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Yard-Sale exchange on networks: Wealth sharing and wealth appropriation.

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Yard-Sale (YS) is a stochastic multiplicative wealth-exchange model with two phases: a stable one where wealth is shared, and an unstable one where wealth condenses onto one agent. YS is here studied numerically on 1d rings, 2d square lattices, and random graphs with variable average coordination, comparing its properties with those in mean field (MF). Equilibrium properties in the stable phase are almost unaffected by the introduction of a network. Measurement of decorrelation times in the stable phase allow us to determine the critical interface with very good precision, and it turns out to be the same, for all networks analyzed, as the one that can be analytically derived in MF. In the unstable phase, on the other hand, dynamical as well as asymptotic properties are strongly network-dependent. Wealth no longer condenses on a single agent, as in MF, but onto an extensive set of agents, the properties of which depend on the network. Connections with previous studies of coalescence of immobile reactants are discussed, and their analytic predictions are successfully compared with our numerical results.

I. INTRODUCTION

Wealth exchange models, initially proposed to investigate the emergence of wealth inequality [1] in human societies, have recently became a subject of intense research [2, 3], following the availability of massive amounts of statistical data describing commercial exchange, as well as wealth and income distributions in different contexts [4].
Conservative stochastic exchange models were first used by Angle [5], who considered the spontaneous buildup of wealth differences among equally able agents. In Angle's initial model, wealth concentration is a consequence of an explicit statistical advantage favoring richer agents. In other words, the "rich get richer" phenomenon is assumed explicitely in the exchange rules. Later work showed [6–10] that an explicit advantage favoring the rich is not necessary for wealth concentration to appear. Wealth concentration can develop even if the poor have an explicit statistical advantage. This rather counterintuitive result, which has only recently been stressed [9, 10] in the Econophysics literature, arises when the amount at stake in each transaction is proportional to the poorest agent's wealth, e.g in the so-called Yard-Sale (YS) models [6–12]. Yard Sale is an example of Multiplicative Stochastic Exchange, so named because the wealth of the poorest intervening agent is multiplied by a random number after the exchange [10]. Under YS dynamics, in the long run all wealth may end up in the hands of one lucky agent, even if each pairwise transaction is statistically biased in favor of the poorest of the two intervening agents. Therefore, favoring the poor may not suffice to avoid wealth concentration, if the bias in their favor is not strong enough. Interestingly, YS rules constitute a realistic (although highly simplified) microscopic model for the wealth exchange process occurring during commercial interaction, or trade [6, 7, 12]. This suggests that the conditions for the spontaneous creation of enormous wealth differences for no reason other than luck [13], are built into the commercial exchange rules, even if these rules may superficially appear to favor the poor. Because of the possibility of counterintuitive properties such as this one, and because of their relevance for real world commercial exchange, it is clearly of interest to understand the phenomenology of multiplicative exchange models thoroughly.

In simple versions of YS, pairs of agents 'bet' for a fraction $f \leq 1$ of the wealth of the poorest of them, who has a probability p to win the bet. Depending on p and f, long-term evolution can give rise either to a nontrivial equilibrium wealth distribution $P(w)$ or to *condensation* of the whole wealth in the hands of just one agent. We call the resulting phases, respectively the wealth-sharing (or stable) and the wealth-appropriation (or unstable) phase. To date, most results for this model concern the full-mixture (or Mean-Field) case. However, commercial exchange is often determined by geographical, social or other constraints, which are ignored in the fully mixed approximation. Usually, a given agent can only exchange wealth with a reduced subset of other agents who are "close" to him by some measure of distance. These constraints can be described, at the simplest level, by means of a network in which nodes $i = 1, 2, \ldots, N$ are economic agents and edges ij represent their possible interactions. It is reasonable to expect the topological properties of this network of allowed interactions to have a strong impact upon the general properties of wealth exchange processes occurring on them.

Recent work [14–16] explores the network of commercial interactions among nations, or "World Trade Web" [14]. These studies make it clear that the topology of interaction networks is strongly correlated with the dynamical and static properties of the resulting wealth exchange process taking place on those networks [17]. Numerical investigations [18–20] of the Bouchaud-Mezard [12, 21] model (BMM) on networks, find wealth distributions $P(w)$ that change from lognormal to power-law when the connectivity is increased, for reasons that are easily understood. However, the BMM, while being an interesting solvable model, considers linear exchange (the amount exchanged is proportional to wealth differences), which is not very realistic [12]. On the other hand, the BMM includes both exchange and nonconservative processes (such as investment), and it is precisely from the interplay between the two that the network's connectivity becomes important in the above studies.

The aim of this work is to analyze network effects for a realistic model of pure conservative commercial exchange. Although real economic systems involve nonconservative wealth-modifying processes as well, it is important to first understand the properties of the individual wealth-affecting mechanisms in isolation. In this work, we study the Yard-Sale (YS) model on networks. Our network-restricted version of YS is defined as follows: At each timestep, every agent i interacts (exchanges wealth according to YS rules) with another agent j that is randomly chosen among its neighbors, i.e. among those agents j for which a link ij exists. The interaction network is fixed in time. Therefore, a pair of agents not connected by a link will never interact directly. If the coordination number γ is roughly the same for all nodes, each agent engages in two interactions per timestep, on average. We consider here one-dimensional chains and two-dimensional square lattices with nearest-neighbor links and periodic boundary conditions, as well as Erdős-Rényi Random Graphs [22] with variable coordination γ. In the limit $\gamma \to (N-1)$, the Random Graph becomes a complete graph, and every pair ij has the same probability to interact. This is the full-mixture case.
We focus on the identification of network-specific effects, i.e. the extent to which the static and dynamic properties of the YS model, when implemented on a network, depart from those in full-mixture. Our results show that, while the stable wealth distribution $P(w)$ is mildly network-dependent, the location of the interface $p^*(f)$ that delimits the stable phase remains the same as in the full-mixture case, for all networks considered in this work. The critical line $p^*(f)$ is therefore universal, in the sense defined in the context of the theory of phase transitions. Dynamical properties, as for example decorrelation times, on the other hand, do depend on the network. Decorrelation times, which in this case are a measure of "social mobility" of agents, are found to diverge at the interface with the unstable phase. This divergence is used to locate the critical line with high precision.

In the unstable, or wealth-appropriation, phase, dynamical as well as long-time properties of YS are found to be strongly network-dependent. The most important difference with full-mixture is in this case that, on a network, complete condensation in the hands of one agent no longer occurs. On a network, instead, in the long run an extensive set of "locally rich" agents (LRA) appears, each connected only to extremely impoverished agents. This leads to the effective cessation of all exchange activity, a phenomenon similar to dynamical freezing. This freezing onto a disordered final state is observed in the whole unstable phase. The properties of the final set of LRAs, and their final wealth distribution, depend on the network topology, as well as upon p and f. We discuss the connections between the appearance of a set of locally rich agents and the process of coalescence (or coagulation) of immobile reactants [23–25] on networks. These connections provide analytical predictions for the number of LRAs on Random Graphs, which are consistent with our own numerical results. By increasing the average connectivity γ of a network, the number of LRAs onto which wealth condenses is decreased, until, in the limit $\gamma = N-1$, which is the full-mixture case, only one LRA remains, i.e full condensation is recovered.

This article is organized as follows. Section II recalls some results for YS in full-mixture. In Section III, numerical results on networks in the stable phase are presented and compared with full mixture results. In particular, decorrelation times are used in this section to locate the interface with high precision. Wealth appropriation dynamics on networks is studied in Section IV, where it is found that wealth condenses onto an extensive number of locally rich agents (LRA). Their number and wealth distribution are analyzed in this section. Finally, Section V offers a discussion of our results.

II. YARD-SALE IN FULL-MIXTURE

Consider wealth exchange for a pair of agents i and j with $w_i < w_j$ before interaction, according to the following YS rules. The agents bet for an amount $f \times MIN(w_i, w_j) = fw_i$. The poorest agent ($i$) wins the bet with probability p, in which case $w_i \to w_i(1+f)$ and $w_j \to w_j - fw_i$, or looses the bet with probability $(1-p)$, in which case $w_i \to w_i(1-f)$ and $w_j \to w_j + fw_i$. The wealth of the poor agent is therefore multiplied by a random factor η, which equals $(1+f)$ with probability p and $(1-f)$ with probability $(1-p)$. Long-term evolution under these rules gives rise either to a stable wealth distribution $P(w)$ or to condensation, depending on p and f.

The location of the critical line $p^*(f)$ below which condensation occurs can be derived as follows [9, 10]. The wealth of a very poor agent undergoes a Random Multiplicative Process [26] with multiplier η at each timestep. After a large number t of timesteps, the appropriate central tendency estimator for w is its geometric average $e^{\langle \ln w_t \rangle} = w_0 \; e^{-t\theta}$, where

$$-\theta = \langle \ln(\eta) \rangle = p \ln(1+f) + (1-p) \ln(1-f). \quad (1)$$

If $\theta > 0$, there will be a systematic transference of wealth from poorer to richer agents. This is the wealth-appropriation, or unstable, phase. In this phase, wealth differences among agents are amplified in time, until the whole wealth ends up in the hands of a single (in full-mixture) agent in the long run [9, 10].

If $\theta < 0$, the system is in the wealth-sharing, or stable, phase. Wealth is transferred from richer to poorer agents, which tends to "iron out" wealth fluctuations. In the long run, the distribution of wealth reaches a nontrivial equilibrium form $P(w)$, which depends on p and f.

By the heuristic argument above, the critical interface separating stable and unstable phases is given by $\theta = 0$, or

$$p^*(f) \ = \ \frac{\log(1-f)}{\log(1-f) - \log(1+f)}. \tag{2}$$

A more rigorous analysis [9, 10], involving the master equation for $P(w)$ in the full-mixture approximation, confirms (2).
Notice that the average return of the poorest agent [10] is positive whenever $p > 1/2$. There is thus a region $1/2 < p < p^*(f)$ where complete wealth concentration occurs, i.e. poor agents impoverish further, despite the average return of poor agents being positive.

A. The stable phase

1. *Time correlations*

A dynamical characterization that is useful in the stable phase is the relaxation timescale for equilibrium fluctuations. The excess wealth $\Delta w_i(t) = w_i(t) - \bar{w}$, where $\bar{w}$ is the average per agent wealth, gives the amount by which the wealth w_i of an agent i departs from average at time t. The correlation function $C(\tau)$ at time τ, averaged over T timesteps, is then defined as

$$C(\tau) = \frac{1}{NT}\sum_{t=1}^{T}\sum_{i=1}^{N}\Delta w_i(t)\Delta w_i(t+\tau). \tag{3}$$

We use here the normalized correlation function $c(\tau) = C(\tau)/C(0)$, which equals one for $\tau = 0$ and decays as

$$c(\tau) \sim c_0 e^{-\tau/\tau_0} \tag{4}$$

for large τ. A small value of $c(\tau)$ means that being richer or poorer than average at a given time t has little predictive power τ timesteps later. Therefore, $c(\tau)$ measures the "mobility", in the wealth scale, of a typical agent over a time horizon of τ timesteps. The timescale τ_0 over which $c(\tau)$ converges to zero measures the amount of time needed for full "social mixture" (decorrelation from initial wealths, or loss of memory). In a statistical mechanics context, τ_0 is the relaxation time needed for the decay of equilibrium wealth-fluctuations, or "decorrelation time". Borrowing from the theory of equilibrium phase transitions [27, 28], one expects $\tau_0(p, f)$ to diverge as the critical interface is approached from above, as

$$\tau_0(p,f) \sim (p - p^*(f))^{-z}, \tag{5}$$

where the dynamical exponent z can be a function of f eventually. As shown later, the numerical estimation of $\tau_0(p, f)$ allows a very precise determination of the location $p^*(f)$ of the interface.

B. The unstable phase

1. *Ranked wealths*

When $p < p^*(f)$, the system is in the unstable phase, wealth differences are amplified in time and this eventually leads to condensation in the fully mixed case. In the whole unstable phase, the decorrelation time τ_0 is infinite. Therefore an agents' position in the wealth scale becomes frozen, in the long run. In other words, social mobility is suppressed in the appropriation phase.

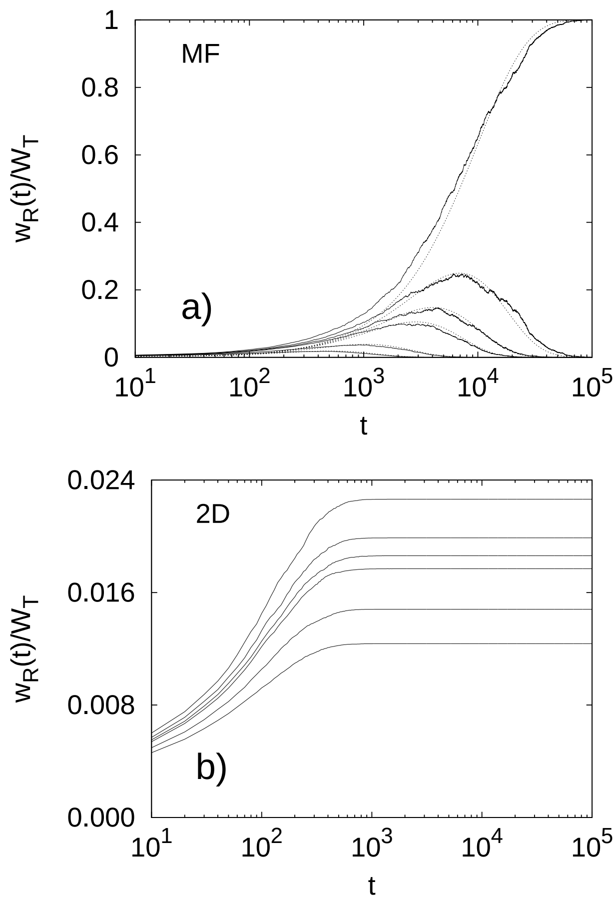

FIG. 1: Time evolution, in the unstable phase, of the relative wealth possessed by the agent whose rank is R, for $R = 1, 2, 3, 4, 10$ and 20 (from top to bottom – full lines) from simulations in: a) full-mixture and, b) on a 2D square lattice with periodic boundaries. The dotted lines in a) show the theoretical prediction (6) for full-mixture. In all cases, $N = 400$ agents, $f = 0.1$, and $p = 0.425$.

A dynamical analysis of the unstable phase [10] for the fully-mixed system shows that the typical wealth $w_R(t)$ of an agent with rank R [29] at time t is $w_R(t) \sim e^{-t\theta\ \frac{(R-1)}{(N-1)}}$, which, after normalizing for a total wealth W_T reads

$$w_R(t) = W_T\ \frac{1 - e^{-t\theta/N}}{1 - e^{-t\theta}} e^{-t\theta\ \frac{(R-1)}{(N-1)}}. \tag{6}$$

This expression is valid at long times, when ranks no longer change as a result of economic exchange. At any fixed time, the ranked-wealth distribution is thus exponential in rank. Accordingly, the transient wealth distribution is of the form $P(w) \sim 1/w$. Fig. 1a compares (6) with numerical results.

2. Condensation criteria

In numerical simulations, a practical criterion is necessary to define wealth condensation within predefined limits. We use for this purpose the ratio $r(t) = w(2^{nd})/w(1^{st})$, involving the wealths of the richest and second-richest agents in the system. This ratio is one for evenly distributed wealth, and goes to zero when all wealth condenses onto a single agent. An alternative useful measure of condensation is the normalized second moment

$$W_2(t) = \frac{\sum_{i=1}^{N} w_i(t)^2}{\left(\sum_{i=1}^{N} w_i(t)\right)^2}, \tag{7}$$

which is similar to the participation ratio in localization studies. W_2 is of order $1/M$ if wealth is more or less evenly distributed among M agents, and goes to one upon condensation onto a single agent. Therefore, $1/W_2$ approximates the number of economically active agents in the system, as much as the inverse participation ratio estimates the number of sites over which a normal mode, or an electron, spreads.

3. Condensation timescales

The timescale $t_0(p, f, N)$ for convergence towards the condensed state is an interesting property that quantifies the dynamics in the unstable phase. This timescale can be estimated theoretically, in the full-mixture case. Using (6) one has that $r(t) = e^{-t/t_0}$, with

$$t_0(p, f, N) \approx \frac{N}{\theta(p, f)}. \tag{8}$$

From (1) and (2), we see that $\theta \sim p^*(f) - p$. Therefore, the condensation timescale t_0 diverges as $(p^*(f) - p)^{-1}$ on approach to the critical interface.

Simple analysis of (6) shows that $w_R(t)$ attains its maximum value at time $T_R = t_0(p, f, N) \log (R/(R-1))$, and goes exponentially fast to zero afterwards for all $R > 1$. This is understood in the following terms. During the condensation process in the unstable phase, an agent with rank R systematically extracts wealth from poorer agents (those with $R' > R$) and transfers some of it to richer agents (those with $R' < R$). As long as the wealth of poorer agents so allows, his balance will be positive, so his wealth will at first increase. But this increase happens at the expense of poorer agents, and for times $t \approx T_R$ these will have exhausted their wealth. Continued transference of wealth upwards (to richer agents) will in turn make the agent with rank R impoverish as well. Therefore, each rank goes bankrupted at a specific timescale. Poorer ranks (of order N) do so at times of order $1/\theta$, while richer ones (of order one) take time $t_0 = N/\theta$. The second-richest agent goes bankrupted at time $T_2 = t_0 \log 2$, leaving a single rich agent to account for most of the wealth. This justifies our identifying of t_0 as the timescale needed for complete condensation. The entire process of enrichment followed by bankruptcy for the different ranks is visualized in Fig. 1.

III. NETWORK YS IN THE STABLE PHASE

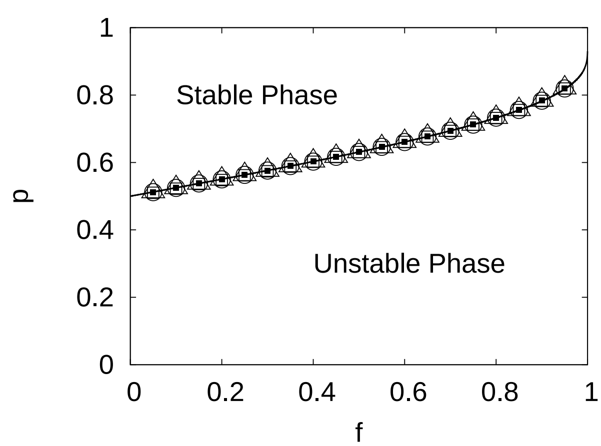

FIG. 2: Critical probability $p_c(f)$ as obtained from fitting (5) to data for $\tau_0(p, f)$ in the full-mixture case (full squares), one-dimensional network (empty squares), two-dimensional networks (circles), and random graphs with $\gamma = 10$ (triangles). Error bars are smaller than the symbols and were not drawn. The full line is the full-mixture theoretical prediction $p^*(f)$ as given by (2). The number of agents was $N = 400$ in all cases.

In this section, results from numerical simulations for 1d rings, 2d square lattices with periodic boundaries, Erdős-Rényi Random Graphs, and full-mixture, are described and compared with analytic predictions for the full-mixture case. Starting from an even distribution of wealth among the N agents, the system is first equilibrated during T_{eq} timesteps before measurements are taken. The required number of equilibration steps is determined by measuring $c(\tau)$ for a series of increasing T_{eq} values, until it is found to no longer depend on T_{eq}. System sizes from $N = 100$ to 1000 agents are considered.

We first describe how the critical line $p_c(f)$ is determined numerically in this work. Firstly, after equilibrating the system as described above, correlation functions $c(\tau)$ are measured in the stable phase for many pairs p, f and for each network considered. Once $c(\tau)$ is known for each pair (f, p) and for each network, (4) is fitted to these data, from where estimates for the relaxation times $\tau_0(p, f)$ are obtained. As expected, τ_0 is found to diverge on approach to a critical value $p_c(f)$ that delimits the stable phase from below. By next fitting (5) to our data for $\tau_0(p, f)$, we can obtain very precise estimates for $p_c(f)$, the location of this divergence. Our results are shown in Fig. 2. The critical values so found are, in all cases, consistent with the full-mixture prediction (Eq. (2)) within numerical errors, suggesting that $p_c(f) = p^*(f)$, for all networks considered.

The above result differs from expectations based on the theory of equilibrium phase transitions. In that case, for a given interacting system, critical parameters as e.g the critical temperature, do depend on the network, i.e. are not universal. The analogous parameter for Yard-Sale is the critical probability p_c, which, within our numerical errors, seems to be network-independent and the same as in the full-mixture, or Mean-Field, case. We therefore propose that the critical interface (2) derived for the full-mixture case is exact on any singly-connected network.

Equilibrium wealth distributions in the stable phase $(p > p^*(f))$ where measured (not shown) for all networks considered in this work, for several pairs (p, f), and compared with full-mixture. We found that $P(w)$ is network-dependent, although differences with full-mixture are minor. Relaxation timescales, on the other hand, are found to be strongly network-dependent, which is reasonable since the paths through which wealth can flow are dictated by network topology. As expected, relaxation to equilibrium takes longer on 1d rings, because there are lesser paths for wealth to flow, and it is fastest in the full-mixture case.

IV. NETWORK YS IN THE UNSTABLE PHASE

In the fully mixed case, whenever the system is in the unstable phase $p < p^*(f)$, where $p^*(f)$ is given by (2), all wealth ends up being owned by a single rich agent in the long run. This process is called wealth condensation [9, 10]. For numerical purposes, in this work we assume that the system is completely condensed when $r(t) = w(2^{nd})/w(1^{st}) \leq 10^{-4}$. The time t_0 needed for this limit to be reached is measured and averaged over 10^3 condensation histories. Results for t_0/N in the complete-graph limit (full mixture) are displayed in Fig. 3a (open and filled circles), and are found to behave as $t_0(p, f) \propto (p^*(f) - p)^{-1}$, in entire accordance with the theoretical result (8) for full-mixture.

For network-restricted Yard-Sale in the unstable phase, complete wealth condensation onto a single agent is no longer observed. Instead, in the long run, the whole wealth condenses onto an extensive set of locally rich agents (LRA). A locally rich agent is *defined* to be one who is richer than any of its neighbors. Agents who are non-LRA impoverish steadily in the long run, because in the unstable phase there is a systematic transference of

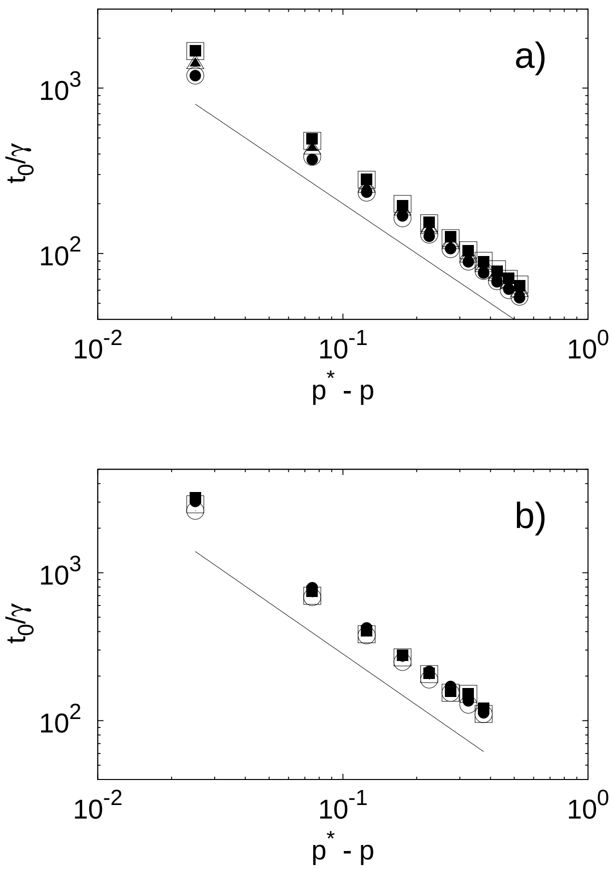

FIG. 3: a) Condensation time t_0 divided by coordination γ, for random graphs with link density $d_l = \gamma/(N-1) = 0.2$ (squares) 0.8 (triangles), and 1.0 (full-mixture case, circles). Empty symbols show data for $N = 400$, filled ones for $N = 900$ agents. The solid line is $\sim 1/(p^* - p)$. b) Condensation time t_0 divided by coordination number $\gamma = 2d$, for periodic rings (squares) and periodic square lattices (circles) with $N = 400$ (empty symbols) and $N = 900$ (filled symbols). The exponent for the divergence at p^* is estimated as 1.15 ± 0.2 (solid line is $\sim 1/(p^* - p)^{1.15}$.).

wealth from poor to rich agents. For long times, each LRA is only connected to agents whose wealth is extremely small. Wealth exchange is then effectively suppressed, leading to dynamical freezing, onto a disordered final state.

A. Condensation times

At long times, there is a clear scale separation between the wealth of each LRAs and those of its neighbors, the latter going exponentially to zero in time. We assume that the final set of LRAs has been irreversibly frozen, and that wealth exchange has effectively stopped, when each LRA is richer than its richest neighbor by a factor of at least 10^{-4}. This criterion generalizes the one

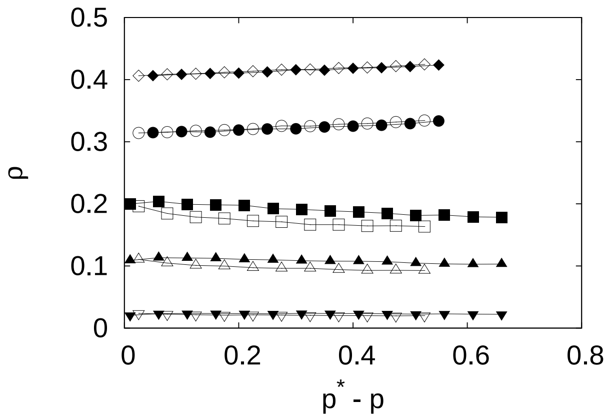

FIG. 4: Final density ρ of locally rich agents, as a function of distance $(p^* - p)$ to the interface, on 1d rings (diamonds), 2d square lattices (circles), and Random Graphs with average coordination $\gamma = 10$ (squares), 20 (triangles), and 100 (inverted triangles). Empty symbols indicate data for for $f = 0.1$ and filled symbols for $f = 0.2$. The number of agents was $N = 400$ in all cases.

we adopted for full-mixture, and reduces to it whenever there is condensation, in which case there is only one LRA.

Fig. 3a shows condensation times divided by coordination γ, for Random Graphs with variable link density $d_l = \gamma/(N-1)$. The case $d_l = 1$ (circles) is the complete graph, or full-mixture case. Condensation times are seen to diverge at the interface as $1/(p^* - p)$, that is, the exponent of this divergence does not depend on γ and is the same as for full-mixture. Our results also show that t_0 is roughly proportional to γ, which is consistent with Eq. (8) for the full-mixture limit, in which case $\gamma = N-1$.

Fig. 3b shows t_0/γ for 1d and 2d networks, where $\gamma = 2d$. The exponent z in $t_0/\gamma \propto (p^*(f) - p)^{-z}$ seems to be slightly larger for these finite-dimensional networks. Although z is arguably dimension-dependent, the quality of our data does not allow us to resolve the difference between z_{1d} and z_{2d}. Our best estimate is $z_{1d,2d} = 1.15 \pm 0.20$ in one and two dimensions.

B. Locally Rich Agents

Clearly, any set of LRAs with arbitrary wealths, surrounded by impoverished agents is a fixed point of the dynamics. There is thus a non-denumerable multiplicity of fixed points, among which the wealth exchange dynamics chooses one stochastically. The statistical properties of these fixed points, as for example the average number of LRAs, and their wealth distribution, depend on the parameters of the model, as well as on the topology of the network, among other things. A detailed study of these properties is beyond the scope of this work. However, some of the most relevant properties of LRAs, namely their number and wealth distribution, will be briefly discussed in the following.

1. *Number of LRAs*

Once the above described criterion for the formation of a set of LRAs is satisfied, the dynamics is stopped, and the properties of LRAs are determined. Measurements are averaged over 10^3 repetitions of the condensation history, for each case.

Condensation of wealth onto a reduced set of agents is a consequence of the unstable nature of the dynamics for $p < p^*(f)$. There is a systematic transfer of wealth from poor to rich, which in turn increases wealth differences. The strength of this instability is given by $\theta(p, f)$ (Eq. (1)), and becomes zero right at the interface $p = p^*(f)$. Close to this interface, where θ is small, wealth appropriation by the richer agents happens very slowly. Wealth has then more time to migrate to richer agents, before the dynamics comes to a halt. One therefore expects the process of wealth concentration onto a single rich agent to happen more completely there, than deep inside the unstable phase, where dynamical arrest takes place in a short time. One could then expect the average number of LRAs to decrease on approach to the interface.

However, our results, displayed in Fig. 4, show that, for a given network, the final density ρ of LRAs depends only very mildly on the exchange parameters p or f. In other words, the number of LRAs is roughly the same in the whole unstable phase, and is only determined by the network's properties (see Section IV B 2). For 1d rings and 2d square lattices, there is even no observable f-dependence in Fig. 4.

2. *Analytical prediction for ρ*

By definition, no two LRAs are connected to each other. Therefore, a set of LRAs constitutes an independent set [22] of the graph. A partition of a graph into independent sets constitutes a coloring. We thus conclude that long-term YS evolution in the appropriation phase identifies colorings of the network.

Since, as our numerical results suggest (see Fig. 4), the number of LRAs is not strongly dependent on f and p, one can obtain useful information by studying the particular case $f = 1$, $p = 1/2$, which is analytically tractable to some extent. In this particular case, whenever two agents interact, the winner is chosen at random. If the richest agent wins, he gets the whole wealth of the loosing agent, who is in turn rendered inactive. A similar process is studied in the context of "coagulation" or "coalescence" $A + A \to A + S$ of immobile reactants on a network [23–25]. Analytical descriptions for the density

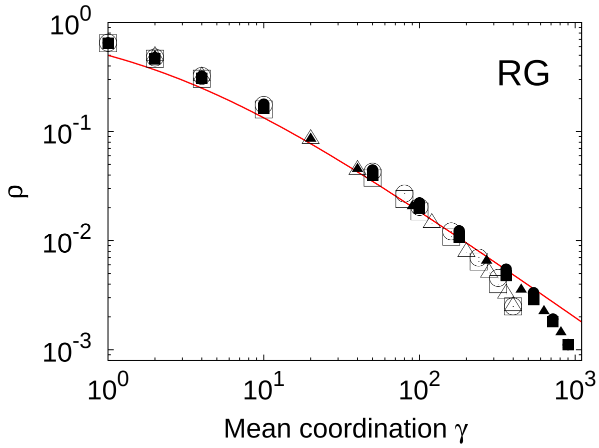

FIG. 5: Final density ρ of locally rich agents versus average coordination γ, on Random Graphs with $N = 400$ agents (empty symbols), and $N = 900$ agents (solid symbols), for $f = 0.1, p = 0$ (squares), $f = 0.6, p = 0$ (circles), and $f = 1.0, p = 1/2$ (triangles). The solid line is a prediction from Abad [25].

of the active species A (S is the inert species), which in our case is the final density of LRA, have been provided for these, as well as for related models [30, 31] that consider "annihilation" $A + A \to S + S$ as well.
In particular, Abad [25] provides explicit expressions for the final density ρ of active agents on 1d and 2d lattices, as well as on Bethe Lattices with coordination γ. If ρ_0 is the initial density of active sites, the final density on a Bethe lattice is

$$\rho = \rho_0 \left(1 + \frac{\gamma - 2}{2}\rho_0\right)^{-\gamma/(\gamma-2)} . \tag{9}$$

For large γ, this gives $\rho = \rho_0$ for $\rho_0 < 2/\gamma$, and $\rho \sim 2/\gamma$ if $\rho_0 > 2/\gamma$.
A comparison between (9) and our own numerical results on Random Graphs (RG) with average coordination γ is shown in Fig. 5. Notice that all sites on a Bethe lattice have γ neighbors, while this is satisfied only on average for Random Graphs. Therefore, a perfect coincidence is not expected. Nevertheless, an acceptable similarity between our numerical results and (9) is found.

The 1d case is obtained from (9) in the limit $\gamma \to 2^+$, and equals $\rho = 1/e$ for $\rho_0 = 1$ as is our case. Our numerical result in 1d is approximately 0.4 (Fig. 4), somewhat larger than this analytic prediction. Abad's two-dimensional approximate result is $\rho = 1/4$ for $\rho_0 = 1$, again slightly smaller than our numerical results on 2d square lattices, shown in Fig. 4.

These results show that the dynamical process of multiplicative wealth concentration on networks has features in common with annihilation and coalescence [23–25, 30, 31] of immobile reactants. Furthermore, it was in the context of those models that the failure of the MF approximation to predict the final density of active species was first noticed. While MF predicts a zero asymptotic density of the active species, on generic networks a finite value is found. This parallels our observation that, while in full mixture wealth condenses onto a single agent, on networks it does so onto an extensive set of agents.

3. *Wealth distribution of LRAs*

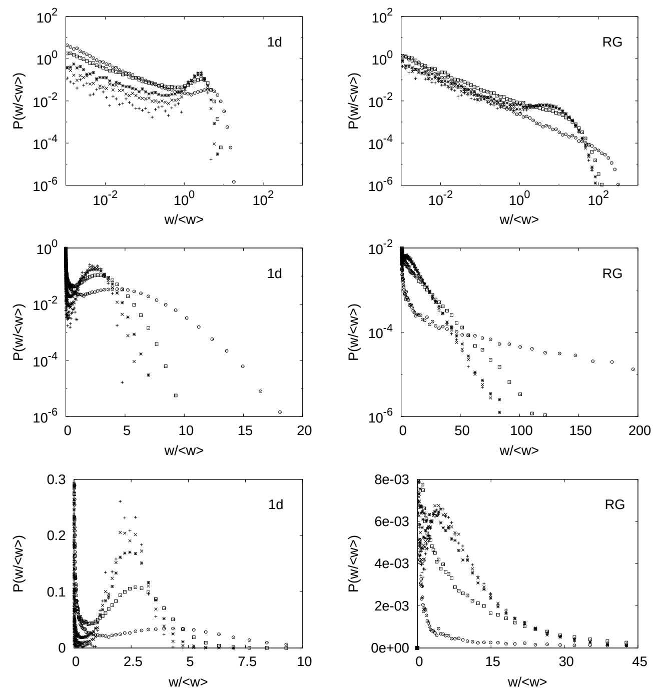

FIG. 6: Log-log (top row), log-lin (middle), and lin-lin (bottom) plots of wealth distribution of locally rich agents (LRA) in the frozen state, with $f = 0.6$ and $p = 0$ (plusses), 0.2 (crosses), 0.4 (asterisks), 0.6 (squares), and 0.659 (circles). The critical probability for this value of f is $p^* = 0.660964$ in MF. The first column is from simulations on 1d rings (similar results were obtained on 2d square lattices) and the second one on Random Graphs with $\gamma = 20$. In all cases, $N = 400$ agents was used.

Wealth distributions of LRAs in the frozen state are displayed in Fig. 6 for 1d and RG with $\gamma = 20$. Wealth distributions on 2d lattices were also measured (are not shown) and found to be qualitatively similar to those in 1d. In all three cases, the minimum in $P(w/\langle w\rangle)$ for $w/\langle w\rangle \approx 1$ suggests the existence of two sets of LRAs with different properties. For the sake of this analysis, the population of LRAs is divided in two groups according to their wealth. Those with wealth $w > \langle w\rangle$, have a roughly normal wealth distribution in 1d and 2d, and an exponential wealth distribution on Random Graphs. We call these "type 1" LRAs. In addition to those, LRAs with $w < \langle w\rangle$, have wealths distributed according to a power-law $P(w) \sim 1/w$ that extends down to zero. These

we call "type 2" LRAs.
We have measured (not shown) the numbers of type 1 and 2 LRAs versus time for all networks with various parameter values. For p values that are not too close to the interface, freezing occurs rapidly, and the final number and cumulative total wealth of type 2 LRAs turns out to be almost negligible compared to those of type 1. In other words, most LRAs are type 1, i.e. have wealths larger than average in the frozen state. Additionally, the wealths of type 1 LRAs are found to have a narrow distribution if not too close to the interface. Very close to the interface, i.e. for $p \to p^*(f)$, on the other hand, a significant amount of conversion from type 1 to type 2 occurs before the frozen state is reached. During this process, the number of type 1 LRAs drops steadily, while the total number of LRAs stays almost constant or increases slowly. Conversion from type 1 to type 2 means that a large number of LRAs, despite being richer than their neighbors, can still loose a significant fraction of their wealth, which in the end goes to the few remaining type 1 LRAs. This is possible because close to the interface θ (Eq. (1)) is small, and therefore being richer does not ensure a strong statistical advantage.
On approach to the interface, wealth distributions of LRAs develop long tails for large wealth, and the power-law behavior $P(w) \sim 1/w$ is seen to extend to the right. Therefore, the distinction between the wealth distributions of type 1 and type 2 LRAs is blurred in this limit. Near the interface, a single LRA ends up owning a significant fraction of the whole wealth. Therefore, even though the total number of LRAs remains approximately constant when the interface is approached, most of them will only have negligible wealth in the end. Therefore, we conclude that wealth condenses onto a single rich agent, on any connected network, when the system is unstable but very close to the critical interface $p^*(f)$.
In the case of Random Graphs, condensation onto a single agent occurs in the whole unstable phase only in the complete graph limit, i.e. in the limit $d_l = \gamma/(N-1) \to 1$. A measure of wealth condensation is provided by W_2 (see Eq. (7)). A plot of W_2 at freezing versus link density d_l is shown in Fig. 7. These data show that full condensation in the whole unstable phase only happens in the complete-graph limit. Inside the unstable phase, wealth is distributed among all LRAs roughly uniformly. Unless the system is really close to the interface, one has $W_2 \sim d_l$. This result can be understood as follows. In the frozen state, as shown in Fig. 6, wealth is distributed exponentially among the resulting $N_{LRA}(\gamma)$ locally rich agents. If there are N_{LRA} locally rich agents and the rest have zero wealth, Eq. (7) can be rewritten as $W_2 = 1/N_{LRA}(1 + \sigma^2/\langle w\rangle^2)$, where σ is the variance of the wealth distribution of the LRAs. For the particular case of an exponential distribution, $\sigma^2 = \langle w\rangle^2$ and therefore $W_2 = 2/N_{LRA}$. As discussed in Section IV B 2, the density ρ of LRA is approximately $2/\gamma$ for large γ. Therefore $N_{LRA} = 2N/\gamma = 2/d_l$, which renders $W_2 \sim d_l$ as observed numerically for (p, f) points not too close to the interface (Fig. 7).

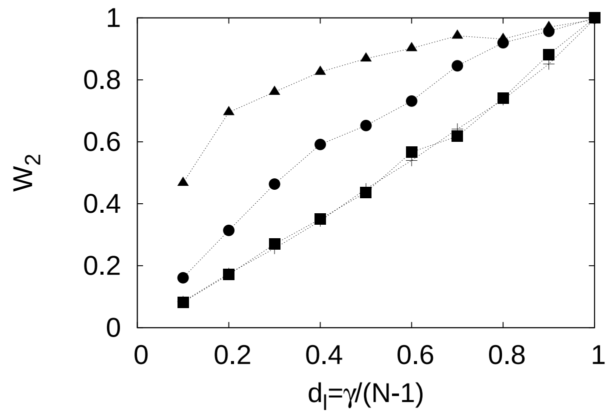

FIG. 7: Second moment W_2 (Eq. (7)) of wealth distribution on Random Graphs at freezing, versus link density $d_l = \gamma/(N-1)$, for $f = 0.1$ and $p = 0.0$ (plusses), 0.45 (squares), 0.52 (circles), and 0.525 (triangles). Dotted lines are guides to the eye. The critical interface for this value of f is located at $p^* = 0.525042$. The number of agents is $N = 200$.

V. DISCUSSION OF RESULTS

Yard-Sale (YS) [6–12] is a simple but realistic model for commercial exchange that presents two phases: a stable (or wealth-sharing) phase where wealth is distributed and an unstable (or wealth-appropriation) one where wealth concentrates in the hands of a few agents. We have numerically studied the static and dynamic properties of YS on several types of networks, comparing them to those in the full-mixture (or mean-field) approximation. Equilibrium wealth distributions $P(w)$ on networks, in the stable phase, are found to be very similar to those in full-mixture. Measuring decorrelation times τ_0, which in the this model can be interpreted as "social mobility" times, we are able to very precisely locate the interface that separates the wealth sharing from the wealth appropriation phases (Fig. 2). Our numerical results strongly suggest that the critical interface $p^*(f)$ derived in the full-mixture approximation (Eq. (2)) is exact on any network.
An important result is the observation that, for network YS in the unstable phase, wealth does not condense onto a single agent as it does in the fully mixed case, but onto an extensive set of locally rich agents (LRA) instead. These LRAs form an independent set [22] in the network, and therefore define a coloring of it. The final density of agents with nonzero wealth is thus finite on networks, while it is zero for full mixture. In recent related work [32], it was proposed that the emergence of many locally rich agents might be due to multiple-connectedness of the network, suggesting that,

on networks made of just one connected component, global condensation onto one agent would eventually occur. This expectation is not confirmed by our results, which show that an extensive number of LRAs remain, in the whole unstable phase, on singly-connected networks as well. It is only in the limit of a complete graph, which is the fully mixed case, or, (on any network) in the limit $\theta \to 0$ (i.e. right at the interface), that condensation onto a single agent is observed.

We have discussed previously unnoticed connections between wealth condensation in YS and earlier studies of annihilation [30, 31] or coalescence [23–25] of immobile reactants, a related statistical problem where the distinction between network results and mean-field ones (i.e. zero vs nonzero final density of LRAs) was first noticed [23, 24]. With the help of these connections, we have been able to compare our own numerical results (Figs. 4 and 5) with analytic predictions [25] for the remaining density of wealth-possessing agents on several networks. A good coincidence is found throughout the entire unstable phase. Furthermore, by using analytical expressions for the density of remaining LRAs on Random Graphs, we were able to explain our numerical results (Fig. 7) showing that $W_2 \propto d_l$ on Random Graphs, deep inside the unstable phase.

Surprisingly, the density of LRAs is essentially constant in the whole unstable phase, although their wealth distribution is not. Their wealth distribution is roughly homogeneous, i.e. has a relatively narrow distribution, except when extremely close to the interface. A particularity that deserves further attention is the fact that the wealth distribution of LRAs in the frozen state is nearly normal in one and two dimensions, but exponential on Random Graphs (Fig. 6)

Very close to the interface that delimits the unstable phase from above, however, wealth is no longer homogeneously distributed among the LRAs, but develops a long right tail of the form $P(w) \sim 1/w$ until, at the interface itself, only one rich agent remains, which owns the whole systems' wealth. Therefore, on the interface itself, condensation onto a single agent is again observed, on any network. However, the time needed for condensation diverges in this limit, in contraposition to the full mixture case, where wealth condenses onto a single agent in finite time.

We have found that YS models only show strongly network-dependent properties when the system's parameters (p, f) are in the unstable phase. In the light of this result, the correlations between topological properties and wealth distributions that have been recently observed in experimental studies of global commercial networks [14–16], may be interpreted as suggesting that the international trade system is itself in the unstable phase. In other words, that the microscopic exchange rules for international trade are such that favor systematic wealth appropriation by larger agents. Other evidences of this possibility have been recently found by analyzing the distribution of per-capita gross domestic products [33].

On the other hand, as already said in the introduction, global wealth distributions depend on processes other than conservative exchange. The generation of wealth by endogenous processes, for example, acts as a source of wealth that would avoid freezing in the unstable phase. A system under unstable exchange, in the presence of wealth creation by endogenous processes, would reach a quasi-stationary state in which wealth is produced everywhere and then channeled towards richer agents by the exchange processes.

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