma + \zeta_T \ :\ (\xi, \zeta) \in \mathfrak{X}^0_\rho \times \mathfrak{X}^0_\sigma,\ \text{for any } \sigma \in \mathcal{T} \text{ s.t. } \sigma \geq \rho\right\}$.

The convexity property holds in most of market models, see [12]. Assumption 2.1.(ii) means that whatever the financial position of the agent is, it is always possible for him to throw away a non-negative quantity of assets at any time, or to do an arbitrarily large transfer of assets allowed by the cone $-\widehat{K}_s$. This last possibility is made for models of markets with convex transaction costs. Finally, the concatenation property also holds in most of market models and often reveals their Markovian behaviour. Note that Assumption (2.1) (i) and (iii) imply that $\mathfrak{X}^0_\rho(T) \subset \mathfrak{X}^0_\tau(T)$ for any $(\rho, \tau) \in \mathcal{T}^2$ such that $\rho \geq \tau$.

2.2 Absence of arbitrage in the financial market

As for any investor on a financial market, we assume that our investor-producer cannot find an arbitrage opportunity. We elaborate below this condition by relying on the core result of Arbitrage Pricing Theory, which resides in the following fact, see the introduction of [8]. Formally, when the financial market prices are represented by a process S, the no-arbitrage property for the market holds if and only if there exists a stochastic deflator, i.e., a strictly positive martingale Γ such that the process $Z := \Gamma S$ is a martingale. The process Z can then be seen as the shadow price or fair price of assets. We assume that such a process Z exists by introducing the following.

Definition 2.4. *Let $\mathcal{M}$ be the set of $\mathbb{F}$-adapted martingales Z on $[0, T]$ taking values in $\widehat{K}^*$, with strictly positive components, such that*

$$\sup\left\{\mathbb{E}\left[Z'_T\xi_T\right] \ :\ \xi \in \mathfrak{X}^0_0 \ \text{ and }\ \exists\kappa \in \mathbb{R}^d_+ \text{ s.t. } \forall\tau \in \mathcal{T}, \xi_\tau \succeq_\tau -\kappa\right\} < +\infty\,. \tag{2.2}$$

In condition (2.2), we apply the pricing measure Z for the subset of $\mathfrak{X}^0_0(T)$ comprising financial wealth processes with a finite credit line κ. We need this basic concept of admissibility for portfolio processes to define $\mathcal{M}$ properly. We will extend admissibility of wealth processes in the next section. Definition 2.4 needs more comment. If the set $\mathfrak{X}^0_0(T)$ is a cone, the left hand of (2.2) is null for any $Z \in \mathcal{M}$, according to Assumption 2.1 (i). In the general non conical case, see Section 2.4.2, the support function in equation (2.2) might be positive, justifying the more general condition. If it is equal to 0 then, for any $Z \in \mathcal{M}$ and any $\xi \in \mathfrak{X}^0_0$ with a finite credit line, according to Assumption 2.1 (iii), $Z'\xi$ is a supermartingale. We then meet the common no arbitrage condition, see especially Section 2.4.1 below. We thus express absence of arbitrage on the financial market by the following assumption.

Assumption 2.2. $\mathcal{M} \neq \emptyset$.

Note that defining $\mathcal{M}$ as above is tailor-made for separation arguments, see the proof of Theorem 1.1.

2.3 Admissible portfolios and closedness property

If $d = 1$, a financial position ξ_t is naturally solvable if $\xi_t \geq 0$ $\mathbb{P}-$a.s. In the general setting with $d \geq 1$, we use the partial order on $\mathbb{R}^d$ induced by the process $\widehat{K}$. Defining solvency allows to define admissibility which is central in continuous time: the closedness property concerns the subset of $\mathfrak{X}_0^0(T)$ constituted of admissible portfolios, see [5, 3, 8, 7] and the various definitions provided therein. From a financial point of view, it imposes realistic constraints on portfolios and avoids doubling strategies. Here, we use a definition close to the one proposed in [3].

Definition 2.5. *For some constant vector $\kappa \in \mathbb{R}_+^d$, a portfolio $\xi \in \mathfrak{X}_0^0$ is said to be κ-*admissible *if $Z'_\tau \xi_\tau \geq -Z'_\tau \kappa$ for all $\tau \in \mathcal{T}$ and all $Z \in \mathcal{M}$, and $\xi_T \succeq_T -\kappa$.*

Given $\mathcal{M} \neq \emptyset$, the concept of admissibility allows to consider a wider class of terminal wealth than those considered in equation (2.2). According to Definition 2.4, a wealth process ξ is κ-admissible in the sense of Definition 2.5 if ξ verifies $\xi_\tau \succeq_\tau -\kappa$ for all $\tau \in \mathcal{T}$ and some $\kappa \in \mathbb{R}_+^d$. The reciprocal is not always true, and is the object of the so-called **B** assumption investigated in [8]. We can finally define the set of admissible elements of $\mathfrak{X}_t^0$:

Definition 2.6. *We define $\mathfrak{X}_{t,adm}^0 := \left\{\xi \in \mathfrak{X}_t^0,\ \xi \text{ is } \kappa\text{-admissible for some } \kappa \in \mathbb{R}_+^d\right\}$, and $\mathfrak{X}_{t,adm}^0(T) := \left\{\xi_T \ : \ \xi \in \mathfrak{X}_{t,adm}^0\right\}$.*

The closedness property will be assigned to the sets $\mathfrak{X}_{t,adm}^0(T)$, and is conveyed under the following technical and standing assumption:

Assumption 2.3. *For $t \in [0, T]$, let $(\xi^n)_{n\geq 1} \subset \mathfrak{X}_{t,adm}^0$ be a sequence of admissible portfolios such that $\xi_T^n \succeq_T -\kappa$ for some $\kappa \in \mathbb{R}_+^d$ and all $n \geq 1$. Then there exists a sequence $(\zeta^n)_{n\geq 1} \subset \mathfrak{X}_{t,adm}^0$ constructed as a convex combination (with strictly positive weights) of $(\xi^n)_{n\geq 1}$, i.e., $\zeta^n \in \operatorname{conv}(\xi^k)_{k\geq n}$, such that ζ_T^n converges a.s. to $\zeta_T^\infty \in \mathfrak{X}_t^0(T)$ with n.*

The above assumption calls for the notion of Fatou-convergence. Recall that a sequence of random variables is Fatou-convergent if it is bounded by below and almost surely convergent. According to Assumption 2.1 (i), $\mathfrak{X}_{t,adm}^0(T)$ is a convex set, which ensures that the new sequence lies in the set. In Arbitrage Pricing Theory, the Fatou-closedness of $\mathfrak{X}_0^0(T)$ often relies on a convergence lemma. Schachermayer [19] introduced a version of Komlos Lemma that is fundamental in [5], while Campi and Schachermayer [3] proposed another version for markets with proportional transaction costs. Assumption 2.3 expresses a synthesis of this result, see Sections 2.4.1 and 4 for applications.

2.4 Illustration of the framework by examples of financial markets

We illustrate here the theoretical framework. We treat two examples, based on [5, 6] and [16, 11] respectively. In section 4, we also apply our results to a continuous time market with càdlàg price processes and proportional transaction costs, as studied in [3].

2.4.1 A multidimensional frictionless market in continuous time

Consider a filtered stochastic basis $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ on $[0,T]$, satisfying the usual assumptions. Let S be a locally bounded $(0,\infty)^d$-valued $\mathbb{F}$-adapted càdlàg semimartingale, representing the price process of d risky assets. We suppose the existence of a non risky asset which is taken constant on $[0,T]$ without loss of generality. Let Θ be the set of $\mathbb{F}$-predictable S-integrable processes and Π the set of $\mathbb{F}$-predictable increasing processes on $[0,T]$. We define

$$\mathfrak{X}^0_\rho := \left\{ \xi = (\xi^1, 0, \ldots, 0) \ : \ \xi^1_s = \int_\rho^s \vartheta_u . dS_u - (\ell_s - \ell_{\rho^-}) \ : \ (\vartheta, \ell) \in \Theta \times \Pi, \ s \in [\rho, T] \right\} , \quad \forall \rho \in \mathcal{T} .$$

Observe that the set $\mathfrak{X}^0_\rho(T)$ is a convex cone of $\mathbb{R} \times \{0\}^{d-1}$ containing 0. The set Θ defines the financial strategies. The set Π represents possible liquidation or consumption in the portfolio. The introduction of the latter ensures Assumption (2.1) (ii), but does not infer on the mathematical treatment of [5] where Π is not considered. The set $\mathfrak{X}^0_0(T)$ also verifies Assumption (2.1) (i) and (iii).

In this context, Delbaen and Schachermayer introduced the No Free Lunch with Vanishing Risk condition (NFLVR) and proved that it is equivalent to

$$\mathcal{Q} := \{\mathbb{Q} \sim \mathbb{P} \text{ such that } S \text{ is a } \mathbb{Q} - \text{local martingale}\} \neq \emptyset \quad \text{(Theorem 1.1 in [5])} .$$

To relate the NFLVR condition to Definition 2.4, we define $\mathcal{M}$ as the set of $\mathbb{P}$-equivalent local martingale measure processes $\frac{d\mathbb{Q}}{d\mathbb{P}}\big|_{\mathcal{F}_\cdot}$ for $\mathbb{Q} \in \mathcal{Q}$. If S is a locally bounded martingale, elements of $\mathfrak{X}^0_0$ are local martingales. We now apply Definition 2.5 of admissibility. We take without ambiguity $\widehat{K} = \widehat{K}^* = \mathbb{R}^d_+$. As a consequence, a portfolio $\xi \in \mathfrak{X}^0_{0,adm}$ is κ-admissible only if $\xi^1_t \geq -\kappa$ for all $t \in [0,T]$, and we retrieve the definition of admissibility of [5]. Therefore, any admissible portfolio is a true supermartingale under $\mathbb{Q} \in \mathcal{Q}$.

By Theorem 4.2 in [5], NFLVR implies that $\mathfrak{X}^0_{0,adm}(T)$ is Fatou-closed. The proof uses the following convergence property: for any 1-admissible sequence $\xi^n \in \mathfrak{X}^0_0$, it is possible to find $\zeta^n \in \text{conv}(\xi^k)_{k \geq n}$ such that ζ^n converges in the semimartingale topology (Lemmata 4.10 and 4.11 in [5]). Hence, ζ^n_T Fatou-converges in $\mathfrak{X}^0_0(T)$. This can be easily extended to $\mathfrak{X}^0_\tau(T)$ for any $\tau \in \mathcal{T}$ and for any bound of admissibility. In this case, Definition 2.5 and the martingale property of ξ^1 imply uniform admissibility in the sense of [5]. Assumption 2.3 then holds in this context.

2.4.2 A physical market with convex transaction costs in discrete time

Let $(t_i)_{0 \leq i \leq N} \subset [0,T]$ be an increasing sequence of deterministic times with $t_N = T$. Let us consider the discrete filtration $\mathbb{G} := (\mathcal{F}_{t_i})_{0 \leq i \leq N}$. Here, the market is modelled by a $\mathbb{G}$-adapted sequence $C = (C_{t_i})_{0 \leq i \leq N}$ of closed-valued mappings $C_{t_i} : \Omega \mapsto \mathbb{R}^d$ with $\mathbb{R}^d_- \subset C_{t_i}(\omega)$ and $C_{t_i}(\omega)$ convex for every $0 \leq i \leq N$ and $\omega \in \Omega$. We define the recession cones $C^\infty_t(\omega) = \bigcap_{\alpha > 0} \alpha C_t(\omega)$ and their dual cones $C^{\infty,*}_t(\omega) = \left\{ y \in \mathbb{R}^d \ : \ xy \geq 0, \ \forall x \in C^\infty_t(\omega) \right\}$, see also [16] for a freestanding definition

This setting has been introduced in [16] to model markets with convex transaction costs, such as currency markets with illiquidity costs, in discrete time. Every financial position is labelled in physical units of the d assets, and the sets C_{t_i} denote the possible self financing changes of position at time

t_i, so that

$$\mathfrak{X}^0_{t_i}(T) := \left\{ \sum_{k=i}^{N} \xi_{t_k} \ : \ \xi_{t_k} \in L^0(C_{t_k}, \mathcal{F}_{t_k}),\ \forall i \le k \le N \right\} \text{ for all } 0 \le i \le N\ .$$

In this context, Assumption 2.1 trivially holds. If $C_{t_i}(\omega)$ is a cone in $\mathbb{R}^d$ for all $0 \le i \le N$ and $\omega \in \Omega$, i.e., $C = C^\infty$, we retrieve a market with proportional transaction costs as described in [11]. In the latter, Kabanov and al. show that the Fundamental Theorem of Asset Pricing can be expressed with respect to the *robust no-arbitrage* property, see [11] for a definition. This condition is equivalent to the existence of a martingale process Z such that $Z_{t_i} \in L^\infty(\mathrm{ri}(C^{\infty,*}_{t_i}), \mathcal{F}_{t_i})$, where $\mathrm{ri}(C^{\infty,*}_{t_i})$ denotes the relative interior of $C^{\infty,*}_{t_i}$. The super replication theorem, see Lemma 3.3.2 in [12], allows $\mathcal{M}$ given by Definition 2.4 to be characterized by such elements Z. In that case, the reader can see that C^∞ replaces our conventional cone process $\widehat{K}$.

As mentioned in [16], the case of general convex transaction costs leads to two possible definitions of arbitrage. One of them is based on the recession cone. Following the terminology of [16], the market represented by C satisfies the *robust no-scalable arbitrage* property if C^∞ satisfies the robust no-arbitrage property. This definition implies that arbitrages might exist, but they are limited for elements of $\mathfrak{X}^0_0(T)$ and even not possible for the recession cone. Pennanen and Penner [16] proved that the set $\mathfrak{X}^0_0(T)$ is closed in probability under this condition. Hence, it is Fatou-closed. The convergence result used in this context is a different argument than the one of Assumption 2.3. However, the latter can be applied, see [2] in which Assumption 2.3 has been applied in a very similar context. The notion of admissibility can also be avoided in the discrete time case.

2.5 Addition of production possibilities

The previous introduction of a financial market comes from the possibility to interpret the available assets on the market as raw material or saleable goods for a producer. Therefore, we model the production as a function transforming a consumption of the d assets in a new wealth in $\mathbb{R}^d$. Other observations from the situation of an electricity provider lead to our upcoming setting. On a deregulated electricity market, power is provided with respect to an hourly time grid. Production control can thus be fairly approximated by a discrete time framework. We also introduce a delay in the control, as a physical constraint in the production process. See [14] for a monograph illustrating these concerns.

Definition 2.7. *Let $(t_i)_{0 \le i \le N} \subset [0, T]$ be a deterministic collection of strictly increasing times. We then define a* production regime *as an element β in $\mathcal{B}$, where*

$$\mathcal{B} := \left\{ (\beta_{t_i})_{0 \le i < N} \ : \ \beta_{t_i} \in L^0(\mathbb{R}^d_+, \mathcal{F}_{t_i}),\ 0 \le i < N \right\}\ .$$

A production function *is then a collection of maps $R := (R_{t_i})_{0 < i \le N}$ such that for $0 < i \le N$, R_{t_i} is a $\mathcal{F}_{t_i}$-measurable map from $\mathbb{R}^d_+$ to $\mathbb{R}^d$, in the sense that $R_{t_i}(\beta_{t_{i-1}}) \in L^0(\mathbb{R}^d, \mathcal{F}_{t_i})$ for $\beta_{t_{i-1}} \in L^0(\mathbb{R}^d_+, \mathcal{F}_{t_{i-1}})$.*

Without loss of generality, it is also possible to consider an increasing sequence of stopping times in $\mathcal{T}$ instead of the $(t_i)_{0 \le i \le N}$. The set $\mathcal{B}$ can also be defined via sequences $(\beta_{t_i})_{0 \le i < N}$ such that β_{t_i} takes values in a convex closed subset of $\mathbb{R}^d_+$. The proofs in section 5 would be identical and we refrain from doing this. Notice also that it has no mathematical cost to consider separate times of injection and times of production, i.e., a non-decreasing sequence $\{t_0, s_0, t_1, s_1, \ldots, t_N, s_N\} \subset [0, T]$ with $t_i < s_i$, $(t_i)_{0 \le i < N}$ and $(s_i)_{0 < i \le N}$ allowing to define $\mathcal{B}$ and R respectively. As invoked in the introduction, we add fundamental assumptions on the production function.

Assumption 2.4. *The production function has the three following properties:*

(i) Concavity*: for all* $0 < i \le N$*, for all* $(\beta^1, \beta^2) \in L^0(\mathbb{R}_+^{2d}, \mathcal{F}_{t_{i-1}})$ *and* $\lambda \in L^0([0,1], \mathcal{F}_{t_{i-1}})$*,*

$$R_{t_i}(\lambda\beta^1 + (1-\lambda)\beta^2) - \lambda R_{t_i}(\beta^1) - (1-\lambda)R_{t_i}(\beta^2) \in \mathbb{R}_+^d \ \mathbb{P} - a.s.$$

(ii) Boundedness*: there exists a constant* $\mathfrak{K} \in \mathbb{R}_+^d$ *such that for all* $0 < i \le N$*,*

$$\mathfrak{K} - |R_{t_i}(\beta) - \beta| \in \mathbb{R}_+^d \ \mathbb{P} - a.s. \ , \quad \text{for all } \beta \in \mathbb{R}_+^d \ .$$

(iii) Continuity*: For any* $0 < i \le N$*, we have that* $\lim_{\beta^n \to \beta^0} R_{t_i}(\beta^n) = R_{t_i}(\beta^0)$ *.*

These assumptions are fundamental for the continuous time setting. Assumption 2.4 (i) keeps the convexity property for the set $\mathfrak{X}_0^R(T)$, see Proposition 5.1 in the proofs section. Assumption 2.4 (ii) does not only ensure the admissibility of investment-production portfolios when we add production. From the economical point of view, it affirms that the net production income is bounded, which forbids infinite profits. It thus provides a realistic framework for physical production systems. Finally, Assumption 2.4.(iii) is a technical assumption in order to use Assumption 2.3. It is only needed to ensures upper semicontinuity on the boundary of $\mathbb{R}_+^d$, since continuity comes from (i) inside the domain. See corollary 4.3 in [2], where convexity is not needed and upper semicontinuity is sufficient. Notice that concavity and the upper bound $\mathfrak{K}$ for the production incomes are given with respect to $\mathbb{R}_+^d$ and not $\widehat{K}$. This is a useful artefact in the proofs, but also a meaningful expression of a physical bound of production, which has nothing to do with a financial model. With Assumption 2.4, it is possible to fairly approximate a generation asset, see Section 4.

Definition 2.8. *The set of investment-production wealth processes starting at time t is given by*

$$\mathfrak{X}_t^R := \left\{ V \ : \ V_s := \xi_s + \sum_{i=1}^{N} R_{t_i}(\beta_{t_{i-1}} \mathbb{1}_{\{t_{i-1} \ge t\}}) \mathbb{1}_{\{t_i \le s\}} - \beta_{t_{i-1}} \mathbb{1}_{\{t \le t_{i-1} \le s\}}, \ (\xi, \beta) \in \mathfrak{X}_{t,adm}^0 \times \mathcal{B} \right\} .$$

The set of terminal possible outcomes for the investor-producer is given by $\mathfrak{X}_t^R(T) := \{V_T \ : \ V \in \mathfrak{X}_t^R\}$*.*

The agent manages his production system as follows. Assume that he starts an investment-production strategy at time t. On one hand, he performs a financial strategy given by $\xi \in \mathfrak{X}_{t,adm}^0$. On the other hand, he can decide to put a quantity of assets $\beta_{t_{i-1}}$ at time t_i into the production system if $t_{i-1} \ge t$. The latter returns a position $R_{t_i}(\beta_{t_{i-1}})$ labelled in assets at time t_i. At this time, the agent also decides the regime of production β_{t_i} for the next step of time, and so on until time reaches t_N.

The generalization to continuous time controls raises mathematical difficulties. When coming to a continuous time control, we have to make a distinction between the continuous and the discontinuous part of the control, i.e., between a regime of production as a rate and an instantaneous consumption of assets put in the production system. This natural distinction has already been observed for liquidity matters in financial markets, see [4]. This implies a separate treatment of consumption in the function R. With a continuous control and as in [4] the production becomes a linear function of that control, which is very restrictive and similar to the polyhedral cone setting of markets with proportional transaction costs. With a discontinuous control, non linearity can appear but we face two difficulties. If the number of discontinuities is bounded, it is easy to see that the set of controls is not convex. On the contrary, if it is not bounded, the set is not closed. This problem typically appears in impulse control problems and is not easy to overcome, see Chapter 7 in [15]. We ought to focus on that difficulty in another paper.

3 The conditional sure profit condition

In the situation of our agent, even if we accept no arbitrage on the financial market, there is no economical justification for the interdiction of profits coming from the production. This is the reason why the concept of *no marginal arbitrage for high production regime* has been introduced in [2] (**NMA** for short). The **NMA** condition expresses the possibility to make sure profits coming from the production possibilities, but that marginally tend to zero if the production regime β is pushed toward infinity. This condition relied on an affine bound for the production function, introducing then an auxiliary linear production function for which sure profits are forbidden. We propose another parametric condition based on the idea of possibly making solvable profits for a small regime of production. It is stronger than **NMA** under Assumption 2.4, see Remark 2.5 in [2], but we express directly the new condition with the production function.

Definition 3.1. *We say that there are only* conditional sure profits *for the production function R,* **CSP***(R) holds for short, if there exists $C > 0$ such that for all $0 \le k < N$ and for all $(\xi, \beta) \in \mathfrak{X}^0_{t_k,adm} \times \mathcal{B}$ we have:*

$$\xi_T + \sum_{i=k}^{N-1} R_{t_{i+1}}(\beta_{t_i}) - \beta_{t_i} \succeq_T \sum_{i=k}^{N-1} R_{t_{i+1}}(0)\ \mathbb{P} - a.s. \quad \Longrightarrow \quad \|\beta_{t_i}\| \le C \text{ for } k \le i < N\,.$$

The condition **CSP**(R) thus reads as follows. If the agent starts an investment-production strategy at an intermediary date $t \in (t_{k-1}, t_k]$ for some k (whatever his initial position is at t), then he can start his production at index k. We then assess that he can do better than the strategy $(0,0) \in \mathfrak{X}^0_0 \times \mathcal{B}$ only if the regime of production is bounded. **CSP**(R) comes from the following observation: coming back to the usual case of a financial market, a possible interpretation of a no-arbitrage condition is that there is no strategy which is better than the null strategy $\mathbb{P}$-a.s. We thus transpose this interpretation to production with a slight modification. Here, doing nothing implies that the agent is subject to possible fixed costs expressed by $R(0)$. Since we do not specify portfolios by an initial holding, we can focus on portfolios starting at any time before T with any initial position.

The terminology **CSP**(R) refers to the *no sure profit* property introduced by Rasonyi [17] (which became the *no sure gain in liquidation value* condition in the final version), since it is formulated in a very similar way and expresses the interdiction for sure profit if some condition is not fulfilled. The **CSP**(R) property is indeed very flexible. It is possible to change the condition "$\|\beta_{t_i}\| \le C$ for $k \le i \le N$" by any restriction of the form:

"There exists a value $c_i \in (0, +\infty)$ s.t. $\|\beta_{t_i}\| \ne c_i$ for all $0 \le i < N$ ".

This can convey the condition that the regime of production shall be null or greater than a threshold to allow profits, or observe a more precise condition on its components as long as it also constrains the norm of β. Posing **CSP**(R) implies that the closedness property on the financial market alone transmits to the market with production possibilities. Theorem 1.1 given in introduction then follows as a corollary to the following proposition.

Proposition 3.1. *The set $\mathfrak{X}_0^R(T)$ is Fatou-closed under* **CSP***(R).*

4 Application to the pricing of a power future contract

We illustrate Theorem 1.1 by an application to an electricity producer endowed with a generation system converting a raw material, e.g. fuel, into electricity and who has the possibility to trade that asset on a market. We address here the question of a possible price of a term contract a producer can propose on power when he takes into account his the generation asset. We assume that the financial market is submitted to proportional transaction costs. For this reason, we place ourselves in the financial framework developed by Campi and Schachermayer [3].

4.1 The financial market

We consider a financial market on $[0,T]$ composed of two assets, cash and fuel, which are indexed by 1 and 2 respectively. The market is represented by a so-called bid-ask process π, see [3] for a general definition.

Assumption 4.1. *The process $\pi = (\pi_t^{12}, \pi_t^{21})_t$ is a $(0,+\infty)^2$-valued $\mathbb{F}$-adapted càdlàg process verifying efficient frictions, i.e.,*

$$\pi_t^{12} \times \pi_t^{21} > 1 \text{ for all } t \in [0,T]\ \mathbb{P}-a.s.$$

Here π_t^{12} denotes at time t the quantity of cash necessary to obtain and $(\pi_t^{21})^{-1}$ denotes the quantity of cash that can be obtained by selling one unit of fuel. The efficient frictions assumption conveys the presence of positive transaction costs. The process π generates a set-valued random process which defines the solvency region:

$$\widehat{K}_t(\omega) := \text{cone}(e^1, e^2, \pi_t^{12}(\omega)e^1 - e^2, \pi_t^{21}(\omega)e^2 - e^1) \quad \forall (t,\omega) \in [0,T] \times \Omega \ .$$

Here (e^1, e^2) is the canonical base of $\mathbb{R}^2$. The process $\widehat{K}$ is $\mathbb{F}$-adapted and closed convex cone-valued. It provides the partial order on $\mathbb{R}^2$ of Definition 2.1.

Assumption 4.2. *Every $\xi \in \mathfrak{X}_0^0$ is a làdlàg $\mathbb{R}^2$-valued $\mathbb{F}$-predictable process with finite variation verifying, for every $(\sigma,\tau) \in \mathcal{T}_{[0,T]}^2$ with $\sigma < \tau$,*

$$(\xi_\tau - \xi_\sigma)(\omega) \in \overline{\text{conv}} \left(\bigcup_{\sigma(\omega) \le u \le \tau(\omega)} -\widehat{K}_u(\omega) \right) ,$$

the bar denoting the closure in $\mathbb{R}^d$.

Assumption 4.2 implies Assumption 2.1. Admissible portfolios are defined via Definitions 2.5 and 2.1.

Corollary 4.1. *Every $Z \in \mathcal{M}$ is a $\mathbb{R}^2_+$-valued martingale verifying $(\pi^{21}_t)^{-1} \le Z^1_t/Z^2_t \le \pi^{12}_t$ $\mathbb{P} - a.s.$ and:*

- *for all $\sigma \in \mathcal{T}$, $(\pi^{21}_\sigma)^{-1} < Z^1_\sigma/Z^2_\sigma < \pi^{12}_\sigma$;*
- *for all predictable $\sigma \in \mathcal{T}$, $(\pi^{21}_{\sigma^-})^{-1} < Z^1_{\sigma^-}/Z^2_{\sigma^-} < \pi^{12}_{\sigma^-}$.*

Proof The market model is conical, so that $\alpha^0_0(Z) := \sup\left\{\mathbb{E}\left[Z'_T V_T\right] \ : \ V \in \mathfrak{X}^0_{0,adm}\right\} = 0$, for all $Z \in \mathcal{M}$. The fact that Definition 2.4 corresponds to these elements Z follows from the construction of $\widehat{K}$ and is a part of the proof of Theorem 4.1 in [3]. □

Under the assumption that $\mathcal{M} \neq \emptyset$, $Z\xi$ is a supermartingale for all $Z \in \mathcal{M}$ and admissible $\xi \in \mathfrak{X}^0_0$, see Lemma 2.8 in [3]. Finally, Assumption 2.3 is given by Proposition 3.4 in [3]. For a comprehensive introduction of all these objects, we refer to [3].

4.2 The generation asset

We suppose that the agent possesses a thermal plant allowing to produce electricity out of fuel on a fixed period of time. The electricity spot price is determined per hour, so that we define the calendar of production as $(t_i)_{0\le i\le N} \subset [0,T]$, where N represents the number of generation actions for each hour of the fixed period. At time t_i, the agent puts a quantity $\beta_{t_i} = (\beta^1_{t_i}, \beta^2_{t_i})$ of assets in the plant. The production system transforms at time t_{i+1} the quantity $\beta^2_{t_i}$ of fuel, given a fixed heat rate $q_{i+1} \in \mathbb{R}_+$, into a quantity $q_{i+1}\beta^2_{t_i}$ of electricity (in MWh). The producer has a limited capacity of injection of fuel given by a threshold $\Delta_{i+1} \in L^\infty(\mathbb{R}_+, \mathcal{F}_{t_{i+1}})$. This implies that any additional quantity over Δ_{i+1} of fuel injected in the process will be redirected to storage facilities, i.e., as fuel in the portfolio. The electricity is immediately sold on the market via the hourly spot price. On most of electricity markets, the spot price is legally bounded. It can also happen to be negative. It is thus given by $P_{i+1} \in L^\infty(\mathbb{R}, \mathcal{F}_{t_{i+1}})$. For a given time t_{i+1}, the agent is subject to a fixed cost γ_{i+1} in cash. The agent also faces a cost in fuel in order to maintain the plant activity. This is given by a supposedly non-positive increasing concave function c_{i+1} on $[0, \Delta_{i+1}]$ such that $c'_{i+1}(\Delta_{i+1}) \ge 1$, where c'_{i+1} represents the left derivative. Altogether, we propose the following.

Assumption 4.3. *The production function is given by $R_{t_{i+1}}(\beta_{t_i}) = (R^1_{t_{i+1}}(\beta_{t_i}), R^2_{t_{i+1}}(\beta_{t_i}))$ for $0 \le i < N$, where*

$$\left\{\begin{array}{lcl} R^1_{t_{i+1}}((\beta^1_{t_i}, \beta^2_{t_i})) & = & P_{i+1}q_{i+1}\min(\beta^2_{t_i}, \Delta_{i+1}) - \gamma_{i+1} + \beta^1_{t_i} \\ R^2_{t_{i+1}}((\beta^1_{t_i}, \beta^2_{t_i})) & = & c_{i+1}(\min(\beta^2_{t_i}, \Delta_{i+1})) + \max(\beta^2_{t_i} - \Delta_{i+1}, 0) \end{array}\right. .$$

We can constraint $\beta^1_{t_i}$ to be null at every time t_i without any loss of generality.

Corollary 4.2. *Assumption 2.4 holds under Assumption 4.3.*

Proof For each i, $R_{t_{i+1}}$ verifies Assumption 2.4 (ii):

$$|R^1_{t_{i+1}}((\beta^1_{t_i}, \beta^2_{t_i})) - \beta^1_{t_i}| \le |P_{i+1}q_{i+1}\Delta_{i+1}| + |\gamma_{i+1}| \in L^\infty(\mathbb{R}, \mathcal{F}_{t_{i+1}})$$

and

$$|R^2_{t_{i+1}}((\beta^1_{t_i}, \beta^2_{t_i})) - \beta^2_{t_i}| \le \max(|c_{i+1}(0)|, |c_{i+1}(\Delta_{i+1}) - \Delta_{i+1}|) \in L^\infty(\mathbb{R}_+, \mathcal{F}_{t_{i+1}}) .$$

Notice that since c_{i+1} is concave with $c'_{i+1}(\Delta_{i+1}) \geq 1$, the function $R^2_{t_{i+1}}$ is clearly concave. The function R is then concave in each component with respect to the usual order, so that Assumption 2.4 (i) holds with the partial order induced by $\widehat{K}$. It is also continuous, so that Assumption 2.4 (iii) holds. □

4.3 Super replication price of a power futures contract

We now fix a condition provided by the agent in order to apply Definition 3.1. For example suppose that the agent knows at time t_i that by producing under a typical regime C (a given threshold of fuel to put in his system) and selling the production at the market price, he can refund the quantity of fuel needed to produce. It is a conceivable phenomenon on the electricity spot market. Since the electricity spot price is actually an increasing function of the total amount of electricity produced by the participants, the agent can sell a small quantity of electricity at high price if the total production is high. He can then partially or totally recover his fixed cost and even make sure profit. The constant C can depend on external factors of the model, such as the level of aggregated demand of electricity.

Assumption 4.4. *We assume that there exists $C > 0$ such that*

$$R^1_{t_{i+1}}(\beta^2_{t_i}) + \gamma_{i+1} \geq (\pi^{12}_{t_{i+1}})^{-1}(R^2_{t_{i+1}}(\beta^2_{t_i}) - \beta^2_{t_i} - c_{i+1}(0)) \ \mathbb{P} - a.s. \Longrightarrow \beta^2_{t_i} \leq C \tag{4.1}$$

Here, an immediate transfer $\xi_{t_{i+1}}$ of quantity $R^1_{t_{i+1}}(\beta^2_{t_i})$ of asset 1 brought in asset 2 gives $\xi_{t_{i+1}} + R_{t_{i+1}}(\beta_{t_i}) \succeq R_{t_{i+1}}(0)$, and **CSP**(R) condition holds under Assumption 4.4. The latter thus implies that the set $\mathfrak{X}^R_{0,adm}(T)$ is Fatou-closed, so that we can apply Theorem 1.1.

Now we consider the following contingent claim. We denote by $F(x)$ the price of a power futures contract with physical delivery. Buying this contract at time 0 provides a fixed power x (in MW) for N consecutive hours of a fixed period. Here, the N hours correspond to the $(t_i)_{1\leq i \leq N}$. Theorem 1.1 can be immediately applied to obtain the price at which the investor-producer can sell the contract.

Corollary 4.3. *The price is given by* $F(x) = \sup_{Z\in\mathcal{M}} \left(\frac{1}{Z^1_0} \mathbb{E}\left[\sum_{i=1}^N Z^1_{t_i} P_i x\right] - \alpha^R_0(Z) \right)$ *where*

$$\alpha^R_0(Z) = \sup_{\beta\in\mathcal{B}} \mathbb{E}\left[\sum_{i=1}^N Z^1_{t_i}\left(P_i q_i \min(\beta^2_{t_{i-1}}, \Delta_i) - \gamma_i\right) + Z^2_{t_i}\left(c_i(\min(\beta^2_{t_{i-1}}, \Delta_i)) - \min(\beta^2_{t_{i-1}}, \Delta_i)\right)\right] .$$

The theorem then ensures the existence of a wealth process, involving a financial strategy starting with wealth $F(x)$ and production activities, such that his terminal position is solvent $\mathbb{P}-$a.s..

5 Proofs

5.1 Proof of Proposition 3.1

We define a collection of sets

$$\widetilde{\mathfrak{X}}^k_t := \left\{ V \ : \ V_s := \xi_s + \sum_{i=1}^k R_{t_{N+1-i}}(\beta_{t_{N-i}}) \mathbb{1}_{\{t_{N+1-i}\leq s\}} - \beta_{t_{N-i}} \mathbb{1}_{\{t_{N-i}\leq s\}}, \ (\xi,\beta) \in \mathfrak{X}^0_{t,adm} \times \mathcal{B} \right\}$$

and $\widetilde{\mathfrak{X}}^k_t(T) := \left\{V_T \ : \ V \in \widetilde{\mathfrak{X}}^k_t\right\}$ for $t \in [0,T]$ and $0 \le k \le N$, with the convention that

$$\sum_{i=1}^{0} R_{t_{N+1-i}}(\beta_{t_{N-i}}) - \beta_{t_{N-i}} = 0 \ .$$

Note thus that $\widetilde{\mathfrak{X}}^0_t(T)$ corresponds precisely to the set $\mathfrak{X}^0_{t,adm}(T)$. We are conducted by the following guideline. According to Assumption 2.3, $\widetilde{\mathfrak{X}}^0_{t_N}(T)$ is Fatou closed. We then proceed by induction in two steps: we first show that $\widetilde{\mathfrak{X}}^k_{t_{N-(k+1)}}(T)$ is closed if $\widetilde{\mathfrak{X}}^k_{t_{N-k}}(T)$ is closed. Then we prove that $\widetilde{\mathfrak{X}}^{k+1}_{t_{N-(k+1)}}(T)$ is closed if $\widetilde{\mathfrak{X}}^k_{t_{N-(k+1)}}(T)$ is closed.

Proposition 5.1. *For all $0 \le k \le N$, the set $\widetilde{\mathfrak{X}}^k_{t_{N-k}}(T)$ is convex.*

Proof This is a consequence of Assumption 2.4 (i). Indeed take (ξ^1, β^1) and (ξ^2, β^2) in $\mathfrak{X}^0_{t_{N-k},adm} \times \mathcal{B}$ and $\lambda \in [0,1]$. Take $(\kappa^1, \kappa^2) \in \mathbb{R}^{2d}_+$ the respective bounds of admissibility for ξ^1 and ξ^2. Note that $\lambda\xi^1 + (1-\lambda)\xi^2$ is clearly $(\lambda\kappa^1 + (1-\lambda)\kappa^2)$-admissible since $\widehat{K}$ is a cone-valued process. By Assumption 2.4 (i), there exists $(\ell_{t_{N+1-i}})_{1 \le i \le k}$ with $\ell_{t_{N+1-i}} \in L^0(\mathbb{R}^d_-, \mathcal{F}_{t_{N+1-i}})$ such that

$$R_{t_{N+1-i}}(\lambda\beta^1_{t_{N-i}} + (1-\lambda)\beta^2_{t_{N-i}}) + \ell_{t_{N+1-i}} = \lambda R_{t_{N+1-i}}(\beta^1_{t_{N-i}}) + (1-\lambda) R_{t_{N+1-i}}(\beta^2_{t_{N-i}}) \ , \quad 1 \le i \le k \ .$$

Notice that $\mathbb{R}^d_- \subset \widehat{K}_t$ for any $t \in [0,T]$, so that $\ell_{t_{N+1-i}} \in L^0(-\widehat{K}_{t_{N+1-i}}, \mathcal{F}_{t_{N+1-i}})$. We will use this fact throughout the proof. Notice also that, according to Assumption 2.4 (ii), each $\ell_{t_{N+1-i}}$ is bounded by below by $2\mathfrak{K}$ for $1 \le i \le k$, where $\mathfrak{K}$ is the bound of net production incomes. By relation (2.1) and the above fact, $\lambda\xi^1_T + (1-\lambda)\xi^2_T + \sum_{i=1}^k \ell_{t_{N+1-i}} \in \mathfrak{X}^0_{t_{N-k},adm}(T)$. Assembling the parts gives the proposition. □

Proposition 5.2. *If $\widetilde{\mathfrak{X}}^k_{t_{N-k}}(T)$ is Fatou-closed, then the same holds for $\widetilde{\mathfrak{X}}^k_{t_{N-(k+1)}}(T)$.*

Proof Let $(V^n_T)_{n \ge 1} \subset \widetilde{\mathfrak{X}}^k_{t_{N-(k+1)}}(T)$ be a sequence such that V^n_T Fatou-converges to some V^0_T. Let $(\xi^n)_{n \ge 1} \subset \mathfrak{X}^0_{t_{N-(k+1)},adm}$ and $(\beta^n_{t_{N-i}})_{1 \le i \le k, n \ge 1}$ with $(\beta^n_{t_{N-i}})_{n \ge 1} \subset L^0(\mathbb{R}^d_+, \mathcal{F}_{t_{N-i}})$ for $1 \le i \le k$, and $\kappa \in \mathbb{R}^d_+$, such that

$$V^n_T = \xi^n_T + \sum_{i=1}^{k} R_{t_{N+1-i}}(\beta^n_{t_{N-i}}) - \beta^n_{t_{N-i}} \succeq_T -\kappa \quad \forall n \ge 1 \ .$$

According to Assumption 2.4 (ii), and since $\mathbb{R}^d_+ \subset \widehat{K}_T$, we have that for any $n \ge 1$,

$$-k\mathfrak{K} \preceq_T \sum_{i=1}^{k} R_{t_{N+1-i}}(\beta^n_{t_{N-i}}) - \beta^n_{t_{N-i}} =: \widehat{V}^n_T \in \widetilde{\mathfrak{X}}^k_{t_{N-k}}(T) \ .$$

Due to Assumption 2.4 (ii) also, we have that $\xi^n_T \succeq_T -(\kappa + k\mathfrak{K})$ for all $n \ge 1$. According to Assumption 2.3, we can then find a sequence of convex combinations $\widetilde{\xi}^n$ of ξ^n, $\widetilde{\xi}^n \in \mathrm{conv}(\xi^m)_{m \ge n}$, such that $\widetilde{\xi}^n_T$ Fatou-converges to some $\widetilde{\xi}^0_T \in \mathfrak{X}^0_{t_{N-(k+1)},adm}(T)$. The convergence of $\widetilde{\xi}^n_T$ implies, by using the same convex weights, that there exists a sequence $(\widetilde{V}^n_T)_{n \ge 1}$ of convex combinations of $\widehat{V}^m_T$, $m \ge n$, converging $\mathbb{P}$ − a.s. to some $\widetilde{V}^0_T$. By Proposition 5.1 above, the sequence $(\widetilde{V}^n_T)_{n \ge 1}$ lies in $\widetilde{\mathfrak{X}}^k_{t_{N-k}}(T)$. Recall that it is also bounded by below. Since $\widetilde{\mathfrak{X}}^k_{t_{N-k}}(T)$ is Fatou-closed, $\widetilde{V}^0_T \in \widetilde{\mathfrak{X}}^k_{t_{N-k}}(T)$ and moreover, $\widetilde{V}^0_T$ is of the form $\sum_{i=1}^k R_{t_{N+1-i}}(\beta^0_{t_{N-i}}) - \beta^0_{t_{N-i}} + \ell^0_{t_{N+1-i}}$ for some $\beta^0 \in \mathcal{B}$ and $(\ell^0_{t_{N+1-i}})_{1 \le i \le k}$ with

$\ell^0_{t_{N+1-i}} \in L^\infty(-\widehat{K}_{t_{N+1-i}}, \mathcal{F}_{t_{N+1-i}})$ for $1 \le i \le k$. This is due to Assumption 2.4 (i)-(ii). If we let $(\lambda_m)_{m\ge n}$ be the above convex weights, we can always write for $1 \le i \le k$ and $n \ge 1$

$$\sum_{m\ge n} \lambda_m \left(R_{t_{N+1-i}}(\beta^m_{t_{N-i}}) - \beta^m_{t_{N-i}} \right) = R_{t_{N+1-i}}(\sum_{m\ge n} \lambda_m \beta^m_{t_{N-i}}) - \sum_{m\ge n} \lambda_m \beta^m_{t_{N-i}} + \ell^n_{t_{N+1-i}} \,.$$

The sets $L^0(-\widehat{K}_{t_{N+1-i}}, \mathcal{F}_{t_{N+1-i}})$ and $L^0(\mathbb{R}^d_+, \mathcal{F}_{t_{N-i}})$ are closed convex cones for $1 \le i \le k$, so that $\ell^n_{t_{N+1-i}}$ and $\sum_{m\ge n} \lambda_m \beta^m_{t_{N-i}}$ and their possible limits stay in those sets respectively. From the boundedness condition of Assumption 2.4 (ii), the vectors $\ell^n_{t_{N+1-i}}$ are uniformly bounded by below by $2\mathfrak{K}$ for any $1 \le i \le k$ and $n \ge 1$, and so are $\ell^0_{t_{N+1-i}}$ for $1 \le i \le k$. According to (2.1), $\widetilde{\xi}^n_T + \sum_{i=1}^k \ell^0_{t_{N+1-i}} \in \mathfrak{X}^0_{t_{N-(k+1)},adm}(T)$. We then have that $\widetilde{\xi}^n_T + \widetilde{V}^N_T$ converges to $\widetilde{\xi}^0_T + \widetilde{V}^0_T = V^0_T \in \widetilde{\mathfrak{X}}^k_{t_{N-(k+1)}}(T)$. □

Proposition 5.3. *If $\widetilde{\mathfrak{X}}^k_{t_{N-(k+1)}}(T)$ is Fatou-closed, then the same holds for $\widetilde{\mathfrak{X}}^{k+1}_{t_{N-(k+1)}}(T)$.*

Proof Let $(V^n_T)_{n\ge1} \subset \widetilde{\mathfrak{X}}^{k+1}_{t_{N-(k+1)}}(T)$ such that there exists $\kappa \in \mathbb{R}^d_+$ verifying $V^n_T \succeq_T -\kappa$ for $n \ge 1$, and V^n_T converges $\mathbb{P}-$a.s. toward $V_T \in L^0(\mathbb{R}^d, \mathcal{F}_T)$ when n goes to infinity. We let $(\bar{V}^n_T, \bar{\beta}^n)_{n\ge1} \subset \widetilde{\mathfrak{X}}^k_{t_{N-(k+1)}}(T) \times L^0(\mathbb{R}^d_+, \mathcal{F}_{t_{N-(k+1)}})$ be such that $V^n_T = \bar{V}^n_T + R_{t_{N-k}}(\bar{\beta}^n) - \bar{\beta}^n$. Define $\eta^n = |\bar{\beta}^n|$ and the $\mathcal{F}_{t_{N-(k+1)}}$-measurable set $E := \{\limsup_{n\to\infty} \eta^n < +\infty\}$. We consider two cases.

1. First assume that $E = \Omega$. Then $(\bar{\beta}^n)_{n\ge1}$ is $\mathbb{P}-$a.s. uniformly bounded. According to Lemma 2 in [13], we can find a $\mathcal{F}_{t_{N-(k+1)}}$-measurable random subsequence of $(\bar{\beta}^n)_{n\ge1}$, still indexed by n for sake of clarity, which converges $\mathbb{P}-$a.s. to some $\bar{\beta}^0 \in L^\infty(\mathbb{R}^d_+, \mathcal{F}_{t_{N-(k+1)}})$. By Assumption 2.4 (iii), $R_{t_{N-k}}(\bar{\beta}^n)$ converges to $R_{t_{N-k}}(\bar{\beta}^0)$, Recall that $\bar{V}^n_T \succeq -\kappa - \mathfrak{K}$ for $n \ge 1$. Since it is $\mathbb{P}$-almost surely convergent to $V_T - R_{t_{N-k}}(\bar{\beta}^0) + \bar{\beta}^0 =: \bar{V}^0_T$ and that $\widetilde{\mathfrak{X}}^k_{t_{N-(k+1)}}(T)$ is Fatou-closed, the limit $\bar{V}^0_T$ lies in that set. This implies that $V_T \in \widetilde{\mathfrak{X}}^{k+1}_{t_{N-(k+1)}}(T)$.

2. Assume now that $\mathbb{P}[E^c] > 0$. Since E^c is $\mathcal{F}_{t_{N-(k+1)}}$-measurable, we argue conditionally to that set and suppose without loss of generality that $E^c = \Omega$. We then know that there exists a $\mathcal{F}_{t_{N-(k+1)}}$-measurable subsequence of $(\eta^n)_{n\ge1}$ converging $\mathbb{P}$-almost surely to infinity with n by an argument similar to the one of Lemma 2 in [13]. We overwrite n by the index of this subsequence. We write V^n_T as follows:

$$V^n_T = \xi^n_T + R_{t_{N-k}}(\beta^n_{t_{N-(k+1)}}) - \beta^n_{t_{N-(k+1)}} + \sum_{i=1}^k R_{t_{N+1-i}}(\beta^n_{t_{N-i}}) - \beta^n_{t_{N-i}} \,, \tag{5.1}$$

with $(\xi^n)_{n\ge1} \subset \mathfrak{X}^0_{t_{N-(k+1)},adm}$ and $(\beta^n_{t_{N-i}})_{1\le i\le k+1, n\ge1}$ with $(\beta^n_{t_{N-i}})_{n\ge1} \subset L^0(\mathbb{R}^d_+, \mathcal{F}_{t_{N-i}})$ for $1 \le i \le k+1$, and with the natural convention that for all $n \ge 1$, $\beta^n_{t_{N-(k+1)}} = \bar{\beta}^n$. We then define

$$(\widetilde{V}^n_T, \widetilde{\xi}^n_T, \widetilde{\beta}^n_{t_{N-(k+1)}}, \ldots, \widetilde{\beta}^n_{t_N}) := \frac{2\|C\|}{1+\eta^n}(V^n_T, \xi^n_T, \beta^n_{t_{N-(k+1)}}, \ldots, \beta^n_{t_N}) \,. \tag{5.2}$$

Now that $(\widetilde{\beta}^n_{t_{N-(k+1)}})_{n\ge1}$ is a bounded sequence, we can extract a random subsequence, still indexed by n, such that $(\widetilde{\beta}^n_{t_{N-(k+1)}})_{n\ge1}$ converges $\mathbb{P}-$a.s. to $\beta^0_{t_{N-(k+1)}} \in L^0(\mathbb{R}^d_+, \mathcal{F}_{t_{N-(k+1)}})$. Notice for later that $\|\widetilde{\beta}^n_{t_{N-(k+1)}}\|$ converges to $\|\beta^0_{t_{N-(k+1)}}\| = 2\|C\|$. It is clear that Assumption 2.4 (i) allows to write

$$\frac{2\|C\|}{1+\eta^n}\left(R_{t_{N+1-i}}(\beta^n_{t_{N-i}}) - \beta^n_{t_{N-i}} \right) = R_{t_{N+1-i}}(\widetilde{\beta}^n_{t_{N-i}}) - \widetilde{\beta}^n_{t_{N-i}} - \left(1 - \frac{2\|C\|}{1+\eta^n}\right) R_{t_{N+1-i}}(0) + \ell^n_{t_{N+1-i}} \,, \tag{5.3}$$

with $(\ell^n_{t_{N+1-i}})_{n\geq 1} \subset L^\infty(-\widehat{K}_{t_{N+1-i}}, \mathcal{F}_{t_{N+1-i}})$ for $1 \leq i \leq k+1$. Note that, according to Assumption 2.4 (iii), the particular case $i = k+1$ gives

$$\lim_{n\uparrow\infty} R_{t_{N-k}}(\widetilde{\beta}^n_{t_{N-(k+1)}}) - \widetilde{\beta}^n_{t_{N-(k+1)}} = R_{t_{N-k}}(\beta^0_{t_{N-(k+1)}}) - \beta^0_{t_{N-(k+1)}} \ . \tag{5.4}$$

The general case $i \leq k$ follows from Assumption 2.4 (ii) applied to equation (5.3): the left hand term converges to 0 and $(1 - \frac{2\|C\|}{1+\eta^n})$ converges to 1, so that

$$\lim_{n\uparrow\infty} R_{t_{N+1-i}}(\widetilde{\beta}^n_{t_{N-i}}) - \widetilde{\beta}^n_{t_{N-i}} + \ell^n_{t_{N+1-i}} = R_{t_{N+1-i}}(0) \ . \tag{5.5}$$

By construction of the subsequence, the convexity of $\mathfrak{X}^0_{t_{N-(k+1)},adm}(T)$ and the belonging of 0 to that set, $\widetilde{\xi}^n_T \in \mathfrak{X}^0_{t_{N-(k+1)},adm}(T)$. By using property of Assumption 2.1 (ii) and since the sequence $(\ell^n_{t_{N+1-i}})_{n\geq 1}$ is uniformly bounded for any $1 \leq i \leq k+1$, see proof of Proposition 5.2 above, we define

$$\widehat{V}^n_T := \widetilde{\xi}^n_T + \ell^n_{t_{N-k}} + \sum_{i=1}^{k} \Big(R_{t_{N+1-i}}(\widetilde{\beta}^n_{t_{N-i}}) - \widetilde{\beta}^n_{t_{N-i}} + \ell^n_{t_{N+1-i}} \Big) \in \widetilde{\mathfrak{X}}^k_{t_{N-(k+1)}}(T) \ ,$$

which converges by definition and equations (5.4) and (5.5) to $\widehat{V}^0_T$ such that

$$\widehat{V}^0_T + R_{t_{N-k}}(\beta^0_{t_{N-(k+1)}}) - \beta^0_{t_{N-(k+1)}} \succeq_T \sum_{i=1}^{k+1} R_{t_{N+1-i}}(0) \ . \tag{5.6}$$

Notice also that by Assumption 2.4 (ii), for all $n \geq 1$

$$\widehat{V}^n_T = \widetilde{V}^n_T - R_{t_{N-k}}(\widetilde{\beta}^n_{t_{N-(k+1)}}) + \widetilde{\beta}^n_{t_{N-(k+1)}} + \sum_{i=1}^{k+1} \left(1 - \frac{2\|C\|}{1+\eta^n}\right) R_{t_{N+1-i}}(0) \succeq_T -(\kappa + (k+1)\mathfrak{K}) \ .$$

By Fatou-closedness of $\widetilde{\mathfrak{X}}^k_{t_{N-(k+1)}}(T)$, we finally obtain that $\widehat{V}^0_T + R_{t_{N-k}}(\beta^0_{t_{N-(k+1)}}) - \beta^0_{t_{N-(k+1)}} \in \widetilde{\mathfrak{X}}^{k+1}_{t_{N-(k+1)}}(T)$. By equation (5.6) and **CSP**(R), $\|\beta^0_{t_{N-(k+1)}}\| \leq C$ but by construction, $\|\beta^0_{t_{N-(k+1)}}\| = 2\|C\|$, so that we fall on a contradiction. The case **2.** is not possible. □

Remark that the flexibility of the **CSP**(R) condition is reflected in the construction in equation (5.2) used in the last lines of the proof of Proposition 5.3. The choice of a good norm for $\widetilde{\beta}$ can indeed vary according to the condition we aim at. Following Propositions 5.2 and 5.3, $\widetilde{\mathfrak{X}}^{k+1}_{t_{N-(k+1)}}(T)$ is Fatou-closed if $\widetilde{\mathfrak{X}}^k_{t_{N-k}}(T)$ is Fatou-closed. Proposition 5.2 is used a last time to pass from the closedness of $\widetilde{\mathfrak{X}}^N_{t_0}(T)$ to the closedness of $\mathfrak{X}^R_0(T)$.

5.2 Proof of Theorem 1.1

Proof The "⇒" sense is obvious. To prove the "⇐" sense, we take $H \in L^0(\mathbb{R}^d, \mathcal{F}_T)$ such that $H \succeq -\kappa$ for some $\kappa \in \mathbb{R}^d_+$ and such that $\mathbb{E}[ZH] \leq \alpha^R_0(Z)$ for all $Z \in \mathcal{M}$ and $H \notin \mathfrak{X}^R_0(T)$, and work toward a contradiction. Let $(H^n)_{n\geq 1}$ be the sequence defined by $H^n := H\mathbb{1}_{\{\|H\|\leq n\}} - \kappa\mathbb{1}_{\{\|H\|>n\}}$. By Proposition 3.1, $\mathfrak{X}^R_0(T)$ is Fatou-closed, so by Lemma 5.5.2 in [12], $\mathfrak{X}^R_0(T) \cap L^\infty(\mathbb{R}^d, \mathcal{F}_T)$ is weak*-closed. Since $H \notin \mathfrak{X}^R_0(T)$, there exists k large enough such that $H^k \notin \mathfrak{X}^R_0(T) \cap L^\infty(\mathbb{R}^d, \mathcal{F}_T)$ but, because any $Z \in \mathcal{M}$ has positive components, still satisfies

$$\mathbb{E}\left[Z'_T H^k\right] \leq \alpha^R_0(Z) := \sup\left\{\mathbb{E}\left[Z'_T V_T\right] \ : \ V_T \in \mathfrak{X}^R_0(T)\right\} \quad \text{for all } Z \in \mathcal{M}. \tag{5.7}$$

By Proposition 5.1, the set $\mathfrak{X}_0^R(T)$ is convex, so that we deduce from the Hahn-Banach theorem that we can find $z \in L^1(\mathbb{R}^d, \mathcal{F}_T)$ such that

$$\sup\left\{\mathbb{E}\left[z'V_T\right] \ : \ V_T \in \mathfrak{X}_0^R(T) \cap L^\infty(\mathbb{R}^d, \mathcal{F}_T)\right\} < \mathbb{E}\left[z'H^k\right] < +\infty. \tag{5.8}$$

We define $\widetilde{Z}$ by $\widetilde{Z}_t = \mathbb{E}\left[z|\mathcal{F}_t\right]$. By using the same argument as in Lemma 3.6.22 in [12], we have that $\mathfrak{X}_0^R(T) \cap L^\infty(\mathbb{R}^d, \mathcal{F}_T)$ is dense in $\mathfrak{X}_0^R(T)$ and so that the left hand term of equation (5.8) is precisely $\alpha_0^R(\widetilde{Z})$. The process $\widetilde{Z}$ is a non negative martingale and since

$$\left(\mathfrak{X}_0^R(T) - L^\infty(\widehat{K}_t, \mathcal{F}_t)\right) \subset \left(\mathfrak{X}_0^R(T) \cap L^\infty(\mathbb{R}^d, \mathcal{F}_T)\right) \ \forall t \in [0,T] \ ,$$

we have $\widetilde{Z}_t \in L^1(\widehat{K}_t^*, \mathcal{F}_t)$. The contrary would make the left term of equation (5.8) equal to $+\infty$ for suitable sequences $(\xi^m)_{m\geq 1} \subset \mathfrak{X}_0^R$ (see the proof of Proposition 3.4 in [2]). By using the same arguments as above, and since $\mathfrak{X}_{0,adm}^0(T)$ is Fatou-closed too, we have that $\mathfrak{X}_{0,adm}^0(T) \cap L^\infty(\mathbb{R}^d, \mathcal{F}_T)$ is dense in $\mathfrak{X}_{0,adm}^0(T)$. This implies that

$$\begin{aligned}
\alpha_0^0(\widetilde{Z}) &:= \sup\left\{\mathbb{E}\left[\widetilde{Z}_T'V_T\right] \ : \ V_T \in \mathfrak{X}_{0,adm}^0(T)\right\} \\
&= \sup\left\{\mathbb{E}\left[\widetilde{Z}_T'V_T\right] \ : \ V_T \in \mathfrak{X}_{0,adm}^0(T) \cap L^\infty(R^d, \mathcal{F}_T)\right\} \\
&\geq \sup\left\{\mathbb{E}\left[\widetilde{Z}_T'V_T\right] \ : \ V \in \mathfrak{X}_0^0 \ \text{ and } \ V_\tau \succeq_\tau -\kappa \text{ for all } \tau \in \mathcal{T}, \text{ for some } \kappa \in \mathbb{R}_+^d\right\}
\end{aligned}$$

Moreover, according to Assumption 2.4 (ii), $\xi_T + \sum_{i=1}^N R_{t_i}(0) \in \mathfrak{X}_0^R(T) \cap L^\infty(\mathbb{R}^d, \mathcal{F}_T)$ for any $\xi_T \in \mathfrak{X}_{0,ad}^0(T) \cap L^\infty(\mathbb{R}^d, \mathcal{F}_T)$, so that

$$\alpha_0^0(\widetilde{Z}) - N\widetilde{Z}_0'\mathfrak{K} \leq \alpha_0^0(\widetilde{Z}) + \mathbb{E}\left[\widetilde{Z}_T' \sum_{i=1}^N R_{t_i}(0)\right] \leq \sup\left\{\mathbb{E}\left[z'V_T\right] \ : \ V_T \in \mathfrak{X}_0^R(T) \cap L^\infty(\mathbb{R}^d, \mathcal{F}_T)\right\}$$

and then $\alpha_0^0(\widetilde{Z})$ is finite according to equation (5.8). Take $Z \in \mathcal{M}$. Then there exists $\varepsilon > 0$ small enough such that, by taking $\check{Z} = \varepsilon Z + (1-\varepsilon)\widetilde{Z}$,

$$\alpha_0^R(\check{Z}) \leq \varepsilon\alpha_0^R(Z) + (1-\varepsilon)\alpha_0^R(\widetilde{Z}) < \varepsilon\mathbb{E}\left[Z_T'H^k\right] + (1-\varepsilon)\mathbb{E}\left[\widetilde{Z}_T'H^k\right] = \mathbb{E}\left[\check{Z}_T'H^k\right] \ .$$

It is easy to see that $\check{Z} \in \mathcal{M}$, so that the above inequality contradicts (5.7). □

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