The LeChatelier Principle

By PAUL MILGROM AND JOHN ROBERTS *

The LeChatelier principle, in the form introduced into economics by Paul A. Samuelson, asserts that at a point of long-run equilibrium, the derivative of long-run compensated demand with respect to own price is larger in magnitude than the derivative of short-run compensated demand. We introduce an extended LeChatelier principle that applies also to large price changes and to uncompensated demand as well as to a wide range of concave and nonconcave maximization problems outside the scope of demand theory. This extension also clarifies the intuitive basis of the principle. (*JEL* C60, D10, D20).

The idea that long-run demand is typically more elastic than short-run demand is common in economics. The LeChatelier principle expresses this idea mathematically. The principle has its cleanest expression in the neoclassical theory of the firm, where it applies to input demand. Let there be two inputs, say capital and labor, and suppose that the price of labor falls. In the short run, if the capital input is fixed, the direct effect of the change will be to lead to (weakly) more labor being employed. In the long run, changes in capital usage may occur which alter the productivity of labor. The first formal analysis to conclude that such changes would *increase* the use of labor was offered by Paul A. Samuelson (1947), who returned to the subject frequently (Samuelson 1949, 1960a, 1960b, 1972). His original treatment analyzed the properties of the regular maxima of smooth functions and established that the derivative of long-run demand was (weakly) more negative than that of short-run demand at a point of long-run equilibrium.[1]

Various intuitive arguments have been offered to explain why labor demand should become (weakly) more elastic when capital is adjusted, the most accurate of which goes as follows. First suppose capital and labor are substitutes in the sense that increasing the use of one reduces the marginal product of the other. (This implies that the two are also substitutes in the demand-theoretic sense that lowering the price of one decreases the demand for the other.) Then in the long run the firm will reduce its use of capital in response to the lower price of labor. Because the inputs are substitutes, reducing the amount of capital raises the marginal product of labor, and this results in a further increase in labor's employment. Thus, the long-run adjustment is greater than the short-run one. On the other hand, if capital and labor are complements, the firm will respond to the lower wage and resultant short-run increase in labor by employing more capital in the long-run. This raises the marginal product of labor, again leading to an increase in the employment of labor beyond the short-run equilibrium level.

These intuitive arguments amount to observing that the extra adjustments involved in long-run demand create a positive feedback that is missing from short-run demand. The feedback argument makes no appeal to small price changes, divisible inputs, convex production technologies, or competitive markets

* Department of Economics, Stanford University, Stanford CA 94350-6072, and Graduate School of Business, Stanford University, Stanford, CA 94305-5015. We thank Preston McAfee, Hal Varian and an anonymous referee for comments and Milgrom thanks the National Science Foundation for financial support.

[1] Not all existing treatments are limited to infinitesimal price changes in smooth problems. Samuelson (1949, 1960a) showed that the principle also applies to problems in linear programming, where the changes can be discrete. His theory, however, is still a local one in the appropriate sense for linear programming. Eugene Silberberg (1974) established a global version of the LeChatelier principle for smooth, concave optimization problems.

for the fixed input. One might therefore suspect that it supports a more general conclusion, eschewing such restrictions. For example, when real wages fall, a firm might change its organization to accommodate training and supervising a larger labor force. That, one might expect, would lead to a greater increase in employment than if there were no possibility of such a reorganization.

It is not true, however, that discrete adaptations always make long-run labor demand more elastic than short-run demand. Nor is it true in general that for any discrete price increase, the quantity of input demanded falls more in the long run than in the short run. A simple example will suffice to show this.[2] Let the set of feasible triples of capital, labor, and output (k, l, q) be the convex hull of the following set of three points: $\{(0, 0, 0), (0, 2, 1), (1, 1, 1)\}$, expanded to allow for free disposal. If the initial price of output is 2 and the initial wage and capital rental rates satisfy $w < r < 1$, then the initial optimal input mix is $(k, l) = (0, 2)$. If the wage then rises to satisfy $1 < w < 2 - r$, the new short-run optimal input mix is $(0, 0)$, but the new long-run optimum is $(1, 1)$. In this example, the demand for labor following a wage increase falls more in the short run than in the long run.

On account of such examples, the usual formalization of the LeChatelier principle can rank derivatives of demand at just one point.

Our contributions in this paper are to identify a new, global LeChatelier principle that applies to arbitrarily large price changes, nondifferentiable demands, and even discrete choice variables, and to provide an argument that directly formalizes the intuitive logic of positive feedbacks. In its most general form, the new principle dispenses with any assumptions that the choice variables are quantities and that the changing parameter is a price. In the special case where the problem involves input demands with price as the parameter, we find that long-run adjustments are larger than short-run adjustments whenever the two inputs are either uniformly substitutes or uniformly complements, because long-run demand then entails an additional positive feedback. Long-run choices can fail to change by more than short-run choices only when the inputs are substitutes over some portion of the domain and complements over another portion.[3] Thus, in the counterexample given above, suppose the output price is $p = 2$ and the capital rental rate is $r = .9$. Then, as the wage rises from .85 to 1.05 to 1.15, the demand for capital rises from 0 to 1 and then falls back to 0: capital is a substitute for labor over one part of the price domain and a complement over another.

We illuminate the relation between our global LeChatelier principle and Samuelson's local principle by deriving the local principle from the global one. This might seem impossible because the global principle entails a restrictive assumption—that the pair of inputs be everywhere substitutes or everywhere complements—which is not required by the local principle. Intuitively, this extra assumption is implicit in the local versions of the principle because these results are theorems about *linear approximations* of the input demand system. With linear demand, the condition that a pair of inputs be either everywhere substitutes or everywhere complements is always satisfied.

The argument we use to establish the global LeChatelier principle involves simply composing monotonic functions, where the monotonicity follows from the assumption that the goods are either substitutes or complements. This highlighting of the role of monotonicity suggests an approach to establishing a LeChatelier principle for *uncompensated* consumer demands.

In Section I, we state and prove a simple, global LeChatelier principle that is adequate for most applications in the neoclassical and modern theories of the firm. In that section, we also show how the global principle implies the local LeChatelier principle as a special case. As well, we discuss the case of uncompensated consumer demand. Section II provides our most general statement of the LeChatelier

[2] Additional examples can be found in Samuelson (1960a, 1960b) and Milgrom and Roberts (1994).

[3] Samuelson (1960a) noted the reverse implication for concave problems, namely, that if the variables switched between being complements and substitutes, then a counterexample to a global LeChatelier principle could be found.

principle, going beyond the needs of the demand theory and using the concepts of lattice theory. For a LeChatelier principle that applies to fixed point problems, see Milgrom and Roberts (1994).

I. A Global LeChatelier Principle

The global LeChatelier principle compares changes in the maximizers in two different maximization problems. To state it precisely, we make the following definitions. Let $\mathcal{A}$, $\mathcal{B}$ be subsets of $\mathbb{R}$ and let

$$x^*(\theta, y) = \text{Argmax} f(x, y, \theta)$$

$$\text{subject to } x \in \mathcal{A}$$

$$y^*(\theta) = \text{Argmax} f(x^*(\theta, y), y, \theta)$$

$$\text{subject to } y \in \mathcal{B}.$$

Assuming that $\mathcal{A}$ and $\mathcal{B}$ are compact and that f is (upper semi-) continuous in (x, y), at least one maximum exists in each problem. If multiple maxima exist in the first problem, then, for definiteness, let $x^*(\theta, y)$ be the largest maximizer. Similarly, in the second problem, let $y^*(\theta)$ be the largest y-maximizer.

This formulation includes the neoclassical theory of the firm as one important case. To see this, take $f(x, y, \theta) = pg(x, y) - ry + \theta x$, where p is the price of output, g is the production function, x is the labor input to production, y is the capital input, r is the rental rate of capital, and θ is the negative of the wage. (Why we take θ to be the negative of the wage will become clear shortly.) For analytical convenience, we treat the prices p and r as unchanging. Then, the short-run demand for labor $x^*(\theta, y^*(\theta'))$ depends both on the current wage $-\theta$ and on the previous wage level $-\theta'$ on which the choice of capital was based. The long-run demand for labor is $x^*(\theta, y^*(\theta))$.

Recall that a twice continuously differentiable function $f(x, y, \theta)$ is *supermodular* if its mixed partial derivatives f_{xy}, $f_{x\theta}$, and $f_{y\theta}$ are all nonnegative everywhere. The general definition of supermodular functions is given and interpreted in Milgrom and Roberts (1990): it states that the change in f resulting from a given increase in one argument is increasing in the other arguments.

THEOREM 1: *Suppose that f is supermodular and $\theta \le \theta'$. Then,*

$$x^*(\theta, y^*(\theta)) \le x^*(\theta, y^*(\theta')) \le x^*(\theta', y^*(\theta')).$$

PROOF OF THEOREM 1:

By Topkis's theorem (Donald Topkis, 1978; Milgrom and Roberts, 1990), $y^*(\cdot)$ and $x^*(\cdot, \cdot)$ are monotonically nondecreasing. Hence, $y^*(\theta) \le y^*(\theta')$ and $x^*(\theta, y^*(\theta)) \le x^*(\theta, y^*(\theta')) \le x^*(\theta', y^*(\theta'))$.

Theorem 1 applies neatly to the problem of the neoclassical firm. The condition that $\theta \le \theta'$ means that the wage has increased from $-\theta'$ to $-\theta$. In this very special application, $f_{y\theta} \equiv 0$ and $f_{x\theta} \equiv 1$, so those two conditions of supermodularity are satisfied. (This was why we set $\theta = -w$.) The condition that $f_{xy} \ge 0$ reduces to the requirement that $g_{xy} \ge 0$, that is, that the production function g be supermodular. That is the textbook condition under which capital and labor are complements. Thus, with this assignment of variables, the theorem asserts that if capital and labor are complements, then labor demand falls more in response to a wage increase in the long run than in the short run. We emphasize again that this is a global conclusion—not restricted to infinitesimal price changes—and that it assumes no divisibility of the capital and labor inputs and no concavity of the production function g (or of the objective f). Indeed, $\mathcal{A}$ and $\mathcal{B}$ are arbitrary compact sets, so the theorem still applies, for example, when capital or labor or both are restricted to be integer valued.

The case where capital and labor are substitutes is also covered by Theorem 1, as can be seen by relabeling the variables. Recall from standard price theory that capital and labor are substitutes exactly when the production function g is *submodular*, that is (for smooth g), when $g_{xy} \le 0$ everywhere. (In general, a function h is submodular if $-h$ is supermodular.) As before, let x be the labor input but now let y be the negative of the capital input. Then the objective is $f(x, y, \theta) = pg(x, -y) + \theta x + ry$. It is routine to check that when g is submodular (in the amounts of capital and labor), the function f defined in this way is supermodular, so Theorem 1 applies. We summarize the two cases together in the following result:

THEOREM 2: *Consider the objective $pg(x, y) - ry - wx$. If g is either supermodular ($g_{xy} \geq 0$ everywhere) or submodular ($g_{xy} \leq 0$ everywhere) and $w \geq w'$, then*

$$x^*(w, y^*(w)) \leq x^*(w, y^*(w')) \leq x^*(w', y^*(w')).$$

Note that production function corresponding to the example given earlier is neither supermodular nor submodular: an increase in the capital input from 0 to 1 raises the marginal product of the first unit of labor from $\frac{1}{2}$ to 1 but reduces that of the second unit of labor from $\frac{1}{2}$ to 0. Theorem 2 identifies this changing effect of capital on the marginal product of labor as the condition the makes the LeChatelier principle inapplicable in the example.

Our next task is to derive Samuelson's local LeChatelier principle as a corollary of Theorem 2. To do this, let g be the firm's (twice continuously differentiable) production function, let $(\bar{x}, \bar{y})$ and $(p, \bar{w}, r)$ be the relevant input quantities and price levels around which the analysis is to be conducted and let $\theta = -w$. We assume that inputs to production are divisible and the relevant input demand functions are smooth. As we have seen, the relevant input demand functions for the analysis are $x^*(w, y)$ and $y^*(w)$, from which both the long- and short-run demand for input x can be derived. The demand function x^* satisfies the first order condition: $pg_x(x, y) + \theta = 0$. Similarly, y^* satisfies the other first-order condition: $pg_y(x^*(\theta, y), y) - r = 0$. Assuming that $(\bar{x}, \bar{y})$ is a regular value of the input demand functions, the implicit function theorem applies to determine the first derivatives of x^* and y^* in terms of the second derivatives of g in the relevant neighborhood.

Let $\hat{g}$ be the quadratic production function derived from g by matching the function value and the first and second derivatives at the point $(\bar{x}, \bar{y})$. By the preceding analysis, the input demand function derived from $\hat{g}$ has the same derivatives at $(\bar{x}, \bar{y})$ as the demand function derived from g. Since every quadratic function $\hat{g}(x, y)$ of two variables is either supermodular or submodular (because $\hat{g}_{xy}$ is a constant), Theorem 2 applies to it. Restating our conclusion about $\hat{g}$ in terms of the derivative of labor demand with respect to the wage, we have Samuelson's local LeChatelier principle:

Corollary: Suppose the pair $(\bar{x}, \bar{y})$ is a regular value of the demand functions x^* and y^* corresponding to the wage $\bar{w}$. Then the derivative of long-run demand with respect to the wage rate at $\bar{w}$ is more negative than the corresponding derivative of short-run demand:

$$\left.\frac{dx^*(w, y^*(w))}{dw}\right|_{w=\bar{w}} \leq \left.\frac{\partial x^*(w, y^*(\bar{w}))}{\partial w}\right|_{w=\bar{w}} \leq 0.$$

So far, we have limited attention to the input demands of firms, but the idea that an extra positive feedback is the source of the additional price-sensitivity of long-run demand applies also to consumer demands. We give consumer demand theory a condensed treatment, since no fundamentally new ideas are involved.

Suppose there are N goods and that we are considering an individual's (uncompensated) demand for some good, say gasoline, when the quantity of a second good, say the stock of automobiles, and the prices of the other $N - 2$ goods are held fixed. The quantity of gasoline demanded can be written as $x^*(p, y^*(p))$, where p is the price of gasoline and $y^*(p)$ is the demand for autos when the price of gasoline is p. If automobiles are complementary to gasoline in the twin senses that (1) a decrease in the price of gasoline raises the long-run demand for autos (y^* is decreasing), and (2) an exogenous increase in the quantity of autos purchased would lead to an increase in the demand for gasoline ($x^*(p, \cdot)$ is nondecreasing), then the logic of positive feedbacks applies. In the long run, a decrease in the price of gasoline will lead to an increase in the quantity of automobiles purchased. This will, in turn, lead to an increase in the consumption of gasoline above its short-run equilibrium level. An analogous argument applies to the case of substitutes. These arguments are global and apply directly to uncompensated demand, which is the empirically relevant case.

Global LeChatelier principles for *compensated* demand involve a difficult conceptual issue, because the required level of compensation for a price change varies between the long run and short run. To derive such a principle, we have to face the problem of whether to specify different levels of compensation for the long run and the short run. Local principles are conceptually simpler, because the local compensation requirements for the long and short runs are identical. (By the Envelope Theorem, the derivative of the required compensation with respect to a change in price P_x is $x^*(P_x)$, which is, by definition, the same for both short- and long-run demand at a position of long-run equilibrium.) Once that issue has been resolved, even to apply our argument locally, one still requires the "extra" condition that the two goods involved are either substitutes or complements in *both* of the two senses described in the preceding paragraph. This would seem to imply that the local LeChatelier principle involves a different mechanism than the positive feedbacks mechanism of the global principle.

As in the theory of the firm, the reconciliation lies in noticing that the local LeChatelier principle can be construed as a global principle that applies to the linear demand system determined by a first-order demand approximation, using the linear compensation function $(P_x - \bar{P}_x)x^*(\bar{P}_x)$, where $\bar{P}_x$ is the initial price. The extra condition required by the global principle is then a logical consequence of the linearity of demand together with the symmetry of the substitution matrix.

The local LeChatelier principle for compensated demand is thus derivable from a more general principle, which applies also, with the stated qualifications, to the empirically important case of finite price changes and uncompensated demands. In Milgrom and Roberts (1994), we show that even demands that do not emerge from optimization by consumers or firms can satisfy a LeChatelier principle if they satisfy conditions like (1) and (2).

II. Additional Generality

There are at least two ways in which Theorem 1 fails to be completely satisfactory. First, it applies only to optimization problems with two real variables. We would prefer a version of the principle that applies when the variables x and y take values in $\mathbb{R}^n$ and $\mathbb{R}^m$.[4] A corollary of Theorem 3 below is that Theorem 1 remains true as stated when the variables x and y take values in any lattice, including $\mathbb{R}^n$ and $\mathbb{R}^m$, provided that the sets $\mathcal{A}$ and $\mathcal{B}$ are restricted to be sublattices.[5]

Second, even when one is interested in problems with two real variables x and y, the condition of Theorem 1 that f be supermodular is unnecessarily restrictive. To see why, let f be a function with argument (x, y, θ) that is not supermodular but such that $\log(f(\cdot))$ is supermodular. It is clear that the maximizers of f are the same as the maximizers of $\log(f)$ and that Theorem 1 applies to $\log(f)$. Then the maximizers of $\log(f)$ and hence of f satisfy the conclusions of Theorem 1, even though f does not satisfy the hypothesis of the Theorem. There is nothing special about the log transformation in this example: the same objection applies when f is not supermodular but $g(f(\cdot))$ is supermodular, for any increasing function g from $\mathbb{R}$ to $\mathbb{R}$.

Our most general version of the LeChatelier principle avoids both of these limitations. To state it, we employ the language and concepts of lattice theory. Readers unfamiliar with the lattice theory approach to comparative statics are directed to Milgrom and Chris Shannon (1994) for the relevant definitions and background.[6]

[4] One might also want the objective to take the form $f(x, y, z, \theta)$, where the additional variable z is neither fixed nor part of the x whose variations are being investigated. However, by setting $\hat{f}(x, y, \theta) = \text{Max}_{z \in Z} f(x, y, z, \theta)$, the problems of maximizing f can be transformed into one of maximizing $\hat{f}(x, y, \theta)$, which is the form studied in this paper. This is what was done in the treatment of uncompensated consumer demand given above.

[5] Every subset of $\mathbb{R}$ is also a sublattice of $\mathbb{R}$ with the usual greater than or equal to order, so the extension does include Theorem 1 as a special case.

[6] Additional hints for checking the single crossing condition are found in Milgrom (1994).

Let $\mathcal{X}$ and $\mathcal{Y}$ be lattices and S a sublattice of the product lattice $\mathcal{X} \times \mathcal{Y}$. Define $(x^*(\theta), y^*(\theta))$ to be the least upper bound of $\text{Argmax}_{(x,y) \in S} f(x, y, \theta)$ and $x^*(\theta, y)$ to be the least upper bound of $\text{Argmax}_{x \in \{x|(x,y) \in S\}} f(x, y, \theta)$.

THEOREM 3: *Suppose that $f(x, y, \theta)$ is continuous (in the order-interval topology) and quasisupermodular in (x, y) and has the single crossing property in $(x, y; \theta)$. Let $\theta \leq \theta'$. Then, $x^*(\theta, y^*(\theta)) \leq x^*(\theta, y^*(\theta')) \leq x^*(\theta', y^*(\theta'))$.*

PROOF.

By a theorem of Milgrom and Shannon (1994), $y^*(\cdot)$ and $x^*(\cdot, \cdot)$ are monotonically nondecreasing. Hence, $y^*(\theta) \leq y^*(\theta')$ and $x^*(\theta, y^*(\theta)) \leq x^*(\theta, y^*(\theta')) \leq x^*(\theta', y^*(\theta'))$.

III. Discussion

The global LeChatelier principle has important applications in economic analysis. First, it identifies conditions under which economic variables may respond slowly to changing conditions (when complementary or substitutable inputs or institutions are restricted from changing quickly). In addition, it is useful in interpreting and applying the conclusions of empirical studies. When price or income elasticities are estimated from time-series data, the principle provides a reason why the estimates may be too low (because complementary changes have not had time to occur). The principle also illuminates why estimated elasticities based on cross-sectional data may underestimate the effects of economy-wide shifts in certain variables (because economy-wide shifts may be accompanied by adaptations in complementary institutions, as for example when an increase in the number of working mothers led U.S. schools to make available in-school lunches). Some of these effects may be more naturally analyzed using equilibrium models rather than optimization models, but there are global LeChatelier principles for equilibrium models as well (Milgrom and Roberts, 1994).

Our new formulation is clearer than previous ones about the assumptions needed to justify the conclusions of the LeChatelier principle. Previous formulations have all incorporated one or more assumptions about the choice variables being quantities, the quantities being infinitely divisible, the parameters being prices, the price changes being small, the objective functions being concave, the demand functions being differentiable, and the fixed inputs being traded in perfectly competitive markets. Such assumptions eliminate many natural applications of the principle—especially ones where the complementary changes involve new institutions or changing business practices. Our new version avoids all these restrictions, which have no actual bearing on the global analysis. Rather, we identify the important condition underlying the principle as the condition that the choice variables be uniformly substitutes or uniformly complements on the entire relevant domain. As we have shown, this condition is still restrictive. Nevertheless, clarity about the relevant restriction is a crucial precondition for correct application of the principle.

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