A MULTI-AGENT NONLINEAR MARKOV MODEL OF THE ORDER BOOK

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Abstract. We introduce and treat rigorously a new multi-agent model of the limit order book. Our model is designed to explain a behavior of the market when new information affecting the market arrives. Our model has two major features. First, it constitutes a nonlinear Markov process. Second, it has two additional parameters which we call slow parameters. These parameters measure mood of two groups of investors, namely, bulls and bears. We explain the intuition behind the equations and present numerical simulations which show that behavior of the model is similar to the behavior of the real market.

Contents

1. Introduction. COB and volume.

Starting from Louis Jean-Baptiste Alphonse Bachelier people tried to model behavior of the market using various stochastic processes. Bachelier himself in 1900 used a Brownian motion, [3]. Later people began to use Markovian diffusions, diffusions with jumps and even processes determined by infinite divisible distributions. Models of the price change are numerous as well as the behavior of the market which changes its statistics over time.

In the last 15-20 years all US equity and futures exchanges made the order book electronic and information about active orders in the book is available to market participants. The term market *microstructure* was born though the definition varies depending on the author, [7].

In the recent years numerous models of the market microstructure were introduced and studied. Here we have to mention papers [11, 19] in financial literature on various models of the book and limit order markets. At the same time in physics literature can be found a large number of the so called multi-agent models, see reviews [2, 12, 13] and original paper [17]. In these models idealized market participants or agents submit buy or sell orders (limit or market) to the matching mechanism and these orders are executed according rules of exchange. Since limit orders are waiting for execution in FIFO queues these multi-agent models should be treated as a part of queuing theory. Also we have to mention the papers of [1, 21] where stochastic models of the order book dynamics are considered.

In this paper we introduce and study a new multiagent nonlinear Markov model of the order book. Our model is designed to simulate a real life phenomena well known to traders in equity and futures market. The phenomena is depicted on the Fig 1 which contains a graph of the price of the index S&P500 future contract ESM08 (HLOC one minute bars) and another graph of trading volume on Friday April 4 of the year 2008. This is the first Friday of the month and information about employment is released at 8:30 am.

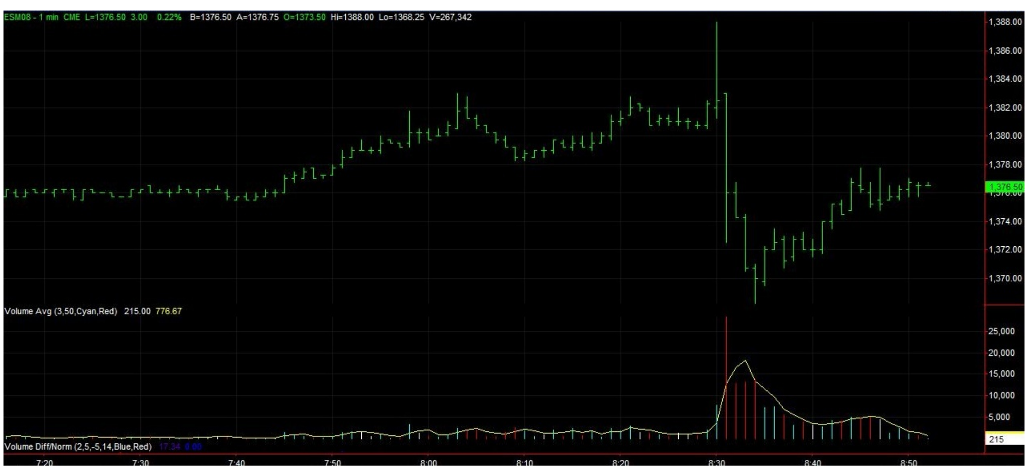

Fig 1.

In absence of the news before 8:30 the price changes in a diffusive manner and volume of trading is fairly low. When news hit the market the price changes by a jump and then slowly returns to the initial range. At the beginning of this process the volume of trading increases significantly. This can be seen from the second graph. When time goes by the volume decreases and the price returns to the vicinity of the price before the announcement. Such mean reverting behavior is well known to intraday traders who follow the standard calendar of announcements of economic indicators, [16].

The price change in our new nonlinear Markov multiagent model is produced by interaction of market participants with different roles through the matching mechanism. This stochastic process is a nonlinear discrete analog of the classical Ornstein-Uhlenbeck process. Our approach is conceptually similar to the attempt to explain classical Brownian motion through mechanical model of a heavy particle interacting with the ideal gas, see [20].

Our model has two major features. First, it uses ideas of kinetic theory. Specifically the ideas from the theory of Vlasov equation. Originally this equation was written for plasma where ions interact with long range Coulomb forces. Therefore an interaction between ions can not be neglected. The force acting on an ion can be computed by averaging the potential over the distribution of other ions in the configuration space. A mathematical model of this phenomena was introduced by H.McKean in the form of nonlinear Markov process, [15]. In our model the dynamics of parameters at any moment of time is also determined through the quantities which are obtained by averaging over the distribution of price at this moment.

Second, the important feature of our model is that it introduces two "slow" parameters $L(t)$ and $M(t)$ which are functions of time. It is known that there are various time scales in time series or data generated by financial markets. For concise review see [4]. Functions $L(t)$ and $M(t)$ measure sentiment of the market (bulls and bears) about fair price of the security. They change relatively slow compare to the price which is the fast parameter.

The numerical simulations of our model produce the graphs of functions $L(t)$, $M(t)$ price and volume which are depicted on Fig 2.

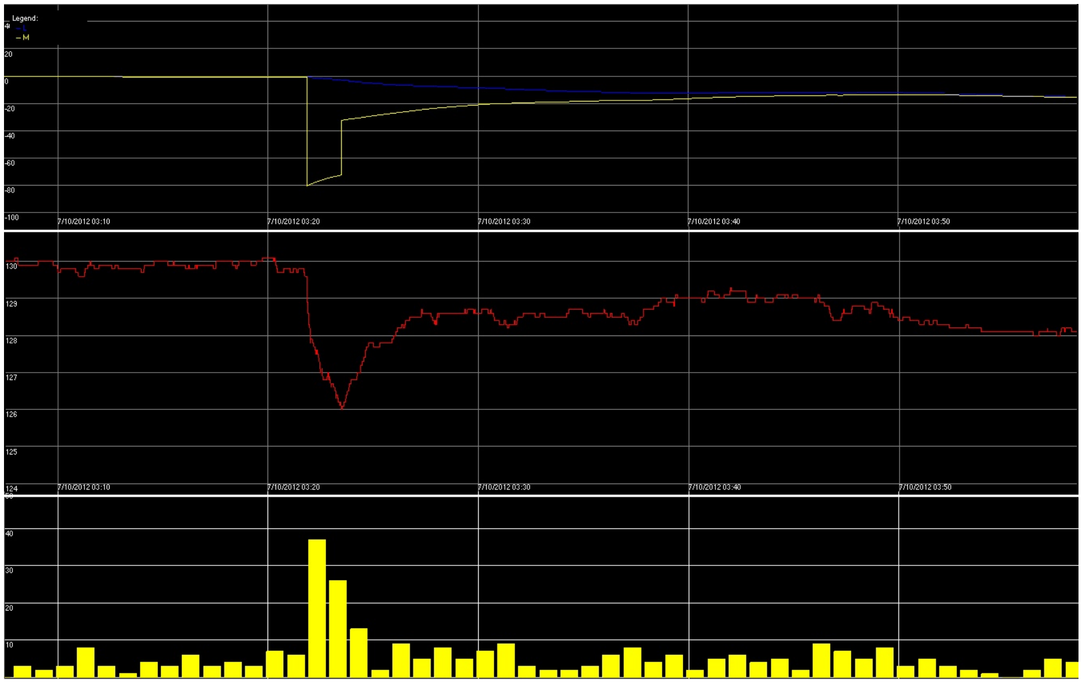

Fig 2.

We let the reader to compare two graphs of price and volume.

Now we proceed to the description of our model.

2. Description of the model.

2.1. **Matching mechanism.** Here we will describe matching mechanism. We adopt the following conventions. A queue is represented by a half axis as shown on Figure 3. Orders contain integer number of elementary unites (contracts/stocks) and they are executed by FIFO rule. Orders can be executed partially.

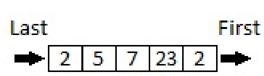

Fig 3.

Order book is a structured list of interacting queues as shown on Figure 4. Price levels are indexed by integers numbers. Each price level has two queues. One queue contains orders to buy not higher then this price level, the second contains orders to sell at price not lower then the price level.

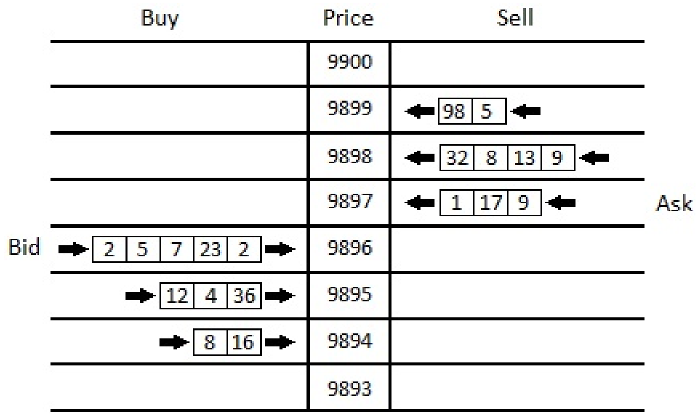

Fig 4.

The book has this form due to the following rule. If some price level X has k orders to buy and some price level Y has p orders to sell then if X greater or equal to Y them min(k,p) orders are executed immediately and removed from the system. This is shown on Figure 5.

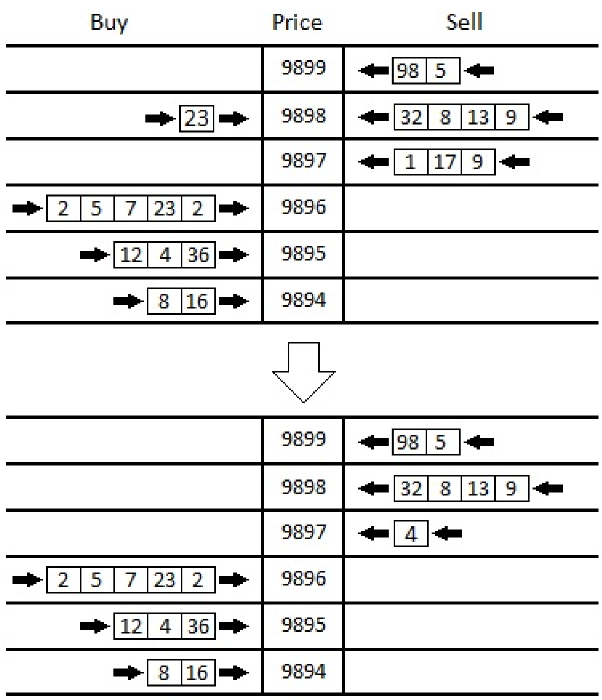

Fig 5.

The real book on NYSE and NASDAQ shows all orders at all levels. On CME the book shows only aggregated levels.

2.2. **Market participants.** In the real market traders submit market and limit orders. The behavior of market participants determine the state of the book and evolution of the price.

In our model these trading activities are separated between four groups of traders.

Ask maker - Ask(p,s).

Bid maker - Bid(p,s).

Ask taker - Buy(s).

Bid taker - Sell(s).

We assume that each group consists of one trader. Ask/Bid makers fill each level of the book with just one contract as shown on Fig. 6.

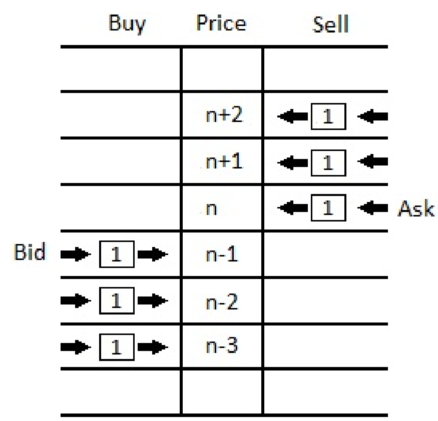

Fig 6.

If Ask taker submit buy order then she buy at the price n. If Bid taker submit market sell order then she sells at the price n-1. If ask/bid taker send the order and take contract on the ask/bid then immediately bid/ask maker fill the emptied level. Therefore dynamics of the book is determined by ask/bid takers.

Ask taker (buy market orders) sends the order with exponential rate

$$\lambda_n = \lambda_n^L = e^{-c(n-L)}.$$

Bid takers (sell market orders) sends the orders with exponential rate

$$\mu_n = \mu_n^M = e^{c(n-M)}.$$

These rates produce a continuous time Markov chain X_t on Z^1. The functions $L(t)$ and $M(t)$ vary in time according to the differential equations

$$\frac{dL(t)}{dt} = -(E_{\pi_{X_t}} \lambda_{X_t}^L - E_{\pi^s} \lambda_X^s) + \sum A_k \delta(t - \tau_k), \tag{2.1}$$

$$\frac{dM(t)}{dt} = +(E_{\pi_{X_t}} \mu_{X_t}^M - E_{\pi^s} \mu_X^s) - \sum B_k \delta(t - \theta_k).$$

The expectation $E_{\pi_{X_t}}$ with respect to the distribution of the process X_t in the moment of time t. The expectation E_{π^s} with respect to the equilibrium distribution of the process X_t with $s = L = M$. The quantity $E_{\pi^s} \lambda_X^s = E_{\pi^s} \mu_X^s$ is a constant. The variables A's and B's are random. They arrive at random times τ's and θ's. We can assume that these times are exponentially distributed. We call terms incorporated under the sum signs an exterior forces. The triple

$$(X_t, L(t), M(t))$$

forms a *nonlinear Markov process*. Its transition probabilities depend on the distribution π_{X_t} and the distribution of the exterior forces. Without exterior forces it can be described as a nonlinear discrete analog of the classical Ornstein-Uhlenbeck process.

Questions regarding proofs of existence and convergence to equilibrium are considered in [18]. In the following sections we explain intuition behind these equations and some of their properties.

Apparently behavior of idealized market participants can be defined differently. In some cases the book can be analyzed only numerically, [5].

3. Ehrenfest model and continuous OU.

We start with the simplest discreet analog of the classical Ornstein-Uhlenbeck process which also exhibits mean reverting behavior. In 1907 P. and T. Ehrenfest [6] introduced a model which later became known as the "Ehrenfest urn model". Fix an integer N and imagine two urns each containing a number of balls, in a such way that the total number of balls in the two urns is 2N. At each moment of time we pick one ball at random (each with probability $1/2N$) and move it to the other urn. If Y_t denote number of balls in the first urn minus N then $Y_t,\ t = 0, 1, 2...;$ form a Markov chain with the state space $\{-N, ..., N\}$.

This Markov chain is reversible with binomial distribution as a stationary measure

$$\pi_k^E = \frac{2N!}{(N+k)!(N-k)!} \left(\frac{1}{2}\right)^{2N}, \qquad k = -N, ..., N;$$

and the generator

$$\mathfrak{A}^E f(k) = \frac{1}{2}\left(1 - \frac{k}{N}\right) f(k+1) + \frac{1}{2}\left(1 + \frac{k}{N}\right) f(k-1) - f(k).$$

In 1930 Ornstein and Uhlenbeck [10] introduced a model of Brownian particle moving under linear force. It is a Markov process with the state space R^1 and the generator

$$\mathfrak{A}^{OU} = \frac{d^2}{dx^2} - 2cx\frac{d}{dx}, \qquad c > 0.$$

The OU process is reversible with invariant density

$$\pi^{OU}(x) = \sqrt{\frac{c}{\pi}}\, e^{-cx^2}.$$

The two Markov processes are related. Under scaling

$$\begin{aligned} k &\sim x, \\ \text{spatial } 1 &\sim \frac{1}{\sqrt{n}}, \\ N &\sim \frac{\sqrt{n}}{c}; \end{aligned}$$

the generator $\mathfrak{A}^E$ takes the form

$$\mathfrak{A}^E f(x) = \frac{1}{2}\left(1 - \frac{xc}{\sqrt{n}}\right) f(x + \frac{1}{\sqrt{n}}) + \frac{1}{2}\left(1 + \frac{xc}{\sqrt{n}}\right) f(x - \frac{1}{\sqrt{n}}) - f(x).$$

Using Teylor expansion after simple algebra we have

$$\mathfrak{A}^E f(x) = \frac{1}{2n}\left[f''(x) - 2cxf'(x)\right] + ... = \frac{1}{2n}\mathfrak{A}^{OU} + ...$$

That shows on a formal level that $\mathfrak{A}^E$converges to the generator $\mathfrak{A}^{OU}$. Also binomial distribution π_k^E converges to a Gaussian measure with the density $\pi^{OU}(x)$.

4. Discrete OU process. Speed and jump measure.

Now we introduce a different discretization of the classical OU process. In its simplest form X_t is a continuous time Markov process on Z^1. The intensity of a jump up is

$$\lambda_n = \lambda_n^L = e^{-c(n-L)},$$

and of a jump down is

$$\mu_n = \mu_n^M = e^{c(n-M)},$$

where $c > 0$ and L and M are real numbers. Later we will make L and M time dependent.

Now we want to study behavior of the process specifically the probability of getting to infinity and expectation of time of returning from infinity. We assume that $L = M = 0$. Since both spatial infinities are identical we consider the process on right semi-axis. A method of study such processes by means of an auxiliary space(X, E) was introduced in the paper [9]. The set E consists of points $\{x_n\}_{n=0}^{\infty} \subset \mathbb{R}_{\geqslant 0}$, where

$$x_0 = \frac{1}{\mu_0} = 1,$$

$$x_1 = x_0 + \frac{1}{\lambda_0} = 2, \ldots$$

$$x_{n+1} = x_n + \frac{\mu_1\mu_2\dots\mu_n}{\lambda_0\lambda_1\dots\lambda_n} = x_n + e^{cn(n+1)},$$

and

$$x_\infty = \lim_{n\to\infty} x_n = \infty.$$

This fact implies that the process is recurrent, see [1: 16.b].

The measure μ is defined by the rule

$$\mu_n = \mu(\{x_n\}) = \frac{\lambda_0\lambda_1\dots\lambda_{n-1}}{\mu_1\mu_2\dots\mu_n} = e^{-cn^2}.$$

The ideal point x_∞ is an entrance point, *i.e.* the expectation of time to come to the finite part of the phase space starting at infinity is finite. This follows from the estimate

$$\sum_{n=0}^{\infty} x_n\mu_n = \sum_{n=0}^{\infty}\sum_{k=0}^{n} e^{c(k(k-1)-n^2)} \leqslant \sum_{n=0}^{\infty}(n+1)e^{-cn} < \infty.$$

Whence, X_t is very close to a Markov chain with finite number of states.

The detailed balance equations

$$\pi(n)\lambda_n = \pi(n+1)\mu_{n+1}, \qquad \pi(n)\mu_n = \pi(n-1)\lambda_{n-1},$$

are satisfied for the distribution

$$\pi^s(n) = \frac{1}{\Xi}e^{-c(n-s)^2}, \qquad s = \frac{L+M}{2}.$$

The normalization factor $\Xi = \Xi(s,c)$ is given by

$$\Xi(s,c) = e^{-cs^2}\Theta\left(\frac{cs}{i\pi}, \frac{ci}{\pi}\right),$$

where Jacobi theta function, [8], is

$$\Theta(v,\tau) = \sum e^{2\pi ivn + \pi i\tau n^2}.$$

It is interesting that invariant measure π which should depend on both parameters L and M depends on s only. The chain is ergodic and reversible.

For $L = M = 0$ the process X_t has generator

$$\mathfrak{A}f(n) = e^{-cn}f(n+1) + e^{cn}f(n-1) - \left(e^{-cn} + e^{cn}\right)f(n).$$

Let us show how on formal level $\mathfrak{A}^{OU}$ can be obtained from the generator of discrete process. If one scales

$$\begin{aligned} n &\sim x, \\ \text{spatial } 1 &\sim \frac{1}{\sqrt{n}}, \\ c &\sim \frac{c}{\sqrt{n}}; \end{aligned}$$

then

$$\mathfrak{A}f(x) = e^{-\frac{c}{\sqrt{n}}x}\left[f(x+\frac{1}{\sqrt{n}}) - f(x)\right] + e^{\frac{c}{\sqrt{n}}x}\left[f(x-\frac{1}{\sqrt{n}}) - f(x)\right].$$

Using Taylor expansion we have

$$\mathfrak{A}f(x) = \frac{1}{n}\left[f''(x) - 2cxf'(x)\right] + ... = \frac{1}{n}\mathfrak{A}^{OU}f(x) + ...$$

Apart from the factor $\frac{1}{n}$ this is a generator of the classical model. Also under this scaling discrete Gaussian distribution $\pi^s(n)$ converges to the Gaussian distribution with the density $\pi^{OU}(x)$. It is naturally to call X_t a discrete Ornstein-Uhlenbeck process.

In addition to s it is natural to introduce another variable $d = \frac{L-M}{2}$. The case when $d = 0$ we call agreement and $d \neq 0$ we call disagreement. The invariant measure does not depend on parameter d but the intensity of jumps does. Simple arguments show that at the equilibrium for any d

$$E_{\pi^s}\lambda_X^L = E_{\pi^s}\mu_X^M.$$

In the case of disagreement at the equilibrium expectations are changed by exponential factor e^{cd}

$$E_{\pi^s}\lambda_X^L = e^{cd}E_{\pi^s}\lambda_X^s, \qquad E_{\pi^s}\mu_X^M = e^{cd}E_{\pi^s}\mu_X^s. \tag{4.1}$$

Apparently when $d > 0$ the expectation increases and if $d < 0$ decreases.

5. Slow functions L and M. Nonlinear Markov process.

Let us assume that parameters L and M are not constants but are some functions of time. They change according to differential equations

$$\frac{dL(t)}{dt} = -(E_{\pi_{X_t}}\lambda_{X_t}^L - E_{\pi^s}\lambda_X^s) + \sum A_k\delta(t-\tau_k), \tag{5.1}$$

$$\frac{dM(t)}{dt} = +(E_{\pi_{X_t}}\mu_{X_t}^M - E_{\pi^s}\mu_X^s) - \sum B_k\delta(t-\theta_k).$$

The variables A's and B's are random and positive. They arrive at random times τ's and θ's. We can assume that these times are exponentially distributed. We call terms incorporated under the sum signs an exterior forces. The triple

$$(X_t, L(t), M(t))$$

forms a *nonlinear Markov process*. Its transition probabilities depend on the distribution π_{X_t} of the first component and distribution of the exterior forces.

Lets us explain the intuition behind these equations. In reality there are two groups of market participants. They are called bulls and bears. Consider the bulls which try to make the price higher all the time and submit buy market orders with intensity λ_n^L. Buying a contract is an expense for them. Bulls have two quantities in mind. One is the quantity $E_{\pi^s}\lambda^s_X$ which is how much they are willing to pay per unit of time. Another quantity is the function $L(t)$ which measures their expectations about fair price. If their expectations about fair price are too high then the quantity $E_{\pi_{X_t}}\lambda^L_{X_t} - E_{\pi^s}\lambda^s_X$ is positive and the bulls decrease their expectations according to the differential equation

$$\frac{dL(t)}{dt} = -(E_{\pi_{X_t}}\lambda^L_{X_t} - E_{\pi^s}\lambda^s_X).$$

When positive news arrive the bulls change their expectations by a jump. That is contribution of the sum corresponding exterior sources. Exactly the same arguments apply to the bears which submit sell orders with intensity μ^M_n.

The situation resembles random walk in random environment. The functions $L(t)$ and $M(t)$ determine transition probabilities $P^{L,M}$ of X_t and expectations $E^{L,M}$ with respect to these probabilities. These functions are hidden parameters of the system and it is possible to condition on them. Instead we condition on the exterior forces A's and B's which through differential equations determine the functions L and M. These transition probabilities are denoted by $P^{A,B}$ and corresponding expectations by $E^{A,B}$. We refer to them as quenched probabilities and expectations.

The Poisson measure on exterior forces A's and B's we denote by P. The product measure $P^{A,B} \times P$ determines averaged transition probabilities of the process X_t. The corresponding expectation we denoted by E_{X_t}.

6. Hydrodynamic limit.

In this case one assumes that the variables L and M change change with macroscopic time t while the process X_τ moves with microscopic time $\tau = t/\epsilon$ where $\epsilon > 0$. In the limit $\epsilon \to 0$ the process X_τ is always at equilibrium *i.e.* it has stationary distribution π^s for all moments of time. Therefore, using 4.1

$$\frac{dL(t)}{dt} = -(e^{cd} - 1)E_{\pi^s}\lambda_X^s + \sum A_k\delta(t - \tau_k),$$
$$\frac{dM(t)}{dt} = +(e^{cd} - 1)E_{\pi^s}\mu_X^s + \sum B_k\delta(t - \theta_k).$$

Subtraction one equation from another and discarding the "news" we obtain closed equation for the function $d(t)$:

$$d^\bullet(t) = -(e^{cd} - 1)V,$$

where $V = E_{\pi^s}\lambda_X^s = E_{\pi^s}\mu_X^s$. This can be solved explicitly using substitution $d = \frac{1}{c}\log(\pm\varphi)$.

If a solution is negative for some moment of time then it stays negative for all moments and it is given by

$$d(t) = \frac{1}{c}\log\frac{Ae^{cVt}}{Ae^{cVt}+1}, \qquad A > 0.$$

The solution increases monotonously from negative infinity to zero when time runs the whole axis.

On the opposite, if a solution is positive for some moment of time then it stays positive for all moments of time when its defined. The solution is given by the formula

$$d(t) = \frac{1}{c}\log\frac{Ae^{cVt}}{Ae^{cVt}-1}, \qquad A > 0.$$

The solution is defined and decreases monotonously from positive infinity to zero for $t > t_0 = -\frac{1}{cV}\log A$.

7. Continuum limit.

Now we consider a continuum limit of the discrete model. The continuous time process $\mathcal{X}(t)$ with values in R^1 is defined as a solution of the stochastic differential equation

$$d\mathcal{X}(t) = e^{\mathfrak{c}(\text{Ł}-\mathcal{M})}\left[dw(t) - 2\mathfrak{c}\left(\mathcal{X}(t) - \frac{\text{Ł}+\mathcal{M}}{2}\right)dt\right],$$

where $\mathfrak{c} > 0$ and the functions $\text{Ł}(t)$ and $\mathcal{M}(t)$ are

$$\frac{d\text{Ł}(t)}{dt} = -(e^{\mathfrak{c}(\text{Ł}-\mathcal{M})} - 1),$$
$$\frac{d\mathcal{M}(t)}{dt} = +(e^{\mathfrak{c}(\text{Ł}-\mathcal{M})} - 1).$$

There is a simple formal scaling relation between two models. The generator of the discrete model has the form

$$\mathfrak{A}f(n) = e^{-c(n-L(t))}f(n+1) + e^{c(n-M(t))}f(n-1) - \left(e^{-c(n-L(t))} + e^{c(n-M(t))}\right)f(n)$$

Introducing new variables

$$s = \frac{L+M}{2}, \qquad d = \frac{L-M}{2},$$

we have

$$\mathfrak{A}f(n) = e^{cd}\left[e^{-c(n-s)}f(n+1) + e^{c(n-s)}f(n-1) - \left(e^{-c(n-s)} + e^{c(n-s)}\right)f(n)\right].$$

If one scales

$$\begin{aligned} n &\sim x \\ \text{spatial } 1 &\sim \frac{1}{\sqrt{n}} \\ c &\sim \frac{\mathfrak{c}}{\sqrt{n}} \\ L &\sim \frac{Ł + \mathcal{M}}{2} + \sqrt{n}(Ł - \mathcal{M}) \\ M &\sim \frac{Ł + \mathcal{M}}{2} - \sqrt{n}(Ł - \mathcal{M}) \end{aligned}$$

then

$$s \sim \frac{Ł + \mathcal{M}}{2}, \qquad d \sim \sqrt{n}(Ł - \mathcal{M});$$

and for the generator we have

$$\mathfrak{A}f(x) \sim \frac{1}{n}e^{\mathfrak{c}(Ł - \mathcal{M})}\left[f''(x) - 2\mathfrak{c}(x-s)f'(x)\right] + \ldots.$$

Apart from the factor $\frac{1}{n}$ this is the generator of the continuous model. Time scaling $t \sim nt$ removes this factor. The differential equations for L and M when they scaled the same way produce differential equations for functions Ł and $\mathcal{M}$.

If $Ł(t) = \mathcal{M}(t) = s$, then $\mathcal{X}(t)$ is just a classical Ornstein-Uhlenbeck process with mean s. It can be proved similar to [18] that if the process starts with some $Ł(0) \neq \mathcal{M}(0)$, then $\mathcal{X}(t)$ converges to OU and $Ł(t)$ and $\mathcal{M}(t)$ converge to the same constant s.

8. Propagation of chaos and Monte-Carlo simulation.

For the purpose numerical simulation we consider multi-particle approximation of the process $(X_t, L(t), M(t))$.

Everywhere below we will denote by $\mathrm{c}(\bullet)$ any nonnegative nondecreasing function of some parameters. Define a banach space B_α consisted of all vector-functions $p = \{p_n\}_{n\in\mathbb{Z}}$ with the Banach norm

$$\|p\|_\alpha = \sum_{n\in\mathbb{Z}} |p_n| \exp\left(\frac{cn^2}{2} + \alpha|n|\right), \quad \alpha \in \mathbb{R}.$$

Also let $p(0)$ denote the initial distribution of X_t. In [18] we have proved the following theorem:

Theorem 8.1. *For any initial probability measure $p(0) \in B_\alpha$, the process X_t is defined correctly on the half-line $[0, \infty)$ and for any $t \geq 0$ its distribution $p(t)$ belongs to B_α.*

To simplify notations we assume $c = 1$. Fix some integer N and consider the distribution of N particles on $\{\xi_i\}$: $\xi_1, \xi_2, \ldots, \xi_N$; $\xi_i \in \mathbb{Z}$. The positions of particles change with time $\xi_i = \xi_i(t)$. We define a stochastic process $X_N = (p_N, L_N(t), M_N(t))$, where $p_N(\cdot) = p_N(\bullet, t)$ is empirical distribution of $\{\xi_i\}$ at the moment t. All particles $\{\xi_i\}$ jump with the same rates

$$\lambda_N(\xi_i) = e^{-(\xi_i - L_N)}, \quad \mu_N(\xi_i) = e^{(\xi_i - M_N)}.$$

The functions $L_N(t)$, $M_N(t)$ change according to differential equations:

$$\begin{cases} \frac{dL_N}{dt} = & -\left(\sum_{n\in\mathbb{Z}} p_N(n)\lambda_N(n) - E_{\pi^s}\lambda_X^s\right) + \sum A_k\delta(t-\tau_k); \\ \frac{dM_N}{dt} = & +\left(\sum_{n\in\mathbb{Z}} p_N(n)\mu_N(n) - E_{\pi^s}\mu_X^s\right) - \sum B_k\delta(t-\theta_k). \end{cases} \tag{8.1}$$

The random variables A, B, τ, θ's have the same distributions as before.

We also define an auxiliary inner product on the space of infinite sequences $\xi(n)$ and $\eta(n)$, $n \in \mathbb{Z}$; as

$$\langle \xi, \eta \rangle_\alpha = \sum_{n\in\mathbb{Z}} \xi(n)\eta(n) \exp\left(\frac{n^2}{2} + \alpha|n|\right), \quad \alpha \in \mathbb{R}.$$

This inner product produces the euclidean norm denoted by $|\cdot|_\alpha$. Now we are ready to prove the following theorem.

Theorem 8.2. *Let $p(0) \in B_{\alpha+1}$, where α is an arbitrary real number. Assume that $E\|p_N(0) - p(0)\|_{\alpha+1} \to 0$, as $N \to \infty$. Then for all $t \geq 0$*

$$\sup_{s\leq t} E|p_N(s) - p(s)|_\alpha^2 \to 0, \qquad as \qquad N \to \infty.$$

Proof. We will prove the theorem in assumption of absence of the news. For our purposes it will be convinient to decompose the inifitisemal matrices of any participants corresponding to the processes X and X_N into the sum of their diagonal and off-diagonal parts:

$$H = H_0 + V; \quad H_N = H_{N0} + V_N.$$

Lemma 8.3. *(auxilary results) Let $t \geq 0$, then for all $N \in \mathbb{N}$*

(1) $e^{L(t)} \leq \mathrm{c}(t)$; $e^{-M(t)} \leq \mathrm{c}(t)$; $e^{L_N(t)} \leq \mathrm{c}(t)$; $e^{-M_N(t)} \leq \mathrm{c}(t)$.

(2) $|e^{\pm L} - e^{\pm L_N}| \leq \mathrm{c}(t)\Gamma(t)$ *and* $|e^{\pm M} - e^{\pm M_N}| \leq c(t)\Gamma(t)$, *where*

$$\Gamma(t) = \int_0^t \sum_{n\in\mathbb{Z}} |p(n,s) - p_N(n,s)| e^{|n|} ds.$$

(3) $E\Gamma^2(t) \le c(t) \sup_{s\le t} E|p_N - p|_\alpha^2$.
(4) *If* $p(0) \in B_\alpha$, *then* $E|p(H_{N0} - H_0)|_\alpha^2 \le \mathrm{c}(t) \sup_{s\le t} E|p_N - p|_\alpha^2$.
(5) *If* $p(0) \in B_\alpha$, *then* $E|p(V_N - V)|_\alpha^2 \le \mathrm{c}(t) \sup_{s\le t} E|p_N - p|_\alpha^2$.
(6) *If* $p(0) \in B_\alpha$, *then* $E\langle p_N - p, (p_N - p)H_N\rangle_\alpha \le \mathrm{c}(t) \sup_{s\le t} E|p_N - p|_\alpha^2$.
(7) *Let* $E\|p_N(0)\|_{\alpha+1} < \infty$, *then*

$$\sum_n \alpha_n E Q_n \le c(t, E\|p_N(0)\|_{\alpha+1}),$$

where

$$Q_n = p_N(n-1)\lambda_N(n-1) + p_N(n+1)\mu_N(n+1) + p_N(n)\lambda_N(n) + p_N(n)\mu_N(n).$$

Proof. The first statement follows directly from (5.1) and (8.1). Let $\xi = e^{-L} - e^{-L_N}$, then (5.1) and (8.1) imply

$$\xi' + E_{\pi^s}\lambda_X^s \xi = \sum_{n\in\mathbb{Z}} (p(n) - p_N(n))e^{-n}.$$

This ODE can be solved explicitly and we get

$$|e^{-L} - e^{-L_N}| = |e^{-C_\lambda t}\int_0^t e^{C_\lambda s} \sum_{n\in\mathbb{Z}} (p(n) - p_N(n))e^{-n} ds| \le c(t)\Gamma(t),$$

and therefore using the first statement

$$|e^{L} - e^{L_N}| = e^{L+L_N}|e^{-L} - e^{-L_N}| \le c(t)\Gamma(t).$$

The other statements of the second point can be obtained by analogy. Applying Cauchy– Schwarz inequality twice we get the third statement:

$$E\Gamma^2(t) \le c(t)\int_0^t E\left(\sum_n (p_N - p)(n)e^{|n|}\right)^2 ds$$
$$\le c(t)\int_0^t E|p_N - p|_\alpha^2 \left(\sum_n e^{-\frac{n^2}{2} + (2-\alpha)|n|}\right) ds \le \mathrm{c}(t)\sup_{s\le t} E|p_N - p|_\alpha^2.$$

We get the fourth statement in the following way:

$$E|p(H_{N0} - H_0)|_\alpha^2 = E\sum_{n\in\mathbb{Z}} e^{\frac{n^2}{2} + \alpha|n|} p(n)^2 |H_{N0}(n,n) - H_0(n,n)|^2 \le$$
$$\le E\sum_{n\in\mathbb{Z}} e^{\frac{n^2}{2} + (\alpha+2)|n|} p(n)^2 (|e^L - e^{L_N}| + |e^{-M} - e^{-M_N}|)^2 \le$$
$$\le \underbrace{\left(\sum_{n\in\mathbb{Z}} e^{\frac{n^2}{2} + (\alpha+2)|n|} p(n)^2\right)}_{=\mathrm{c}(t),\ \text{since}\ \|p\|_\alpha \le c(t)} \cdot \mathrm{c}(t) E\Gamma^2(t) \le \mathrm{c}(t)\sup_{s\le t} E|p_N - p|_\alpha^2.$$

We get the fifth statement from the next estimation:

$$E|p(V_N - V)|_\alpha^2 = E\sum_{n\in\mathbb{Z}} e^{\frac{n^2}{2}+\alpha|n|}|e^{-n+1}(e^L - e^{L_N})p(n-1) + e^{n+1}(e^{-M} - e^{-M_N})p(n+1)|^2 \le$$

$$\le \text{const}\sum_{n\in\mathbb{Z}} e^{\frac{n^2}{2}+\alpha|n|}\left[e^{-2n+2}p(n-1)^2E(e^L - e^{L_N})^2 + e^{2n+2}p(n+1)^2E(e^{-M} - e^{-M_N})^2\right] \le$$

$$\le \text{const}\underbrace{\sum_{n\in\mathbb{Z}}(e^{\frac{(n+1)^2}{2}+\alpha|n+1|-2n} + e^{\frac{(n-1)^2}{2}+\alpha|n-1|+2n})p(n)^2}_{=\mathrm{c}(t),\ \text{since}\ \|p\|_\alpha\le c(t)} \cdot c(t)E\Gamma^2(t) \le \mathrm{c}(t)\sup_{s\le t} E|p-p_N|_\alpha^2.$$

Denoting for brevity $x = p_N - p$, we get the sixth statement

$$E\langle x, xH_N\rangle_\alpha = \sum_n e^{\frac{n^2}{2}+\alpha|n|}Ex_n\left(x_{n-1}e^{-n+1+L_N} + x_{n+1}e^{n+1-M_N} - x_ne^{-n+L_N} - x_ne^{n-M_N}\right) =$$

$$= \sum_n e^{\frac{n^2}{2}+\alpha|n|-n}Ee^{L_N}\left(ex_{n-1}x_n - x_n^2\right) + \sum_n e^{\frac{n^2}{2}+\alpha|n|+n}Ee^{-M_N}\left(ex_nx_{n+1} - x_n^2\right) \le$$

$$\le c(t)\sum_n e^{\frac{n^2}{2}-n+\alpha|n|}Ex_{n-1}^2 + c(t)\sum_n e^{\frac{n^2}{2}-n+\alpha|n|}Ex_{n+1}^2 \le c(t)E|x|_\alpha^2 \le \mathrm{c}(t)\sup_{s\le t} E|p-p_N|_\alpha^2.$$

Note that $\sum_n \alpha_n EQ_n \le c(t)E\|p_N(t)\|_{\alpha+1}$. Fix $t \ge 0$ and consider the Markov chain $\mathcal{X}$ on $\mathbb{Z}$ with intensities of jumps

$$n \to n+1 : \lambda_n = \exp\{-|n| + \max(+L(0), -M(0)) + E_{\pi^s}\lambda_X^s t\} \cdot I\{n \ge 0\}$$
$$n \to n-1 : \mu_n = \exp\{-|n| + \max(+L(0), -M(0)) + E_{\pi^s}\lambda_X^s t\} \cdot I\{n < 0\}.$$

By analogy with lemma 2.4 in [18] it is easy to prove $\mathcal{X}$ defines a semigroup of transitions probabilities $P_{\mathcal{X}}(\cdot,\cdot)$, which are bounded operators in $B_{\alpha+1}$. Also due to the choice of transition probabilities we can always construct a coupling between one participant of $X_N : \xi_i$ and $\mathcal{X}$ such that the absolute value of position of $\mathcal{X}$ is not greater than $|\xi_i|$ for all $s \le t$. Therefore we conclude $E\|p_N(t)\|_{\alpha+1} \le E\|p_{\mathcal{X}}(t)\|_{\alpha+1} = E\|p_N(0)P_{\mathcal{X}}(0,t)\|_{\alpha+1} \le c(t)E\|p_N(0)\|_{\alpha+1}$. □

Now we are ready to prove the theorem. Let us consider the process X_N. During time dt the variable $p_N(t)$ can be changed in the following way:

$$p_N(n) \to \begin{cases} p_N(n) - \frac{1}{N}, & \text{with probability } P_{out}dt = Np_N(n)(\lambda_N(n) + \mu_N(n))dt; \\ p_N(n) + \frac{1}{N}, & \text{with p}-\text{ty } P_{in}dt = Ndt(p_N(n-1)\lambda_N(n-1) + p_N(n+1)\mu_N(n+1)); \\ p_N(n), & \text{with p}-\text{ty } 1 - P_{out}dt - P_{in}dt. \end{cases}$$

The process X is deterministic, so during time dt

$$p(n) \to p(n) + (pH(t))(n)dt =$$
$$= p(n) + (\lambda_{n-1}p(n-1) - (\lambda_np(n) + \mu_np(n)) + \mu_{n+1}p(n+1))dt.$$

Using that we get

$$(p_N(n)-p(n))^2 \to \begin{cases} (p_N(n)-p(n)-\frac{1}{N})^2+O(dt), & P_{out}dt; \\ (p_N(n)-p(n)+\frac{1}{N})^2+O(dt), & P_{in}dt; \\ (p_N(n)-p(n)-(pH(t))(n)dt)^2, & 1-P_{out}dt-P_{in}dt. \end{cases}$$

Applying Markov property and opening brackets we get

$$\frac{d}{dt}E(p_N(n)-p(n))^2 = 2E(p_N(n)-p(n))(p_N H_N(n)-pH(n)) + \frac{1}{N}EQ_n =$$

$$2E(p_N(n)-p(n)) \cdot [(p_N-p)H_N(n)+p(H_{N0}-H_0)(n)+p(V_N-V)(n)] + \frac{1}{N}EQ_n,$$

Summing up, we get

$$\frac{d}{dt}E|p_N-p|^2_\alpha = \frac{d}{dt}\sum_{n\in\mathbb{Z}} e^{\frac{n^2}{2}+\alpha|n|}E(p_N(n)-p(n))^2 \le$$

$$2E\langle p_N-p,(p_N-p)H_N\rangle_\alpha +$$

$$2\left(E|p_N-p|^2_\alpha\right)^{1/2}\cdot\left[\left(E|p(V_N-V)|^2_\alpha\right)^{1/2}+\left(E|p(H_{N0}-H_0)|^2_\alpha\right)^{1/2}\right]+\frac{1}{N}\sum_{n\in\mathbb{Z}}\alpha_n EQ_n.$$

Summarizing results of lemma 8.3 we get the final estimate:

$$\frac{d}{dt}E|p_N-p|^2_\alpha \le \mathrm{c}(t)\sup_{s\le t}E|p_N(s)-p(s)|^2_\alpha+\frac{1}{N}\mathrm{c}(t).$$

The Grownall's lemma implies the result. □

In order to produce simulations depicted on Fig. 2 we used multiparticle approximation with $N = 1000$ particles $\{\xi_i\}_{i=1}^N$ described above. We suppose that the tick is $\varepsilon = 0.1$, so the process lives on the lattice $\varepsilon\mathbb{Z}$. All particles $\{\xi_i\}$ jump with the same rates

$$\lambda_N(\xi_i) = Ke^{-c(\xi_i-L_N)}, \qquad \mu_N(\xi_i) = Ke^{c(\xi_i-M_N)},$$

where K is a regularizing parameter. We put $c = 0.05$ and $K = 0.0333$. There is a jump on the picture at the moment of event $L \to L-A$, where $A = 80$. After that at some exponentially distributed moment of time there is a second jump $L \to L+\frac{A}{2}$.

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