Chapter 34

NEW ECONOMETRIC APPROACHES TO STABILIZATION POLICY IN STOCHASTIC MODELS OF MACROECONOMIC FLUCTUATIONS

JOHN B. TAYLOR*

Stanford University

Contents

*Grants from the National Science Foundation and the Guggenheim Foundation are gratefully acknowledged. I am also grateful to Olivier Blanchard, Gregory Chow, Avinash Dixit, George Evans, Zvi Griliches, Sandy Grossman, Ben McCallum, David Papell, Larry Reed, Philip Reny, and Ken West for helpful discussions and comments on an earlier draft.

Handbook of Econometrics, Volume III, Edited by Z. Griliches and M.D. Intriligator

1. Introduction

During the last 15 years econometric techniques for evaluating macroeconomic policy using dynamic stochastic models in which expectations are consistent, or rational, have been developed extensively. Designed to solve, control, estimate, or test such models, these techniques have become essential for theoretical and applied research in macroeconomics. Many recent macro policy debates have taken place in the setting of dynamic rational expectations models. At their best they provide a realistic framework for evaluating policy and empirically testing assumptions and theories. At their worst, they serve as a benchmark from which the effect of alternative assumptions can be examined. Both "new Keynesian" theories with sticky prices and rational expectations, as well as "new Classical" theories with perfectly flexible prices and rational expectations fall within the domain of such models. Although the models entail very specific assumptions about expectation formation and about the stochastic processes generating the macroeconomic time series, they may serve as an approximation in other circumstances where the assumptions do not literally hold.

The aim of this chapter is to describe and explain these recently developed policy evaluation techniques. The focus is on discrete time stochastic models, though some effort is made to relate the methods to the geometric approach (i.e. phase diagrams and saddlepoint manifolds) commonly used in theoretical continuous time models. The exposition centers around a number of specific prototype rational expectations models. These models are useful for motivating the solution methods and are of some practical interest *per se*. Moreover, the techniques for analyzing these prototype models can be adapted fairly easily to more general models. Rational expectations techniques are much like techniques to solve differential equations: once some of the basic ideas, skills, and tricks are learned, applying them to more general or higher order models is straightforward and, as in many differential equations texts, might be left as exercises.

Solution methods for several prototype models are discussed in Section 2. The effects of anticipated, unanticipated, temporary, or permanent changes in the policy variables are calculated. The stochastic steady state solution is derived, and the possibility of non-uniqueness is discussed. Evaluation of policy rules and estimation techniques oriented toward the prototype models are discussed in Sections 3 and 4. Techniques for general linear and nonlinear models are discussed in Sections 5 and 6.

2. Solution concepts and techniques

The *sine qua non* of a rational expectations model is the appearance of forecasts of events based on information available before the events take place. Many different techniques have been developed to solve such models. Some of these techniques are designed for large models with very general structures. Others are designed to be used in full information estimation where a premium is placed on computing reduced form parameters in terms of structural parameters as quickly and efficiently as possible. Others are short-cut methods designed to exploit special features of a particular model. Still others are designed for exposition where a premium is placed on analytic tractability and intuitive appeal. Graphical methods fall in this last category.

In this section, I examine the basic solution concept and explain how to obtain the solutions of some typical linear rational expectations models. For expositional purposes I feel the method of undetermined coefficients is most useful. This method is used in time series analysis to convert stochastic difference equations into deterministic difference equations in the coefficients of the infinite moving average representation. [See Anderson (1971, p. 236) or Harvey (1981, p. 38)]. The difference equations in the coefficients have exactly the same form as a deterministic version of the original model, so that the method can make use of techniques available to solve deterministic difference equations. This method was used by Muth (1961) in his original exposition of the rational expectations assumption. It provides a general unified treatment of most stochastic rational expectations models without requiring knowledge of any advanced techniques, and it clearly reveals the nature of the assumptions necessary for existence and uniqueness of solutions. It also allows for different viewpoint dates for expectations, and provides an easy way to distinguish between the effects of anticipated versus unanticipated policy shifts. The method gives the solution in terms of an infinite moving average representation which is also convenient for comparing a model's properties with the data as represented in estimated infinite moving average representations. An example of such a comparison appears in Taylor (1980b). An infinite moving average representation, however, is not useful for maximum likelihood estimation for which a finite ARMA model is needed. Although it is usually easy to convert an infinite moving average model into a finite ARMA model, there are computationally more advantageous ways to compute the ARMA model directly as we will describe below.

2.1. *Scalar models*

Let y_t be a random variable satisfying the relationship

$$y_t = \alpha \underset{t}{\mathrm{E}} y_{t+1} + \delta u_t, \tag{2.1}$$

where α and δ are parameters and E_t is the conditional expectation based on all information through period t. The variable u_t is an exogenous shift variable or "shock" to the equation. It is assumed to follow a general linear process with the representation

$$u_t = \sum_{i=0}^{\infty} \theta_i \varepsilon_{t-i}, \tag{2.2}$$

where $\theta_i = 0,1,2,\ldots$ is a sequence of parameters, and where ε_t is a serially uncorrelated random variable with zero mean. The shift variable could represent a policy variable or a stochastic error term as in an econometric equation. In the latter case, δ would normally be set to 1.

The information upon which the expectation in (2.1) is conditioned includes past and current observations on ε_t as well as the values of α, δ, and θ_i. The presence of the expected value of a *future* endogenous variable $\mathrm{E}_t y_{t+1}$ is emphasized in this prototype model because the dynamic properties that this variable gives to the model persist in more complicated models and raise many important conceptual issues. Solving the model means finding a stochastic process for the random variable y_t that satisfies eq. (2.1). The forecasts generated by this process will then be equal to the expectations that appear in the model. In this sense, expectations are consistent with the model, or equivalently, expectations are rational.

A macroeconomic example. An important illustration of eq. (2.1) is a classical full-employment macro model with flexible prices. In such a model the real rate of interest and real output are unaffected by monetary policy and thus they can be considered fixed constants. The demand for real money balances – normally a function of the nominal interest rate and total output – is therefore a function only of the expected inflation rate. If p_t is the log of the price level and m_t is the log of the money supply, then the demand for real money can be represented as

$$m_t - p_t = -\beta\left(\underset{t}{\mathrm{E}} p_{t+1} - p_t\right), \tag{2.3}$$

with $\beta > 0$. In other words, the demand for real money balances depends negatively on the expected rate of inflation, as approximated by the expected first difference of the log of the price level. Eq. (2.3) can be written in the form of eq. (2.1) by setting $\alpha = \beta/(1+\beta)$ and $\delta = 1/(1+\beta)$, and by letting $y_t = p_t$ and $u_t = m_t$. In this example the variable u_t represents shifts in the supply of money, as generated by the process (2.2). Alternatively, we could add an error term v_t to the right hand side of eq. (2.3), to represent shifts in the demand for money. Eq. (2.3) was originally introduced in the seminal work by Cagan (1956), but with adaptive, rather than rational expectations. The more recent rational expectations version has been used by many researchers including Sargent and Wallace (1973).

2.1.1. Some economic policy interpretations of the shocks

The stochastic process for the shock variable u_t is assumed in eq. (2.2) to have a general form. This form includes any stationary ARMA process [see Harvey (1981), p. 27, for example]. For empirical applications this generality is necessary because both policy variables and shocks to equations frequently have complicated time series properties. In many policy applications (where u_t in (2.2) is a policy variable), one is interested in "thought experiments" in which the policy variable is shifted in a special way and the response of the endogenous variables is examined. In standard econometric model methodology, such thought experiments require one to calculate policy multipliers [see Chow (1983), p. 147, for example]. In forward-looking rational expectations models, the multipliers depend not only on whether the shift in the policy variable is temporary or permanent, but also on whether it is anticipated or unanticipated. Eq. (2.2) can be given a special form to characterize these different thought experiments, as the following examples indicate.

Temporary versus permanent shocks. The shock u_t is purely temporary when $\theta_0 = 1$ and $\theta_i = 0$ for $i > 0$. Then any shock u_t is expected to disappear in the period immediately after it has occurred; that is $\mathrm{E}_t u_{t+i} = 0$ for $i > 0$ at every realization of u_t. At the other extreme the shock u_t is permanent when $\theta_i = 1$ for $i > 0$. Then any shock u_t is expected to remain forever; that is $\mathrm{E}_t u_{t+i} = u_t$ for $i > 0$ at every realization of u_t. In this permanent case the u_t process can be written as $u_t = u_{t-1} + \varepsilon_t$. (Although u_t is not a stationary process in this case, the solution can still be used for thought experiments, or transformed into a stationary series by first-differencing.)

By setting $\theta_i = \rho^i$, a range of intermediate persistence assumptions can be modeled as ρ varies from 0 to 1. For $0 < \rho < 1$ the shock u_t is assumed to phase out geometrically. In this case the u_t process is simply $u_t = \rho u_{t-1} + \varepsilon_t$, a first order autoregressive model. When $\rho = 0$, the disturbances are purely temporary. When $\rho = 1$, they are permanent.

Anticipated versus unanticipated shocks. In policy applications it is also important to distinguish between anticipated and unanticipated shocks. Time delays between the realization of the shock and its incorporation in the current information set can be introduced for this purpose by setting $\theta_i = 0$ for values of i up to the length of time of anticipation. For example, in the case of a purely temporary shock, we can set $\theta_0 = 0$, $\theta_1 = 1$, $\theta_i = 0$ for $i > 1$ so that $u_t = \varepsilon_{t-1}$. This would characterize a temporary shock which is anticipated one period in advance. In other words the expectation of u_{t+1} at time t is equal to u_{t+1} because $\varepsilon_t = u_{t+1}$ is in the information set at time t. More generally a temporary shock anticipated k periods in advance would be represented by $u_t = \varepsilon_{t-k}$.

A permanent shock which is anticipated k periods in advance would be modeled by setting $\theta_i = 0$ for $i = 1, \ldots, k-1$ and $\theta_i = 1$ for $i = k, k+1, \ldots$.

Table 1
Summary of alternative policies and their effects.

Model:	$y_t = \alpha \mathop{\mathrm{E}}_t y_{t+1} + \delta u_t,\ \lvert\alpha\rvert < 1.$
Policy:	$u_t = \sum_{i=0}^{\infty} \theta_i \varepsilon_{t-i} \Rightarrow \theta_i = \dfrac{\mathrm{d}u_{t+i}}{\mathrm{d}\varepsilon_t},\ i = 0,1,\ldots.$
Solution Form:	$y_t = \sum_{i=0}^{\infty} \gamma_i \varepsilon_{t-i} \Rightarrow \gamma_i = \dfrac{\mathrm{d}y_{t+i}}{\mathrm{d}\varepsilon_t},\ i = 0,1,\ldots.$

Stochastics: ε_t is serially uncorrelated with zero mean.
Thought Experiment: One time unit impulse to ε_t.
Theorem: For every integer $k \geq 0$.
if

$$\theta_i = \begin{cases} 0 \text{ for } i < k, \\ \rho^{i-k} \text{ for } i \geq k, \end{cases}$$

then

$$\gamma_i = \begin{cases} \dfrac{\delta\alpha^{-(i-k)}}{1-\alpha\rho} \text{ for } i < k, \\ \dfrac{\delta\rho^{i-k}}{1-\alpha\rho} \text{ for } i \geq k. \end{cases}$$

Interpretation:
Policy is *anticipated* k periods in advance,
$k = 0$ means *unanticipated.*
Policy is phased-out at geometric rate ρ, $0 \leq \rho \leq 1$,
$\rho = 0$ means purely *temporary* (N.B. $\rho^0 = 1$ when $\rho = 0$),
$\rho = 1$ means *permanent.*

Similarly, a shock which is anticipated k periods in advance and which is then expected to phase out gradually would be modeled by setting $\theta_i = 0$ for $i = 1,\ldots, k-1$ and $\theta_i = \rho^{i-k}$ for $i = k,\ k+1,\ldots,$ with $0 < \rho < 1$. In this case (2.2) can be written alternatively as $u_t = \rho u_{t-1} + \varepsilon_{t-k}$, a first-order autoregressive model with a time delay.

The various categories of shocks and their mathematical representations are summarized in Table 1. Although in practice, we interpret ε_t in eq. (2.2) as a continually perturbed random variable, for these thought experiments we examine the effect of a one-time unit impulse to ε_t. The solution for y_t derived below can be used to calculate the effects on y_t of such single realizations of ε_t.

2.1.2. *Finding the solution*

In order to find a solution for y_t (that is, a stochastic process for y_t which satisfies the model (2.1) and (2.2)), we begin by representing y_t in the unrestricted infinite moving average form

$$y_t = \sum_{i=0}^{\infty} \gamma_i \varepsilon_{t-i}. \tag{2.4}$$

Finding a solution for y_t then requires determining values for the undetermined coefficients γ_i such that eq. (2.1) and (2.2) are satisfied. Current and past ε_t represent the entire history of the perturbations to the model. Eq. (2.4) simply states that y_t is a general function of all possible events that may potentially influence y_t. The linear form is used in (2.4) because the model (2.2) is linear. Note that the solution for y_t in eq. (2.4) can easily be used to calculate the effect of a one time unit shock to ε_t. The dynamic impact of such a shock is simply $\mathrm{d}y_{t+s}/\mathrm{d}\varepsilon_t = \gamma_s$.

To find the unknown coefficients, the most direct procedure is to substitute for y_t and $\mathrm{E}_t y_{t+1}$ in (2.1) using (2.4), and solve for the γ_i in terms of α, δ and θ_i. The conditional expectation $\mathrm{E}_t y_{t+1}$ is obtained by leading (2.4) by one period and taking expectations, making use of the equalities $\mathrm{E}_t\varepsilon_{t+i} = 0$ for $i > 0$. The first equality follows from the assumption that ε_t has a zero unconditional mean and is uncorrelated; the second follows from the fact that ε_{t+i} for $i < 0$ is in the conditioning set at time t. The conditional expectation is

$$\mathop{\mathrm{E}}_t y_{t+1} = \sum_{i=1}^{\infty} \gamma_i \varepsilon_{t-i+1}. \tag{2.5}$$

Substituting (2.2), (2.4) and (2.5) into (2.1) results in

$$\sum_{i=0}^{\infty} \gamma_i \varepsilon_{t-1} = \alpha \sum_{i=1}^{\infty} \gamma_i \varepsilon_{t-i+1} + \delta \sum_{i=0}^{\infty} \theta_i \varepsilon_{t-i}. \tag{2.6}$$

Equating the coefficients of $\varepsilon_t, \varepsilon_{t-1}, \varepsilon_{t-2}, \ldots$ on both sides of the equality (2.6) results in the set of equations

$$\gamma_i = \alpha\gamma_{i+1} + \delta\theta_i \qquad i = 0,1,2,\ldots. \tag{2.7}$$

The first equation in (2.7) for $i = 0$ equates the coefficients of ε_t on both sides of (2.6); the second equation similarly equates the coefficient for ε_{t-1} and so on.

Note that (2.7) is a deterministic difference equation in the γ_i coefficients with θ_i as a forcing variable. This deterministic difference equation has the same structure as the stochastic difference eq. (2.1). It can be thought of as a deterministic perfect foresight model of the "variable" γ_i. Hence, the problem of solving a *stochastic* difference equation with conditional expectations of future variables has been converted into a problem of solving a *deterministic* difference equation.

2.1.3. *The solution in the case of unanticipated shocks*

Consider first the most elementary case where $u_t = \varepsilon_t$. That is, $\theta_i = 0$ for $i \geq 1$. This is the case of unanticipated shocks which are temporary. Then eq. (2.7) can be written

$$\gamma_0 = \alpha\gamma_1 + \delta. \tag{2.8}$$

$$\gamma_{i+1} = \frac{1}{\alpha}\gamma_i \qquad i = 1,2,\ldots. \tag{2.9}$$

From eq. (2.9) all the γ_i for $i > 1$ can be obtained once we have γ_1. However, eq. (2.8) gives only one equation in the two unknowns γ_0 and γ_1. Hence without further information we cannot determine the γ_i coefficients uniquely. The number of unknowns is one greater than the number of equations. This indeterminacy is what leads to non-uniqueness in rational expectations models and has been studied by many researchers including Blanchard (1979), Flood and Garber (1980), McCallum (1983), Gourieroux, Laffont, and Monfort (1982), Taylor (1977), and Whiteman (1983).

If $|\alpha| \leq 1$ then the requirement that y_t is a stationary process will be sufficient to yield a unique solution. (The case where $|\alpha| > 1$ is considered below in Section 2.1.4.). To see this suppose that $\gamma_1 \neq 0$. Since eq. (2.9) is an unstable difference equation, the γ_i coefficients will explode as i gets large. But then y_t would not be a stationary stochastic process. The only value for γ_1 that will prevent the γ_i from exploding is $\gamma_1 = 0$. From (2.9) this in turn implies that $\gamma_i = 0$ for all $i > 1$. From eq. (2.8) we then have that $\gamma_0 = \delta$. Hence, the unique stationary solution is simply $y_t = \delta\varepsilon_t$. In this case, the impact of a unit shock $\mathrm{d}y_{t+s}/\mathrm{d}\varepsilon_t$ is equal to δ for $s = 0$ and is equal to 0 for $s \geq 1$. This simple impact effect is illustrated in Figure 1a. (The more interesting charts in Figures 1b, 1c, and 1d will be described below).

Example

In the case of the Cagan money demand equation this means that the price $p_t = (1+\beta)^{-1}m_t$. Because $\beta > 0$, a temporary unanticipated increase in the money supply increases the price level by less than the increase in money. This is due to the fact that the price level is expected to decrease to its normal value (zero) next period, thereby generating an expected deflation. The expected deflation increases the demand for money so that real balances must increase. Hence, the price p_t rises by less than m_t. This is illustrated in Figure 2a.

For the more general case of unanticipated shifts in u_t that are expected to phase-out gradually we set $\theta_i = \rho^i$, where $\rho < 1$. Eq. (2.7) then becomes

$$\gamma_{i+1} = \frac{1}{\alpha}\gamma_i - \frac{\delta\rho^i}{\alpha} \qquad i = 0,1,2,3,\ldots. \tag{2.10}$$

Again, this is a standard deterministic difference equation. In this more general case, we can obtain the solution γ_i by deriving the solution to the homogeneous part $\gamma_i^{(H)}$ and the particular solution to the non-homogeneous part $\gamma_i^{(P)}$.

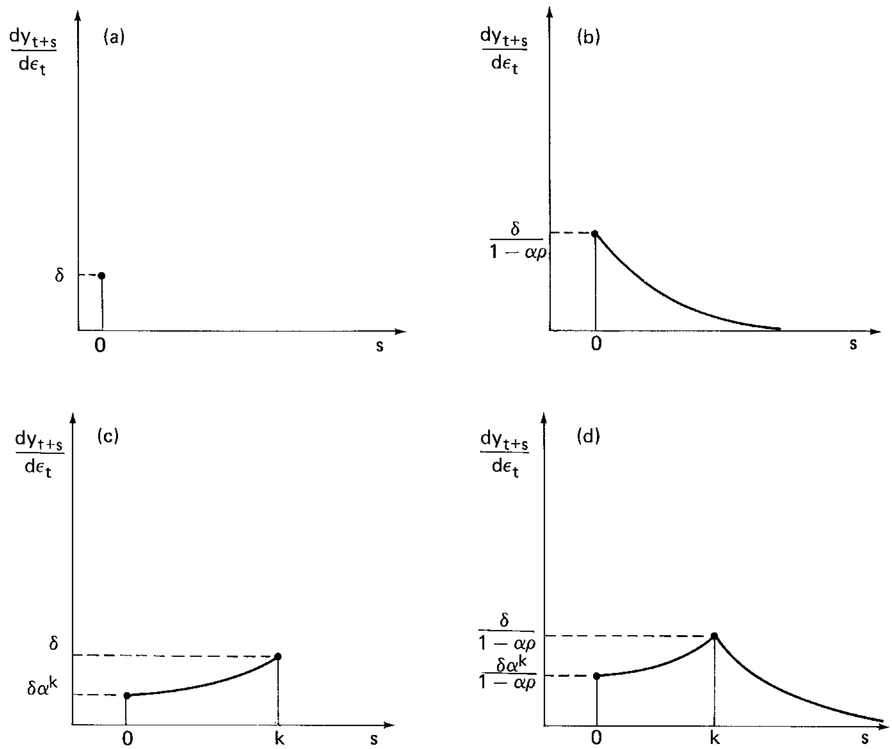


Figure 1(a). Effect on y_t of an unanticipated unit shift in u_t which is temporary ($u_t = \varepsilon_t$). (b). Effect on y_t of an unanticipated unit shift in u_t which is phased-out gradually ($u_t = \rho u_{t-1} + \varepsilon_t$). (c). Effect on y_t of an anticipated unit shift in u_t which is temporary (anticipated at time 0 and to occur at time k) ($u_t = \varepsilon_{t-k}$). (d). Effect on y_t of an anticipated shift in u_t which is phased-out gradually (anticipated at time 0 and to occur at time k) ($u_t = \rho u_{t-1} + \varepsilon_{t-k}$).

The solution to (2.10) is the sum of the homogeneous solution and the particular solution $\gamma_i = \gamma_i^{(H)} + \gamma_i^{(P)}$. [See Baumol (1970) for example, for a description of this solution technique for deterministic difference equations]. The homogeneous part is

$$\gamma_{i+1}^{(H)} = \frac{1}{\alpha}\gamma_i^{(H)} \qquad i = 0,1,2,\ldots, \tag{2.11}$$

with solution $\gamma_{i+1}^{(H)} = (1/\alpha)^{i+1}\gamma_0^{(H)}$. As in the earlier discussion if $|\alpha| < 1$ then for stationarity we require that $\gamma_0^{(H)} = 0$. For any other value of $\gamma_0^{(H)}$ the homogeneous solution will explode. Stationarity therefore implies that $\gamma_i^{(H)} = 0$ for $i = 0,1,2,\ldots$.

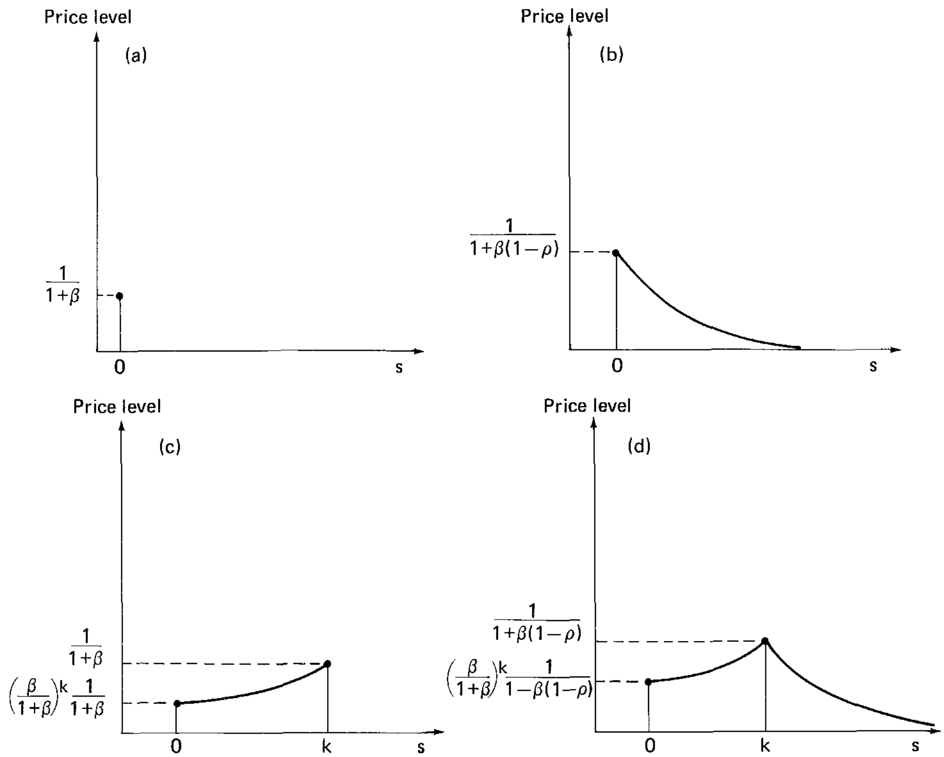


Figure 2(a). Price level effect of an unanticipated unit increase in m_t which lasts for one period. (b). Price level effect of an unanticipated increase in m_t which is phased-out gradually. (c). Price level effect of an anticipated unit increase in m_{t+k} which lasts for one period. The increase is anticipated k periods in advance. (d). Price level of an anticipated unit increase in m_{t+k} which is phased-out gradually. The increase is anticipated k periods in advance.

To find the particular solution we substitute $\gamma_i^{(P)} = hb^i$ into (2.10) and solve for the unknown coefficients h and b. This gives:

$$b = \rho, \qquad (2.12)$$
$$h = \delta(1-\alpha\rho)^{-1}.$$

Because the homogeneous solution is identically equal to zero, the sum of the homogeneous and the particular solutions is simply

$$\gamma_i = \frac{\delta\rho^i}{1-\alpha\rho}, \qquad i = 0,1,2,\ldots. \qquad (2.13)$$

In terms of the representation for y_t this means that

$$\begin{aligned} y_t &= \frac{\delta}{1-\alpha\rho}\sum_{i=0}^{\infty}\rho^i\varepsilon_{t-i} \\ &= \frac{\delta}{1-\alpha\rho}u_t. \end{aligned} \tag{2.14}$$

The variable y_t is proportional to the shock u_t at all t. The effect of a unit shock ε_t is shown in Figure 1b. Note that y_t follows the same type of first order stochastic process that u_t does; that is,

$$y_t = \rho y_{t-1} + \frac{\delta\varepsilon_t}{1-\alpha\rho}. \tag{2.15}$$

Example

For the money demand example, eq. (2.14) implies that

$$\begin{aligned} p_t &= \frac{1}{1-\left(\frac{\beta}{1+\beta}\right)\rho}\left(\frac{1}{1+\beta}\right)m_t \\ &= \left(\frac{1}{1+\beta(1-\rho)}\right)m_t. \end{aligned} \tag{2.16}$$

As long as $\rho < 1$ the increase in the price level will be less than the increase in the money supply. The dynamic impact on p_t of a unit shock to the money supply is shown in Figure 2b. The price level increases by less than the increase in the money supply because of the expected deflation that occurs as the price level gradually returns to its equilibrium value of 0. The expected deflation causes an increase in the demand for real money balances which is satisfied by having the price level rise less than the money supply. For the special case that $\rho = 1$, a permanent increase in the money supply, the price level moves proportionately to money as in the simple quantity theory. In that case there is no change in the expected rate of inflation since the price level remains at its new level.

2.1.4. A digression on the possibility of non-uniqueness

If $|\alpha| > 1$, then simply requiring that y_t is a stationary process will not yield a unique solution. In this case eq. (2.9) is stable, and any value of γ_1 will give a stationary time series. There is a continuum of solutions and it is necessary to place additional restrictions on the model if one wants to obtain a unique solution for the γ_i. There does not seem to be any completely satisfactory approach to take in this case.

One possibility raised by Taylor (1977) is to require that the process for y_t have a minimum variance. Consider the case where u_t is uncorrelated. The variance of y_t is given by

$$\operatorname{Var} y_t = \gamma_0^2 + (\gamma_0 - \delta)^2(\alpha^2 - 1)^{-1}. \tag{2.17}$$

where the variance of ε_t is supposed to be 1. The minimum occurs at $\gamma_0 = \delta_\alpha^{-2}$ from which the remaining γ_i can be calculated. Although the minimum variance condition is a natural extension of the stationarity (finite variance) condition, it is difficult to give it an economic rationale.

An alternative rule for selecting a solution was proposed by McCallum (1983), and is called the "minimum state variable technique". In this case it chooses a representation for y_t which involves the smallest number of ε_t terms; hence, it would give $y_t = \delta\varepsilon_t$. McCallum (1983) examines this selection rule in several different applications.

Chow (1983, p. 361) has proposed that the uniqueness issue be resolved empirically by representing the model in a more general form. To see this substitute eq. (2.8) with $\delta = 1$ and eq. (2.9) into eq. (2.4) for an arbitrary γ_1. That is, from eq. (2.4) we write

$$\begin{aligned} y_t &= \sum_{i=0}^{\infty} \gamma_i \varepsilon_{t-1} \\ &= (\alpha\gamma_1 + 1)\varepsilon_t + \gamma_1\varepsilon_{t-1} + (\gamma_1/\alpha)\varepsilon_{t-2} + (\gamma_1/\alpha^2)\varepsilon_{t-3} + \cdots. \end{aligned} \tag{2.18}$$

Lagging (2.18) by one time period, multiplying by α^{-1} and subtracting from (2.18) gives

$$y_t = \frac{1}{\alpha} y_{t-1} + (\alpha\gamma_1 + 1)\varepsilon_t - \frac{1}{\alpha}\varepsilon_{t-1}, \tag{2.19}$$

which is ARMA (1,1) model with a free parameter γ_1. Clearly if $\gamma_1 = 0$ then this more general solution reduces to the solution discussed above. But, rather than imposing this condition, Chow (1983) has suggested that the parameter γ_1 be estimated, and has developed an appropriate econometric technique. Evans and Honkapohja (1984) use a similar procedure for representing ARMA models in terms of a free parameter.

Are there any economic examples where $|\alpha| > 1$? In the case of the Cagan money demand equation, $\alpha = \beta/(1+\beta)$ which is always less than 1 since β is a positive parameter. One economic example where $\alpha > 1$ is a flexible-price macroeconomic model with money in the production function. To see this consider the following equations:

$$m_t - p_t = az_t - \beta i_t. \tag{2.20}$$

$$z_t = -c\left(i_t - \left(\underset{t}{\mathrm{E}} p_{t+1} - p_t\right)\right), \tag{2.21}$$

$$z_t = d(m_t - p_t). \tag{2.22}$$

where z_t is real output, i_t is the nominal interest rate, and the other variables are as defined in the earlier discussion of the Cagan model. The first equation is the money demand equation. The second equation indicates that real output is negatively related to the real rate of interest (an "*IS*" equation). In the third equation z_t is positively related to real money balances. The difference between this model and the Cagan model (in eq. (2.3)) is that output is a positive function of real money balances. The model can be written in the form of eq. (2.1) with

$$\alpha = \frac{\beta}{1+\beta - d(a+\beta c^{-1})}. \tag{2.23}$$

Eq. (2.23) is equal to the value of α in the Cagan model when $d = 0$. In the more general case where $d > 0$ and money is a factor in the production function, the parameter α can be greater than one. This example was explored in Taylor (1977). Another economic example which arises in an overlapping generation model of money was investigated by Blanchard (1979).

Although there are examples of non-uniqueness such as these in the literature, most theoretical and empirical applications in economics have the property that there is a unique stationary solution. However, some researchers, such as Gourieroux, Laffont, and Monfort (1982), have even questioned the appeal to stationarity. Sargent and Wallace (1973) have suggested that the stability requirement effectively rules out speculative bubbles. But there are examples in history where speculative bubbles have occurred and some analysts feel they are quite common. There have been attempts to model speculative bubbles as movements of y_t along a self-fulfilling nonstationary (explosive) path. Blanchard and Watson (1982) have developed a model of speculative bubbles in which there is a positive probability that the bubble will burst. Flood and Garber (1980) have examined whether the periods toward the end of the eastern European hyperinflations in the 1920s could be described as self-fulfilling speculative bubbles. To date, however, the vast majority of rational expectations research has assumed that there is a unique stationary solution. For the rest of this paper we assume that $|\alpha| < 1$, or the equivalent in higher order models, and we assume that the solution is stationary.

2.1.5. Finding the solution in the case of anticipated shocks

Consider now the case where the shock is anticipated k periods in advance and is purely temporary. That is, $u_t = \varepsilon_{t-k}$ so that $\theta_k = 1$ and $\theta_i = 0$ for $i \neq k$. The difference equations in the unknown parameters can be written as:

$$\gamma_i = \alpha\gamma_{i+1} \qquad i = 0,1,2,\dots k-1. \tag{2.24}$$

$$\gamma_{k+1} = \frac{1}{\alpha}\gamma_k - \frac{\delta}{\alpha}. \tag{2.25}$$

$$\gamma_{i+1} = \frac{1}{\alpha}\gamma_i \qquad i = k+1,\ k+2,\dots. \tag{2.26}$$

The set of equations in (2.26) is identical in form to what we considered earlier except that the initial condition is at $k+1$. For stationarity we therefore require that $\gamma_{k+1} = 0$. This implies from eq. (2.25) that $\gamma_k = \delta$. The remaining coefficients are obtained by working back using (2.24) starting with $\gamma_k = \delta$. This gives $\gamma_i = \delta\alpha^{k-i}$, $i = 0,1,2,\dots k-1$.

The pattern of the γ_i coefficients is shown in Figure 1c. These coefficients give the impact of ε_t on y_{t+s}, for $s > 0$, or equivalently the *impact of the news* that the shock u_t will occur k periods later. The size of γ_0 depends on how far in the future the shock is anticipated. The farther in advance the shock is known (that is, the larger is k), the smaller will be the current impact of the news.

Example

For the demand for money example we have

$$p_t = \delta\left[\alpha^k\varepsilon_t + \alpha^{k-1}\varepsilon_{t-1} + \cdots + \alpha\varepsilon_{t-(k-1)} + \varepsilon_{t-k}\right]. \tag{2.27}$$

Substituting $\alpha = \beta/(1+\beta)$, $\delta = 1/(1+\beta)$, and $\varepsilon_t = u_{t+k} = m_{t+k}$ into (2.27) we get

$$p_t = \left(\frac{1}{1+\beta}\right)\sum_{i=0}^{k}\left(\frac{\beta}{1+\beta}\right)^{k-i} m_{t+k-i}. \tag{2.28}$$

Note how this reduces to $p_t = (1+\beta)^{-1}m_t$ in the case of unanticipated shocks $(k = 0)$, as we calculated earlier. When the temporary increase in the money supply is anticipated in advance, the price level "jumps" at the date of announcement and then gradually increases until the money supply *does* increase. This is illustrated in Figure 2c.

Finally, we consider the case where the shock is anticipated in advance, but is expected to be permanent or to phase-out gradually. Then, suppose that $\theta_i = 0$ for

$i=1,\dots,k-1$ and $\theta_i=\rho^{i-k}$ for $i\geq k$. Eq. (2.7) becomes

$$\gamma_i=\alpha\gamma_{i+1} \qquad i=0,1,2,\dots,k-1, \tag{2.29}$$

$$\gamma_{i+1}=\frac{1}{\alpha}\gamma_i-\frac{\delta\rho^{i-k}}{\alpha} \qquad i=k,k+1,\dots. \tag{2.30}$$

Note that eq. (2.30) is identical to eq. (2.10) except that the initial condition starts at k rather than 0. The homogeneous part of (2.30) is

$$\gamma_{i+1}^{(H)}=\frac{1}{\alpha}\gamma_i^{(H)} \qquad i=k,k+1,\dots. \tag{2.31}$$

In order to prevent the $\gamma_i^{(H)}$ from exploding as i increases it is necessary that $\gamma_k^{(H)}=0$. Therefore $\gamma_i^{(H)}=0$ for $i=k,k+1,\dots$. The unknown coefficients h and b of the particular solution $\gamma_i^{(P)}=hb^{i-k}$ are

$$h=\delta(1-\alpha p)^{-1},$$
$$b=\rho. \tag{2.32}$$

Since the homogeneous part is zero we have that

$$\gamma_i=\frac{\delta\rho^{i-k}}{1-\alpha\rho} \qquad i=k,k+1,\dots. \tag{2.33}$$

The remaining coefficients can be obtained by using (2.29) backwards starting with $\gamma_k=\delta(1-\alpha\rho)^{-1}$. The solution for y_t is

$$y_t=\frac{\delta}{1-\alpha\rho}\big(\alpha^k\varepsilon_t+\alpha^{k-1}\varepsilon_{t-1}+\cdots+\alpha\varepsilon_{t-k+1}+\varepsilon_{t-k}$$
$$+\rho\varepsilon_{t-k-1}+\rho^2\varepsilon_{t-k-2}+\cdots\big). \tag{2.34}$$

After the immediate impact of the announcement, y_t will grow smoothly until it equals $\delta(1-\alpha\rho)^{-1}$ at the time that u_t increases. The effect then phases out geometrically. This pattern is illustrated in Figure 1d.

Example

For the money demand model, the effect on the price level p_t is shown in Figure 2d. As before the anticipation of an increase in the money supply causes the price level to jump. The price level then increases gradually until the increase in money actually occurs. During the period before the actual increase in money, the level of real balances is below equilibrium because of the expected inflation. The initial increase becomes larger as the phase-out parameter ρ gets larger. For the permanent case where $\rho=1$ the price level eventually increases by the same amount that the money supply increases.

2.1.6. *General* ARMA *processes for the shocks*

The above solution procedure can be generalized to handle the case where (2.2) is an autoregressive moving average (ARMA) model. We consider only unanticipated shocks where there is no time delay. Suppose the error process is

$$u_t = \rho_1 u_{t-1} + \cdots + \rho_p u_{t-p} + \varepsilon_t + \psi_1 \varepsilon_{t-1} + \cdots + \psi_q \varepsilon_{t-q}, \tag{2.35}$$

an ARMA (p,q) model. The coefficients in the linear process for u_t in the form of (2.2) can be derived from:

$$\theta_j = \psi_j + \sum_{i=1}^{\min(j,p)} \rho_i \theta_{j-1} \qquad j = 0,1,2,\dots,q,$$

$$\theta_j = \sum_{i=1}^{\min(j,p)} \rho_i \theta_{j-i} \qquad j > q. \tag{2.36}$$

where $\psi_0 = 1$. See Harvey (1981, p. 38), for example.

Starting with $j = M \equiv \max(p, q+1)$ the θ_j coefficients in (2.36) are determined by a pth order difference equation. The p initial conditions $(\theta_{M-1},\dots,\theta_{M-p})$ for this difference equation are given by the p equations that preceed the θ_M equation in (2.36).

To obtain the γ_i coefficients, (2.36) can be substituted into eq. (2.7). As before, the solution to the homogeneous part is $\gamma_i^{(H)} = 0$ for all i. The particular solution to the non-homogeneous part will have the same form as (2.36) for $j \geq M$. That is,

$$\gamma_j = \sum_{i=1}^{p} \rho_i \gamma_{j-1} \qquad j = M, M+1,\dots. \tag{2.37}$$

The initial conditions $(\gamma_{M-1},\dots,\gamma_{M-p})$ for (2.37), as well as the remaining γ values $(\gamma_{M-p-1},\dots,\gamma_0)$ can then be obtained by substitution of θ_i for $i = 0,\dots, M-1$ into (2.37). That is,

$$\gamma_{i+1} = \frac{1}{\alpha}\gamma_i - \frac{\delta}{\alpha}\theta_i \qquad i = 0,1,\dots,M-1. \tag{2.38}$$

Comparing the form of (2.37) and (2.38) with (2.36) indicates that the γ_i coefficients can be interpreted as the infinite moving average representation of an ARMA $(p, M-1)$ model. That is, the solution for y_t is an ARMA $(p, M-1)$ model with an autoregressive part equal to the autoregressive part of the u_t process defined in eq. (2.35). This result is found in Gourieroux, Laffont, and Monfort (1982). The methods of Hansen and Sargent (1980) and Taylor (1980a) can also be used to compute the ARMA representations directly as summarized in Section 2.4 below.

Example : $p = 3, q = 1$

In this case $M = 3$ and eq. (2.36) becomes

$$\begin{aligned} &\theta_0 = 1, \\ &\theta_1 = \psi_1 + \rho_1\theta_0, \\ &\theta_2 = \rho_1\theta_1 + \rho_2\theta_0, \\ &\theta_i = \rho_1\theta_{i-1} + \rho_2\theta_{i-2} + \rho_3\theta_{i-3} \qquad i = 3,4,\dots . \end{aligned} \tag{2.39}$$

The γ coefficients are then given by

$$\gamma_i = \rho_1\gamma_{i-1} + \rho_2\gamma_{i-2} + \rho_3\gamma_{i-3} \qquad i = 3,4,\dots . \tag{2.40}$$

and the initial conditions γ_0, γ_1 and γ_2 are given by solving the three linear equations

$$\begin{aligned} &\gamma_1 = \frac{1}{\alpha}\gamma_0 - \frac{\delta}{\alpha}, \\ &\gamma_2 = \frac{1}{\alpha}\gamma_1 - \frac{\delta}{\alpha}(\psi_1 + \rho_1), \\ &\gamma_2 = \left(\rho_1 - \alpha^{-1}\right)^{-1}\left(\rho_2\gamma_1 + \rho_3\gamma_0 - \frac{\delta}{2}\left(\rho_1^2 + \rho_1\psi_1 + \rho_2\right)\right). \end{aligned} \tag{2.41}$$

Eqs. (2.40) and (2.41) imply that y_t is an ARMA (3,2) model.

2.1.7. *Different viewpoint dates*

In some applications of rational expectation models the forecast of future variables might be made at different points in time. For example, a generalization of (2.1) is

$$y_t = \alpha_1 \underset{t}{\mathrm{E}} y_{t+1} + \alpha_2 \underset{t-1}{\mathrm{E}} y_{t+1} + \alpha_3 \underset{t-1}{\mathrm{E}} y_t + u_t. \tag{2.42}$$

Substituting for y_t and expected y_t from (2.4) into (2.42) results in a set of equations for the γ coefficients much like the equations that we studied above. Suppose $u_t = \rho u_{t-1} + \varepsilon_t$. Then, the equations for γ are

$$\begin{aligned} &\gamma_0 = \alpha_1\gamma_1 + \delta, \\ &\gamma_{i+1} = \left(\frac{1-\alpha_3}{\alpha_1 + \alpha_2}\right)\gamma_i - \frac{\delta\rho^i}{\alpha_1 + \alpha_2} \qquad i = 1,2,\dots . \end{aligned} \tag{2.43}$$

Hence, we can use the same procedures for solving this set of difference equations. The solution is

$$\gamma_0 = \alpha_1 b\rho + \delta,$$
$$\gamma_i = b\rho^i \qquad i = 1,2,\dots.$$

where $b = \delta/(1-\alpha_3-\rho\alpha_2-\rho\alpha_1)$. Note that this reduces to (2.13) when $\alpha_2 = \alpha_3 = 0$.

2.1.8. *Geometric interpretation*

The solution of the difference eq. (2.7) that underlies this technique has an intuitive graphical interpretation which corresponds to the phase diagram method used to solve continuous time models with rational expectations. [See Calvo (1980) or Dixit (1980) for example]. Eq. (2.7) can be written

$$\gamma_{i+1} - \gamma_i = \left(\frac{1}{\alpha} - 1\right)\gamma_i - \frac{\delta}{\alpha}\theta \qquad i = 0,1,\dots. \tag{2.44}$$

The set of values for which γ_i is not changing are given by setting the right-hand side of (2.44) to zero. These values of (γ_i, θ_i) are plotted in Figure 3. In the case where $\theta_i = \rho^i$, for $0 < \rho < 1$ there is a difference equation representation for θ_i of the form

$$\theta_{i+1} - \theta_i = (\rho - 1)\theta_i, \tag{2.45}$$

where $\theta_0 = 1$. The set of points where θ is not changing is a vertical line at $\theta_i = 0$ in Figure 3. The forces which move γ and θ in different directions are also shown in Figure 3. Points above (below) the upward sloping line cause γ_i to increase (decrease). Points to the right (left) of the vertical line cause θ_i to decrease (increase). In order to prevent the γ_i from exploding we found in Section 2.1.3

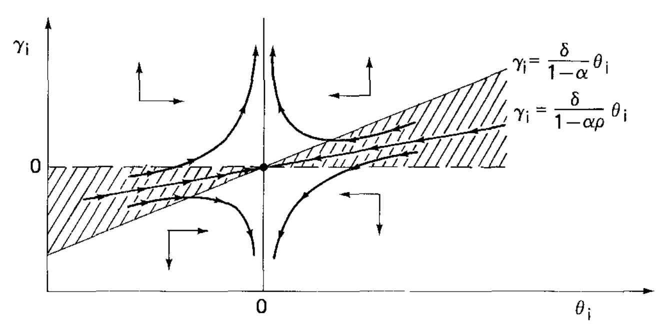


Figure 3. Illustration of the rational expectations solution and the saddle path. Along the saddle path the motion is towards the origin at geometric rate ρ. That is, $\theta_i = \rho\theta_{i-1}$.

that it was necessary for $\gamma_i = (\delta/1 - \alpha\rho)\theta_i$. This linear equation is shown as the straight line with the arrows in Figure 3. This line balances off the unstable vertical forces and uses the stable horizontal forces to bring γ_i back to the values $\gamma_i = 0$ and $\theta_i = 0$ and $i \to \infty$. For this reason it is called a saddle point and corresponds to the notion of a saddle path in differential equation models [see Birkhoff and Rota (1962), for example].

Figure 3 is special in the sense that one of the zero-change lines is perfectly vertical. This is due to the fact that the shock variable u_t is exogenous to y_t. If we interpret (2.1) and (2.2) as a two variable system with variables y_t and u_t as the two variables, then the system is recursive in that u_t affects y_t in the current period and there are no effects of past y_t on u_t. In Section 2.2 we consider a more general two variable system in which u_t is endogenous.

In using Figure 3 for thought experiments about the effect of one time shocks, recall that γ_i is $\mathrm{d}y_{t+i}/\mathrm{d}\varepsilon_i$ and θ_i is $\mathrm{d}u_{t+i}/\mathrm{d}\varepsilon_i$. The vertical axis thereby gives the paths of the endogenous variable y_t corresponding to a shock ε_t to the policy eq. (2.2). The horizontal axis gives the path of the policy variable. The points in Figure 3 can be therefore viewed as displacements of y_t and u_t from their steady state values in response to a one-time unit shock.

The arrows in Figure 3 show that the saddle path line must have a slope greater than zero and a slope less than the zero-change line for γ. That is, the saddle path line must lie in the shaded region of Figure 3. Only in this region is the direction of motion toward the origin. The geometric technique to determine whether the saddle path is upward or downward sloping is frequently used in practice to obtain the sign of an impact effect of policy. [See Calvo (1980), for example].

In Figure 4 the same diagram is used to determine the qualitative movement of y_t in response to a shock to u_t which is anticipated k periods in advance and which is expected to then phase out geometrically. This is the case considered

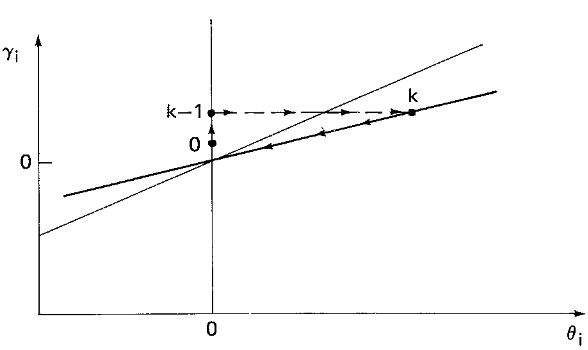


Figure 4. Illustration of the effect of an anticipated shock to u_t which is then expected to be phased out gradually at geometric rate ρ. The shock is anticipated k periods in advance. This thought experiment corresponds to the chart in Figure 1(d).

above in Section 2.1.5. The endogenous variable y initially jumps at time 0 when the future increase in u becomes known; it then moves along an explosive path through period k when u increases by 1 unit. From time k on the motion is along the saddle path as y and u approach their steady state values of zero.

2.1.9. *Nonstationary forcing variables*

In many economic applications the forcing variables are nonstationary. For example the money supply is a highly nonstationary series. One typically wants to estimate the effects of changes in the *growth rate* of the money supply. What happens when the growth rate is reduced gradually? What if the reduction in growth is anticipated? Letting u_t be the log of the money supply m_t, these alternatives can be analyzed by writing the *growth rate* of money as $g_t = m_t - m_{t-1}$ and assuming that

$$g_t - g_{t-1} = \rho(g_{t-1} - g_{t-2}) + \varepsilon_{t-k}.$$

Thus, the *change* in the growth rate is anticipated k periods in advance. The new growth rate is phased in at a geometric rate ρ. By solving the model for the particular solution corresponding to this equation, one can solve for the price level and the inflation rate. In this case, the inflation rate is nonstationary, but the change in the inflation rate is stationary.

2.2. *Bivariate models*

Let y_{1t} and y_{2t} be given by

$$\begin{aligned} y_{1t} &= \alpha_1 \underset{t}{\mathrm{E}} y_{1t+1} + \beta_{10} y_{2t} + \beta_{11} y_{2t-1} + \delta_1 u_t, \\ y_{2t} &= \alpha_2 \underset{t}{\mathrm{E}} y_{1t+1} + \beta_{20} y_{1t} + \beta_{21} y_{2t-1} + \delta_2 u_t, \end{aligned} \tag{2.46}$$

where u_t is a shock variable of the form (2.2). Model (2.46) is a special bivariate model in that there are no lagged values of y_{1t} and no lead values of y_{2t}. This asymmetry is meant to convey the continuous time idea that one variable y_{1t} is a "jump" variable, unaffected by its past while y_{2t} is a more slowly adjusting variable that is influenced by its past values. Of course in discrete time all variables tend to jump from one period to the next so that the terminology is not exact. Nevertheless, the distinction is important in practice. Most commonly, y_{1t} would be a price and y_{2t} a stock which cannot change without large costs in the short run.

We assume in (2.46) that there is only one shock u_t. This is for notational convenience. The generalization to a bivariate shock (u_{1t}, u_{2t}) where u_{1t} appears in the first equation and u_{2t} in the second equation is straightforward, as should be clear below.

Because (2.46) has this special form it can be reduced to a first order 2-dimensional vector process:

$$\begin{pmatrix} 1 & -\beta_{11} \\ -\beta_{20} & -\beta_{21} \end{pmatrix}\begin{pmatrix} y_{1t} \\ y_{2t-1} \end{pmatrix} = \begin{pmatrix} \alpha_1 & \beta_{10} \\ \alpha_2 & -1 \end{pmatrix}\begin{pmatrix} \underset{t}{\mathrm{E}} y_{1t+1} \\ y_{2t} \end{pmatrix} + \begin{pmatrix} \delta_1 \\ \delta_2 \end{pmatrix} u_t. \tag{2.47}$$

This particular way to construct a first order process follows that of Blanchard and Kahn (1980). A generalization to the case of viewpoint dates earlier than time t is fairly straightforward. If y_{1t-1} or $\mathrm{E}_t y_{2t+1}$ also appeared in (2.46) then a first-order model would have to be more than 2 dimensional.

2.2.1. *Some examples*

There are many interesting examples of this simple bivariate model. Five of these are summarized below.

Example 1: Exchange rate overshooting

Dornbusch (1976) considered the following type of model of a small open economy [see also Wilson (1979) and Buiter and Miller (1983)]:

$$m_t - p_t = -\alpha\left(\underset{t}{\mathrm{E}} e_{t+1} - e_t\right),$$

$$p_t - p_{t-1} = \beta(e_t - p_t),$$

where e_t is the log of the exchange rate, and p_t and m_t are as defined in the Cagan model. The first equation is simply the demand for money as a function of the nominal interest rate. In a small open economy with perfect capital mobility the nominal interest rate is equal to the world interest rate (assumed fixed) plus the expected rate of depreciation $\mathrm{E}_t e_{t+1} - e_t$. The second equation describes the slow adjustment of prices in response to the excess demand for goods. Excess demand is assumed to be a negative function of the relative price of home goods. Here prices adjust slowly and the exchange rate is a jump variable. This model is of the form (2.47) with $y_{1t} = e_t$, $y_{2t} = p_t$, $\alpha_1 = 1$, $\beta_{10} = -1/\alpha$, $\beta_{11} = 0$, $\delta_1 = 1/\alpha$, $\alpha_2 = 0$, $\beta_{20} = \beta/(1+\beta)$, $\beta_{21} = 1/(1+\beta)$, $\delta_2 = 0$.

Example 2: Open economy portfolio balance model

Kouri (1976), Rodriquez (1980), and Papell (1984) have considered the following type of rational expectations model which is based on a portfolio demand for foreign assets rather than on perfect capital mobility:

$$e_t + f_t = \alpha\left(\underset{t}{\mathrm{E}} e_{t+1} - e_t\right) + u_t,$$

$$f_t - f_{t-1} = \beta e_t.$$

The first equation represents the demand for foreign assets f_t (in logs) evaluated in domestic currency, as a function of the expected rate of depreciation. Here u_t is a shock. The second equation is the "current account" (the proportional change in the stock of foreign assets) as a function of the exchange rate. Prices are assumed to be fixed and out of the picture. This model reduces to (2.47) with $y_{1t} = e_t$, $y_{2t} = f_t$, $\alpha_1 = \alpha(1+\alpha)$, $\beta_{10} = 1/(1+\alpha)$, $\beta_{11} = 0$, $\delta_1 = 1/1+\alpha$, $\alpha_2 = 0$, $\beta_{20} = \beta$, $\beta_{21} = -1$, $\delta_2 = 0$.

Example 3: Money and capital

Fischer (1979) developed the following type of model of money and capital.

$$y_t = \gamma k_{t-1},$$

$$r_t = -(1-\gamma)k_{t-1},$$

$$m_t - p_t = -a_1 \underset{t}{\mathrm{E}} r_{t+1} - a_2\left(\underset{t}{\mathrm{E}} p_{t+1} - p_t\right) + y_t,$$

$$k_t = b_1 \underset{t}{\mathrm{E}} r_{t+1} + b_2\left(\underset{t}{\mathrm{E}} p_{t+1} - p_t\right) + y_t.$$

The first two equations describe output y_t and the marginal efficiency of capital r_t as a function of the stock of capital at the end of period $t-1$. The third and fourth equations are a pair of portfolio demand equations for capital and real money balances as a function of the rates of return on these two assets. Lucas (1976) considered a very similar model. Substituting the first two equations into the third and fourth we get model (2.47) with

$$y_{1t} = p_t, \qquad y_{2t} = k_t, \qquad \alpha_1 = \frac{a_2}{1+a_2}, \qquad \beta_{10} = \frac{-a_1(1-\gamma)}{1+a_2},$$

$$\beta_{11} = 0, \qquad \delta_1 = \frac{1}{1+a_2}, \qquad \alpha_2 = \frac{b_2}{(1+b_1(1-\gamma))},$$

$$\beta_{20} = \frac{-b_2}{(1+b_1(1-\gamma))}, \qquad \beta_{21} = \frac{\gamma}{(1+b_1(1-\gamma))}.$$

Example 4: Staggered contracts model

The model $y_t = a_1 \mathrm{E}_t y_{t+1} + a_2 y_{t-1} + \delta u_t$ of a contract wage y_t can occur in a staggered wage setting model as in Taylor (1980a). The future wage appears because workers and firms forecast the wage set by other workers and firms. The lagged wage appears because contracts last two periods. This model can be put in the form of (2.47) by stacking the y's into a vector:

$$\begin{pmatrix} 1 & -a_2 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} y_t \\ y_{t-1} \end{pmatrix} = \begin{pmatrix} a_1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \underset{t}{\mathrm{E}} y_{t+1} \\ y_t \end{pmatrix} + \begin{pmatrix} \delta \\ 0 \end{pmatrix} u_t.$$

Example 5: Optimal control problem

Hansen and Sargent (1980) consider the following optimal control problem. A firm chooses a contingency plan for a single factor of production (labor) n_t to maximize expected profits.

$$\underset{t}{\mathrm{E}} \sum_{j=0}^{\infty} \beta^j \left[p_{t+j} y_{t+j} - \frac{\delta}{2} (n_{t+j} - n_{t+j-1})^2 - w_{t+j} n_{t+j} \right],$$

subject to the linear production function $y_t = \gamma n_t$. The random variables p_t and w_t are the price of output and the wage, respectively. The first order conditions of this maximization problem are:

$$\beta \underset{t}{\mathrm{E}} n_{t+1} - (1+\beta) n_t + n_{t-1} = \frac{\beta}{\delta} (w_t - \gamma p_t).$$

This model is essentially the same as that in Example (4) where $u_t = w_t - \gamma p_t$.

2.2.2. Finding the solution

Equation (2.47) is a vector version of the univariate eq. (2.1). The technique for finding a solution to (2.47) is directly analogous with the univariate case.

The solution can be represented as

$$\begin{aligned} y_{1t} &= \sum_{i=0}^{\infty} \gamma_{1i} \varepsilon_{t-i}, \\ y_{2t} &= \sum_{i=0}^{\infty} \gamma_{2i} \varepsilon_{t-i}. \end{aligned} \tag{2.48}$$

These representations for the endogenous variables are an obvious generalization of eqs. (2.4).

Utilizing matrix notation we rewrite (2.47) as

$$Bz_t = C\underset{t}{\mathrm{E}}z_{t+1} + \delta u_t, \tag{2.49}$$

$$\underset{t}{\mathrm{E}}z_{t+1} = Az_t + \mathrm{d}u_t, \tag{2.50}$$

where the definitions of the matrices B and C, and the vectors z_t and δ in (2.49) should be clear, and where $A = C^{-1}B$ and $d = -C^{-1}\delta$. Let $\gamma_i = (\gamma_{1i}, \gamma_{2i-1})'$, $i = 0,1,2,\dots$ and set $\gamma_{2,-1} = 0$. Substitution of (2.2) and (2.48) into (2.50) gives

$$\gamma_{i+1} = A\gamma_i + \mathrm{d}\theta_i \qquad i = 0,1,2,\dots. \tag{2.51}$$

Eq. (2.51) is analogous to eq. (2.7). For $i = 0$ we have three unknown elements of the unknown vectors $\gamma_0 = (\gamma_{10}, 0)'$ and $\gamma_1 = (\gamma_{11}, \gamma_{20})'$. The 3 unknowns are γ_{10}, γ_{11} and γ_{20}. However, there are only two equations (at $i = 0$) in (2.51) that can be used to solve for these three parameters. Much as in the scalar case considering $i = 1$ gives two more equations, but it also gives two more unknowns $(\gamma_{12}, \gamma_{21})$; the same is true for $i = 2$ and so on. To determine the solution for the γ_i process we therefore need another equation. As in the scalar case this third equation comes by imposing stationarity on the process for y_{1t} and y_{2t} or equivalently in this context by preventing either element of γ_i from exploding. For uniqueness we will require that one root of A be greater than one in modulus, and one root be less than one in modulus. The additional equation thus comes from choosing $\gamma_1 = (\gamma_{11}, \gamma_{20})'$ so that γ_i does not explode as $i \to \infty$. This condition implies a unique linear relationship between γ_{11} and γ_{20}. This relationship is the extra equation. It is the analogue of setting the scalar $\gamma_1 = 0$ in model (2.1).

To see this, we decompose the matrix A into $H^{-1}\Lambda H$ where Λ is a diagonal matrix with λ_1 and λ_2 on the diagonal. H is the matrix whose rows are the characteristic vectors of A. Assume that the roots are distinct and that $|\lambda_1| > 1$ and $|\lambda_2| < 1$. Let $\mu_i \equiv (\mu_{1i}, \mu_{2i})' = H\gamma_i$. Then the homogeneous part of (2.51) is

$$\gamma_{i+1} = H^{-1}\Lambda H\gamma_i \qquad i = 1,2,\dots, \tag{2.52}$$

so that

$$\mu_{i+1} = \Lambda\mu_i \qquad i = 1,2,\dots,$$

or

$$\begin{aligned} \mu_{1i+1} &= \lambda_1\mu_{1i} \qquad i = 1,2,\dots, \\ \mu_{2i+1} &= \lambda_2\mu_{2i} \qquad i = 1,2,\dots. \end{aligned} \tag{2.53}$$

For stability of μ_{1i} as $i \to \infty$ we therefore require that $\mu_{11} = 0$ which in turn implies that $\mu_{1i} = 0$ for all $i > 1$. In other words we want

$$\mu_{11} = h_{11}\gamma_{11} + h_{12}\gamma_{20} = 0, \tag{2.54}$$

where (h_{11}, h_{12}) is the first row of H and is the characteristic vector of A corresponding to the unstable root λ_1. Eq. (2.54) is the extra equation. When combined with (2.51) at $i = 0$ we have 3 linear equations that can be solved for γ_{10}, γ_{11} and γ_{20}. From these we can use (2.51) or equivalently (2.53) to obtain the remaining γ_i for $i > 1$. In particular $\mu_{1i} = 0$ implies that

$$\gamma_{1i} = -\frac{h_{12}}{h_{11}}\gamma_{2i-1} \qquad i = 1, 2, \dots, \tag{2.55}$$

From the second equation in (2.53) we have that

$$h_{21}\gamma_{1i+1} + h_{22}\gamma_{2i} = \lambda_2(h_{21}\gamma_{1i} + h_{22}\gamma_{2i-1}).$$

Substituting for γ_{1i+1} and γ_{1i} from (2.55) this gives

$$\gamma_{2i+1} = \lambda_2\gamma_{2i} \qquad i = 0, 1, 2, \dots. \tag{2.56}$$

Given the initial values γ_{21} we compute the remaining coefficients from (2.55) and (2.56).

2.2.3. *The solution in the case of unanticipated shocks*

When the shock u_t is unanticipated and purely temporary, $\theta_0 = 1$ and $\theta_i = 0$ for all $i > 0$. In this case eq. (2.51) for $i = 0$ is

$$\begin{aligned} \gamma_{11} &= a_{11}\gamma_{10} + d_1, \\ \gamma_{20} &= a_{21}\gamma_{10} + d_2, \end{aligned} \tag{2.57}$$

and the difference equation described by (2.51) for $i > 0$ is homogeneous. Hence the solution given by (2.55), (2.56), and (2.57) is the complete solution.

For the more general case where $\theta_i = \rho^i$, eq. (2.57) still holds but the difference equation in (2.51) for $i \geq 1$ has a nonhomogeneous part. The particular solution to the nonhomogeneous part is of the form $\gamma_i^{(P)} = gb^i$ where g is a 2×1 vector. Substituting this form into (2.51) for $i \geq 1$ and equating coefficients we obtain the particular solution

$$\gamma_i^{(P)} = (\rho I - A)^{-1} \mathrm{d}\rho^i, \qquad i = 1, 2, \dots. \tag{2.58}$$

Since eq. (2.55) is the requirement for stability of the homogeneous solution, the complete solution can be obtained by substituting $\gamma_{11}^{(H)} = \gamma_{11} - \gamma_{11}^{(P)}$ and $\gamma_{20}^{(H)} = \gamma_{20} - \gamma_{20}^{(P)}$ into (2.54) to obtain

$$\gamma_{11} - \gamma_{11}^{(P)} = -\frac{h_{12}}{h_{11}}\left(\gamma_{20} - \gamma_{20}^{(P)}\right). \tag{2.59}$$

Eq. (2.59) can be combined with (2.57) to obtain γ_{10}, γ_{11}, and γ_{20}. The remaining coefficients are obtained by adding the appropriate elements of particular solutions (2.58) to the homogeneous solutions of (2.56) and (2.57).

2.2.4. *The solution in the case of anticipated shocks*

For the case where the shock is anticipated k periods in advance, but is purely temporary ($\theta_0 = 0$ for $i = 1,\dots,k-1$, $\theta_i = 0$ for $i = k+1,\dots$), we break up the difference eq. (2.51) as:

$$\gamma_{i+1} = A\gamma_i \qquad i = 0,1,\dots,k-1. \tag{2.60}$$

$$\gamma_{k+1} = A\gamma_k + d. \tag{2.61}$$

$$\gamma_{i+1} = A\gamma_i \qquad i = k+1, k+2,\dots. \tag{2.62}$$

Looking at the equations in (2.62) it is clear that for stationarity, $\gamma_{k+1} = (\gamma_{1k+1}, \gamma_{2k})'$ must satisfy the same relationship that the vector γ_1 satisfied in eq. (2.55). That is,

$$\gamma_{1k+1} = -\frac{h_{12}}{h_{11}}\gamma_{2k}. \tag{2.63}$$

Once γ_{2k} and γ_{1k+1} have been determined the γ values for $i > k$ can be computed as above in eqs. (2.55) and (2.56). That is,

$$\gamma_{1i+1} = -\frac{h_{12}}{h_{11}}\gamma_{2i} \qquad i = k,\dots, \tag{2.64}$$

$$\gamma_{2i+1} = \lambda_2\gamma_{2i} \qquad i = k,\dots. \tag{2.65}$$

To determine γ_{2k} and γ_{1k+1} we solve eq. (2.63) jointly with the $2(k+1)$ equations in (2.60) and (2.61) for the $2(k+1)+1$ unknowns $\gamma_{11},\dots,\gamma_{1k+1}$ and $\gamma_{20},\dots,\gamma_{2k}$. (Note how this reduces to the result obtained for the unanticipated case above when $k = 0$). A convenient way to solve these equations is to first solve the three equations consisting of the two equations from:

$$\gamma_{k+1} = A^{k+1}\gamma_0 + d, \tag{2.66}$$

(obtained by "forecasting" γ_i out k periods) and eq. (2.61) for γ_{2k}, γ_{1k+1} and γ_{10}. Then the remaining coefficients can be obtained from the difference equations in (2.60) starting with the calculated value for γ_{10}.

The case where $\theta_i = 0$ for $i=1,\ldots,k-1$ and $\theta_k = \rho^{k-i}$ for $i=k,\ k-1$ can be solved by adding the particular solution to the nonhomogeneous equation

$$\gamma_{i+1} = A\gamma_i + \mathrm{d}\rho^{(i-k)} \qquad i=k, k+1, k+2,\ldots, \tag{2.67}$$

in place of (2.62) and solving for the remaining coefficients using eqs. (2.60) and (2.61) as above. The particular solution of (2.67) is

$$\gamma_i^{(P)} = (\rho I - A)^{-1}\mathrm{d}\rho^{i-k} \qquad i=k, k+1, k+2,\ldots. \tag{2.68}$$

2.2.5. *The exchange rate overshooting example*

The preceding calculations can be usefully illustrated with Example 1 of Section 2.2.1.: the two variable "overshooting" model in which the exchange rate ($y_{1t} = e_t$) is the jump variable and the price level ($y_{2t} = p_t$) is the slowly moving variable. For this model eq. (2.50) is

$$\begin{pmatrix} \underset{t}{\mathrm{E}} e_{t+1} \\ p_t \end{pmatrix} = A\begin{pmatrix} e_t \\ p_{t-1} \end{pmatrix} + \mathrm{d}m_t, \tag{2.69}$$

where the matrix

$$A = \frac{1}{1+\beta}\begin{pmatrix} 1+\beta\left(1+\frac{1}{\alpha}\right) & \frac{1}{\alpha} \\ \beta & 1 \end{pmatrix}, \tag{2.70}$$

and the vector $d = (-1/\alpha, 0)'$. Suppose that $\alpha = 1$ and $\beta = 1$. Then the characteristic roots of A are

$$\lambda = 1 \pm -0.707. \tag{2.71}$$

The characteristic vector associated with the unstable root is obtained from

$$(h_{11}, h_{12})A = \lambda_1(h_{11}, h_{12}), \tag{2.72}$$

this gives $-h_{12}/h_{11}=-0.414$ so that according to eq. (2.56) the coefficients of the (homogeneous) solution must satisfy

$$\gamma_{1i+1}=-0.414\gamma_{2i} \qquad i=0,1,\dots. \tag{2.73}$$

Using the stable root we have

$$\gamma_{2i+1}=0.293\gamma_{2i} \qquad i=0,1,\dots. \tag{2.74}$$

The particular solution is given by the vector $(\rho I-A)^{-1}\mathrm{d}\rho^{i-k}$ as in eq. (2.68). That is

$$\gamma_{1i}^{(P)}=\frac{(0.5-\rho)\rho^{i-k}}{(1.5-\rho)(0.5-\rho)-0.25} \qquad i=k,k+1,k+2,\dots, \tag{2.75}$$

$$\gamma_{2i-1}^{(P)}=\frac{-0.5\rho^{i-k}}{(1.5-\rho)(0.5-\rho)-0.25} \qquad i=k,k+1,k+2,\dots, \tag{2.76}$$

where k is the number of periods in advance that the shock to the money supply is anticipated ($k=0$ for unanticipated shocks).

In Tables 2, 3, and 4 and in Figures 5, 6, and 7, respectively, the effects of temporary unanticipated money shocks ($k=0$, $\rho=0$), permanent unanticipated money shocks ($k=0$, $\rho=1$), and permanent money shocks anticipated 3 periods

Table 2
Effect of an unanticipated temporary increase in money on the exchange rate and the price level ($k=0$, $\rho=0$).

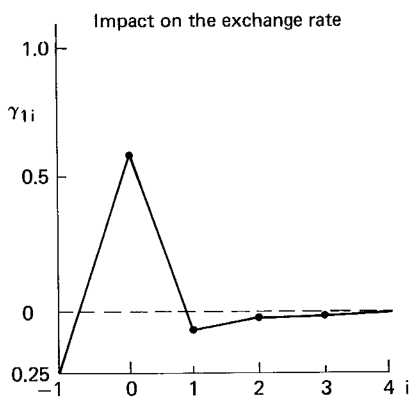
Period after shock:	i	0	1	2	3	4
Effect on exchange rate:	γ_{1i}	0.59	−0.12	−0.04	−0.01	−0.00
Effect on price level:	γ_{2i}	0.29	0.09	0.03	0.01	0.00

Table 3
Effect of unanticipated permanent increase in money on the exchange rate and the price level ($k=0$, $\rho=1$).

Period after shock:	i	0	1	2	3	4
Effect on exchange rate:	γ_{1i}	1.41	1.12	1.04	1.01	1.00
particular solution:	$\gamma_{1i}^{(P)}$	–	1	1	1	1
homogeneous solution:	$\gamma_{1i}^{(H)}$	–	0.12	0.04	0.01	0.00
Effect on price level:	γ_{2i}	0.71	0.91	0.97	0.99	1.00
particular solution:	$\gamma_{2i}^{(P)}$	1	1	1	1	1
homogeneous solution:	$\gamma_{2i}^{(H)}$	−0.29	−0.09	−0.03	−0.01	−0.00

Table 4
Effect of a permanent increase in money anticipated 3 periods in advance on the exchange rate and the price level ($k = 3$, $\rho = 1$).

Period after the shock:	i	0	1	2	3	4	5	6
Effect on the exchange rate:	γ_{1i}	0.28	0.43	0.71	1.21	1.06	1.02	1.00
particular solution:	$\gamma_{1i}^{(P)}$	–	–	–	–	1.00	1.00	1.00
homogeneous solution:	$\gamma_{1i}^{(H)}$	–	–	–	–	0.06	0.02	0.01
Effect on the price level:	γ_{2i}	0.14	0.28	0.50	0.85	0.96	0.99	1.00
particular solution:	$\gamma_{2i}^{(P)}$	–	–	–	1.00	1.00	1.00	1.00
homogeneous solution:	$\gamma_{2i}^{(H)}$	–	–	–	−0.15	−0.04	−0.01	−0.00



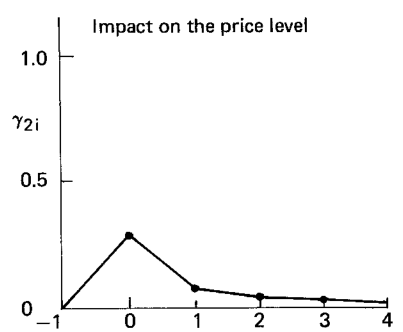


Figure 5. Temporary unanticipated increase in money.

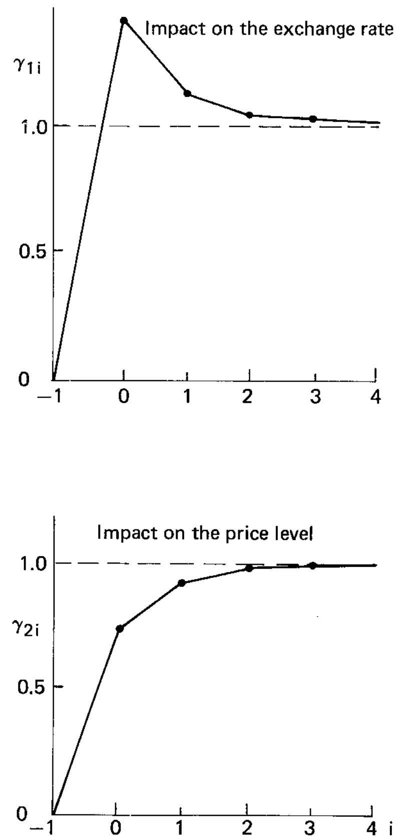


Figure 6. Permanent unanticipated increase in money.

in advance ($k = 3$, $\rho = 1$) are shown. In each case the increase in money is by 1 percent.

A temporary unanticipated increase in money causes the exchange rate to depreciate (e rises) and the price level to increase in the first period. Subsequently, the price level converges monotonically back to equilibrium. In the second period, e falls below its equilibrium value and then gradually rises again back to zero (Table 2 and Figure 5).

A permanent unanticipated increase in money of 1 percent eventually causes the exchange rate to depreciate by 1 percent and the price level to rise by 1 percent. But in the short run e rises above the long-run equilibrium and then gradually falls back to zero. This is the best illustration of overshooting (Table 3 and Figure 6).

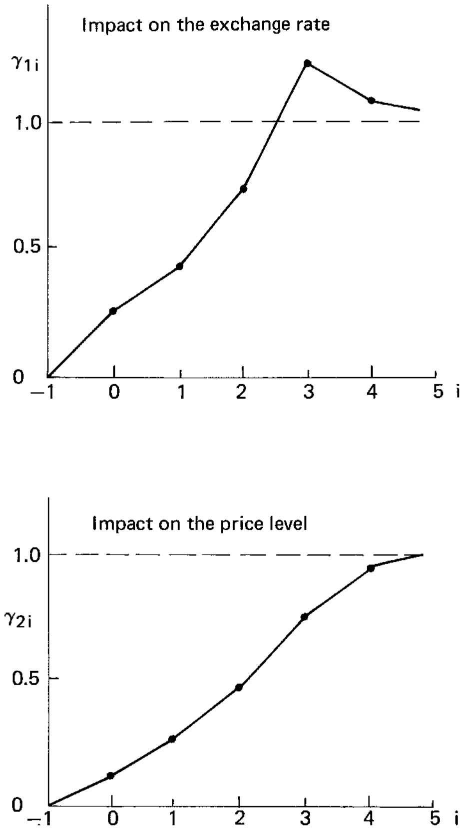


Figure 7. Permanent increase in money, anticipated 3 periods in advance.

If the increase in the money supply is anticipated in advance, then the price level rises and the exchange rate depreciates at the announcement date. Subsequently, the price level and e continue to rise. The exchange rate reaches its lowest value (e reaches its highest value) on the announcement date, and then appreciates back to its new long-run value of 1 (Table 4 and Figure 7). Note that p and e are on explosive paths from period 0 until period 3.

2.2.6. *Geometric interpretation*

The solution of the bivariate model has a helpful geometric interpretation. Writing out eq. (2.51) with $\theta_i = 0$ in scalar form as two different equations and subtracting γ_{1i} and γ_{2i-1} from the first and second equation respectively results in

$$\begin{aligned}\Delta\gamma_{1i+1} &\equiv \gamma_{1i+1}-\gamma_{1i}=(a_{11}-1)\gamma_{1i}+a_{12}\gamma_{2i-1},\\ \Delta\gamma_{2i} &\equiv \gamma_{2i}-\gamma_{2i-1}=a_{21}\gamma_{1i}+(a_{22}-1)\gamma_{2i-1}.\end{aligned} \tag{2.77}$$

According to (2.77) there are two linear relationships between γ_{1i} and γ_{2i-1} consistent with no change in the coefficients: $\Delta\gamma_{1i=1}=0$ and $\Delta\gamma_{2i}=0$. For example, in the exchange rate model in eq. (2.69), the equations in (2.77) become

$$\begin{aligned}\Delta\gamma_{1i+1} &= \frac{\beta}{\alpha(1+\beta)}\gamma_{1i}+\frac{1}{\alpha(1+\beta)}\gamma_{2i-1},\\ \Delta\gamma_{2i} &= \frac{\beta}{1+\beta}\gamma_{1i}-\frac{\beta}{1+\beta}\gamma_{2i-1}.\end{aligned} \tag{2.78}$$

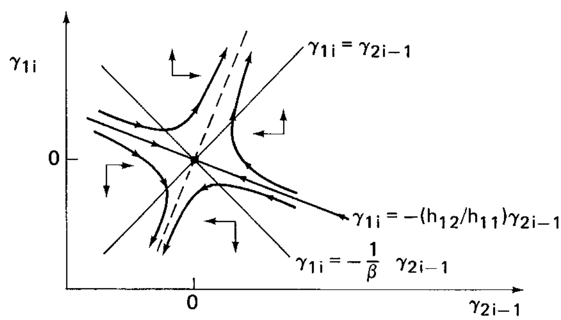


Figure 8. Geometric interpretation of the solution in the bivariate model. The darker line is the saddle point path along which the impact coefficients converge to the equilibrium value of (0,0).

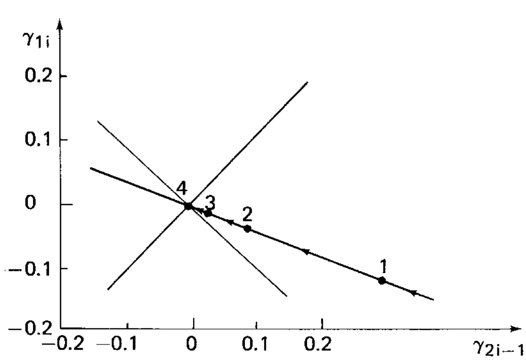


Figure 9. Solution values for the case of temporary-unanticipated shocks. ($k=0,\ \rho=0$). The numbered points are the values of i. See also Table 2 and Figure 5.

The two no-change lines are

$$\gamma_{1i} = -\frac{1}{\beta}\gamma_{2i-1},$$
$$\gamma_{1i} = \gamma_{2i-1}, \tag{2.79}$$

and are plotted in Figure 8. The arrows in Figure 8 show the directions of motion according to eq. (2.78) when the no-change relationships in (2.79) are not satisfied. It is clear from these arrows that if the γ coefficients are to converge to

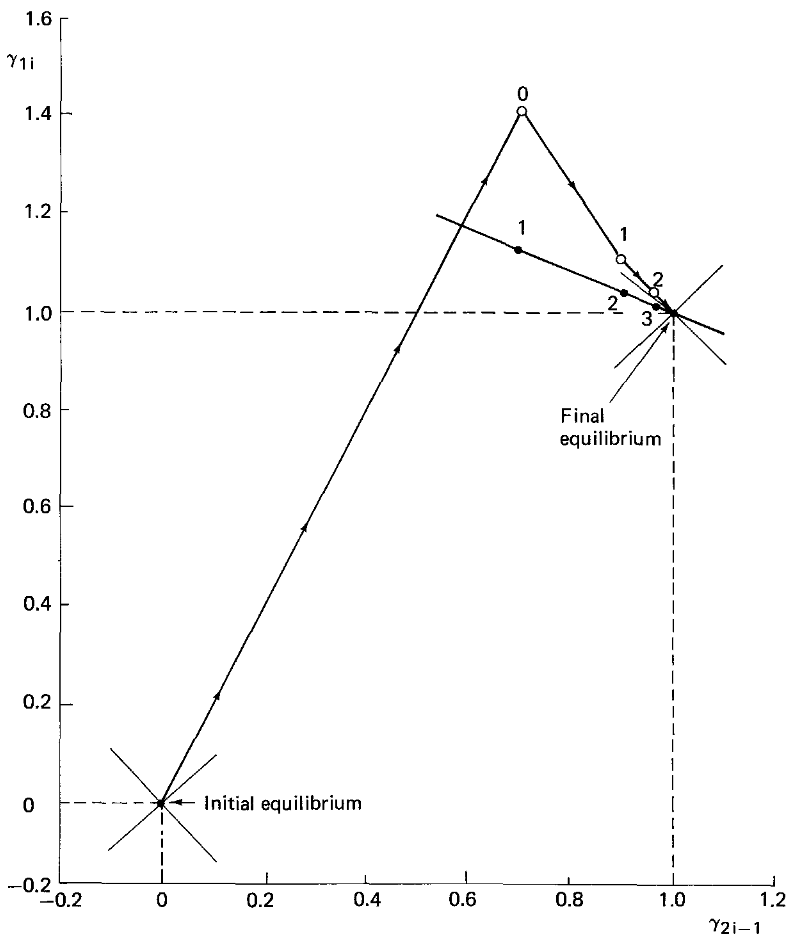


Figure 10. Solution values for a permanent unanticipated increase in the money supply. The open circles give the $(\gamma_{1i}, \gamma_{2i})$ pairs starting with $i = 0$.

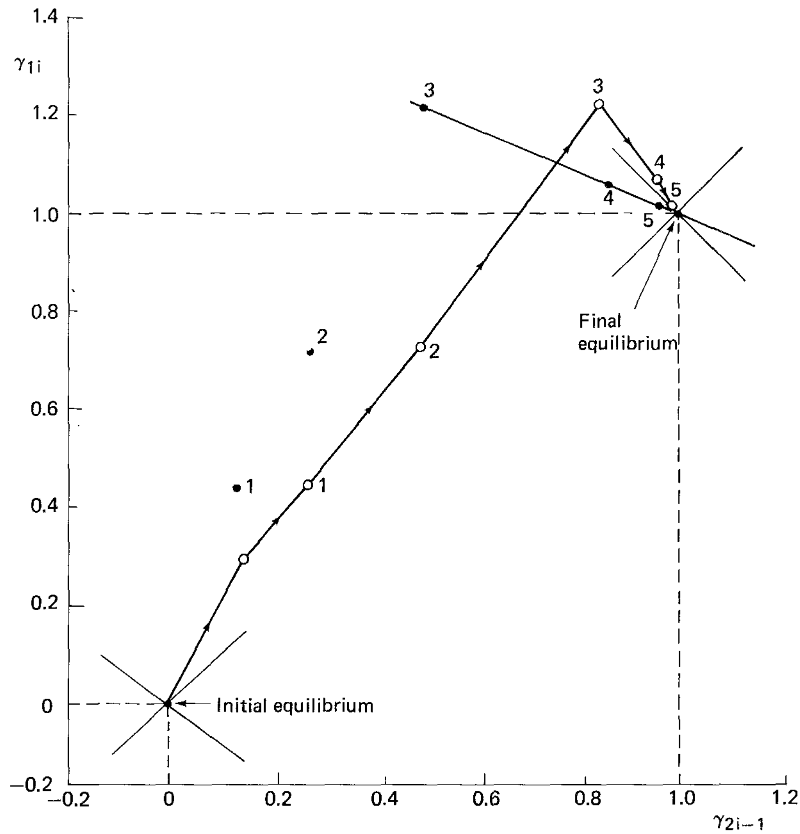


Figure 11. Solution values for an anticipated permanent increase in the money supply. The open circles give the γ_{1i}, γ_{2i} pairs starting with $i = 0$.

their equilibrium value (0,0) they must move along the "saddle point" path shown by the darker line in Figure 8. Points off this line will lead to ever-increasing values of the γ coefficients. The linear combination of γ_{1i} and γ_{2i-1} along this saddle point path is given by the characteristic vector associated with the unstable root λ_1 as given in general by eq. (2.55) and for this example in eq. (2.73). Note how Figure 8 immediately shows that the saddle point path is downward sloping. In Figure 9 the solution values for the impacts on the exchange rate and the price level are shown for the case of a temporary shock as considered in Table 2 and Figure 5. In Figures 10 and 11, the solution values are shown for the case where the increase in money is permanent. The permanent increase shifts the reference point from (0,0) to (1,1). The point (1,1) is simply the value of the particular solution in this case. Figure 10 is the case where the permanent increase is unanticipated; Figure 11 is the anticipated case.

Note that these diagrams do not give the impact on the exchange rate and the price level in the *same* period; they are one period out of synchronization. Hence, the points do not correspond to a scatter diagram of the effects of a change in money on the exchange rate and on the price level. It is a relatively simple matter to deduce a scatter diagram as shown by the open circles in Figures 10 and 11.

2.3. *The use of operators, generating functions, and z-transforms*

As the previous Sections have shown, the problem of solving rational expectations models is equivalent to solving nonhomogeneous deterministic difference equations. The homogeneous solution is obtained simply by requiring that the stochastic process for the endogenous variables be stationary. Once this is accomplished, most of the work comes in obtaining the particular solution to the nonhomogeneous part. Lag or lead operators, operator polynomials, and the power series associated with these polynomials (i.e. generating functions or z-transformations) have frequently been found useful in solving the nonhomogeneous part of difference equations [see Baumol (1970), for economic examples]. These methods have also been useful in rational expectations analysis. Futia (1981) and Whiteman (1983) have exploited the algebra of z-transforms in solving a wide range of linear rational expectations models.

To illustrate the use of operators, let $F^s x_t = x_{t+s}$ be the forward lead operator. Then the scalar equation in the impact coefficients that we considered in eq. (2.7), can be written

$$(1-\alpha F)\gamma_i = \delta\theta_i \qquad i=0,1,2,\dots. \tag{2.80}$$

Consider the case where $\theta_i = \rho^i$ and solve for γ_i by operating on both sides by the inverse of the polynomial $(1-\alpha F)$. We then have

$$\begin{aligned}\gamma_i &= \frac{\delta\rho^i}{1-\alpha F}\\ &= \frac{\delta\rho^i}{1-\alpha\rho} \qquad i=0,1,2\dots.\end{aligned} \tag{2.81}$$

the last equality follows from the algebra of operator polynomials [see for example Baumol (1970)]. The result is identical to what we found in Section 2.1 using the method of undetermined coefficients to obtain the particular solution. The procedure easily generalizes to the bivariate case and yields the particular solution shown in eq. (2.58). It also generalizes to handle other time series specifications of θ_i.

The operator notation used in (2.80) is standard in difference equation analysis. In some applications of rational expectations models, a non-standard operator has been used directly on the basic model (2.1). To see this redefine the operator F as $F\mathrm{E}_t y_t = \mathrm{E}_t y_{t+1}$. That is, F moves the date on the variable but the viewpoint date in the expectation is held constant. Then eq. (2.1) can be written (note that $\mathrm{E}_t y_t = y_t$):

$$(1-\alpha F)\underset{t}{\mathrm{E}} y_t = \delta_t. \tag{2.82}$$

Formally, we can apply the inverse of $(1-\alpha F)$ to (2.82) to obtain

$$\begin{aligned}\underset{t}{\mathrm{E}} y_t &= \delta(1-\alpha F)^{-1}u_t \\ &= \delta\left(1+\alpha F+(\alpha F)^2+\cdots\right)u_t \\ &= \delta\left(u_t+\alpha\underset{t}{\mathrm{E}}u_{t+1}+\alpha^2\mathrm{E}u_{t+2}+\cdots\right) \\ &= \delta\left(u_t+\alpha\rho u_t+(\alpha\rho)^2u_t+\cdots\right) \\ &= \frac{\delta}{1-\alpha\rho}u_t,\end{aligned} \tag{2.83}$$

and where we again assume that $u_t = \rho u_{t-1} + \varepsilon_t$. Eq. (2.83) gives the same answer that the previous methods did (again note that $\mathrm{E}_t y_t = y_t$). As Sargent (1979, p. 337) has discussed, the use of this type of operator on conditional expectations can lead to confusion or mistakes, if it is interpreted as a typical lag operator that shifts all time indexes, including the viewpoint dates. The use of operators on conventional difference operations like (2.6) is much more straightforward, and perhaps it is best to think of the algebra in (2.82) and (2.83) in terms of (2.80) and (2.81).

Whiteman's (1983) use of the generating functions associated with the operator polynomials can be illustrated by writing the power series corresponding to eqs. (2.2) and (2.4):

$$\gamma(z) = \sum_{i=0}^{\infty} \gamma_i z^i,$$

$$\theta(z) = \sum_{i=0}^{\infty} \theta_i z^i.$$

These are the z-transforms [see Dhrymes (1971) for a short introduction to z-transforms and their use in econometrics]. Equating the coefficients of ε_{t-i} in eq. (2.6) is thus the same as equating the coefficients of powers of z. That is, (2.6) means that

$$\gamma(z)=\alpha z^{-1}(\gamma(z)-\gamma_0)+\delta\theta(z). \tag{2.84}$$

Solving (2.84) for $\gamma(z)$ we have

$$\gamma(z)=(1-\alpha^{-1}z)^{-1}(\gamma_0-\delta\alpha^{-1}z\theta(z)). \tag{2.85}$$

As in Section 2.1, eq. (2.85) has a free parameter γ_0 which must be determined before $\gamma(z)$ can be evaluated. For y_t to be a stationary process, it is necessary that $\gamma(z)$ be a convergent power series (or equivalently an analytic function) for $|z|<1$. The term $(1-\alpha^{-1}z)^{-1}$ on the right-hand side of (2.85) is divergent if $\alpha^{-1}>1$. Hence, the second term in parentheses must have a factor to "cancel out" this divergent series. For the case of serially uncorrelated shocks, $\theta(z)$ is a constant $\theta_0=1$ so that it is obvious that $\gamma_0=\delta$ will cancel out the divergent series. We then have $\gamma(z)=\delta$ which corresponds with the results in Section 2.1. Whiteman (1983) shows that in general $\gamma(z)$ will be convergent when $|\alpha|<1$ if $\gamma_0=\delta\theta(\alpha)$. For the unanticipated autoregressive shocks this implies that $\gamma(z)=\delta(1-\rho\alpha)^{-1}(1-\rho z)$ which is the z-transform of the solution we obtained earlier. When $|\alpha|>1$ there is no natural way to determine γ_0, so we are left with non-uniqueness as in Section 2.1.

2.4. *Higher order representations and factorization techniques*

We noted in Section 2.2 that a first-order bivariate model with one lead variable could be interpreted as a second-order scalar model with a lead and a lag. That is,

$$y_t=\alpha_1 \underset{t}{\mathrm{E}} y_{t+1}+\alpha_2 y_{t-1}+\delta u_t, \tag{2.86}$$

can be written as a bivariate model and solved using the saddle point stability method. An alternative approach followed by Sargent (1979), Hansen and Sargent (1980) and Taylor (1980a) is to work with (2.86) directly. That the two approaches give the same result can be shown formally.

Substitute for y_t, y_{t-1}, and $\mathrm{E}_t y_{t+1}$ in eq. (2.86) using (2.4) to obtain the equations

$$\gamma_1=\frac{1}{\alpha_1}(\gamma_0-\delta\theta_0), \tag{2.87}$$

$$\gamma_{i+1}=\frac{1}{\alpha_1}\gamma_i-\frac{\alpha_2}{\alpha_1}\gamma_{i-1}-\frac{\delta}{\alpha_1}\theta_i \qquad i=1,2,\dots. \tag{2.88}$$

As above, we need one more equation to solve for all the γ coefficients. Consider first the homogeneous part of (2.88). Its characteristic polynomial is

$$z^2 - \frac{1}{\alpha_1}z + \frac{\alpha_2}{\alpha_1}, \tag{2.89}$$

which can be *factored* into

$$(\lambda_1 - z)(\lambda_2 - z), \tag{2.90}$$

where λ_1 and λ_2 are the roots of (2.89). The solution to the homogeneous part is $\gamma_i^{(H)} = k_1\lambda_1^i + k_2\lambda_2^i$. As we discussed above, in many economic applications one root, say λ_1, will be larger than 1 in modulus and the other will be smaller than 1 in modulus. Thus, the desired solution to the homogeneous part is achieved by setting $k_1 = 0$ so that $\gamma_i^{(H)} = k_2\lambda_2^i$ where k_2 equals the initial condition $\gamma_0^{(H)}$. Equivalently we can interpret the setting of $k_1 = 0$ as reducing the characteristic polynomial (2.89) to $(z - \lambda_2)$. Thus, the γ coefficients satisfy

$$\gamma_i = \lambda_2\gamma_{i-1} \qquad i = 1,2,\dots. \tag{2.91}$$

Equivalently, we have "factored out" $(z - \lambda_1)$ from the characteristic polynomial.

For the case where u_t is uncorrelated so that $\theta_i = 0$ for $i > 0$, difference equation in (2.88) is homogeneous. We can solve for γ_0 by using $\gamma_1 = \lambda_2\gamma_0$ along with eq. (2.87). This gives $\gamma_0 = \delta(1 - \alpha_1\lambda_2)^{-1}\lambda_2^i \quad i = 0,1,\dots$

To see how this result compares with the saddle-point approach, write (2.88) as

$$\begin{pmatrix} \gamma_{i+1} \\ \gamma_i \end{pmatrix} = \begin{pmatrix} \frac{1}{\alpha_1} & -\frac{\alpha_2}{\alpha_1} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \gamma_i \\ \gamma_{i-1} \end{pmatrix} - \begin{pmatrix} \frac{\delta}{\alpha_1} \\ 0 \end{pmatrix} \theta_i \qquad i = 1,2,\dots. \tag{2.92}$$

The characteristic equation of the matrix A is $\lambda^2 - (1/\alpha_1)\lambda - \alpha_2/\alpha_1 = 0$. Hence, the roots of A are identical to the roots of the characteristic polynomial associated with the second-order difference eq. (2.88). [This is a well-known result shown for the general pth order difference equation in Anderson (1971)].

The characteristic vector of the matrix A associated with the unstable root λ_1 is found from the equation $(h_{11}, h_{12})A = \lambda_1(h_{11}, h_{12})$. Thus, the saddle point path is given by

$$\gamma_i = -\frac{h_{12}}{h_{11}}\gamma_{i-1} = \left(\frac{1}{\alpha_1} - \lambda_1\right)\gamma_{i-1}. \tag{2.93}$$

For the two methods to be equivalent, we need to show that (2.91) and (2.93) are equivalent, or that $\lambda_2 = 1/\alpha_1 - \lambda_1$. This follows immediately from the fact that the sum of the roots $(\lambda_1+\lambda_2)$ of a second-order polynomial equals the coefficients of the linear term in the polynomial: $\lambda_1+\lambda_2=1/\alpha_1$.

For the case where $\theta_i=\rho^i$, we need to compare the particular solutions as well. For the second-order scalar model we guess the form $\gamma_i^{(P)}=ab^i$. Substituting this into (2.88) we find that $b=\rho$ and $a=\delta(1-\alpha_1\rho-\alpha_2\rho^{-1})^{-1}$. To see that this gives the same value for the particular solution that emerges from the matrix formulation in eq. (2.58), note that

$$
\begin{aligned}
(\rho I-A)^{-1}\mathrm{d}\rho^i &= \begin{pmatrix} \rho-\dfrac{1}{\alpha_1} & -\dfrac{\alpha_2^{-1}}{\alpha_1} \\ -1 & \rho \end{pmatrix}\begin{pmatrix} -\dfrac{\delta}{\alpha_1} \\ 0 \end{pmatrix}\rho^i \\
&= \frac{1}{\rho^2-\dfrac{1}{\alpha_1}\rho+\dfrac{\alpha_2}{\alpha_1}}\begin{pmatrix} -\rho\dfrac{\delta}{\alpha_1} \\ -\dfrac{\delta}{\alpha_1} \end{pmatrix}\rho^i .
\end{aligned}
\tag{2.94}
$$

Eq. (2.94) gives the particular solution for the vector $(\gamma_i^{(P)},\gamma_{i-1}^{(P)})$, which corresponds to the vector $\boldsymbol{\gamma}_i^{(P)}$ in eq. (2.58). Hence

$$
\begin{aligned}
\gamma_i^{(P)} &= \frac{-\rho\alpha_1^{-1}\delta\rho^i}{\rho^2-\rho\alpha_1^{-1}+\alpha_2\alpha_1^{-1}} \\
&= \frac{\delta\rho^i}{1-\alpha_1\rho-\alpha_2\rho^{-1}},
\end{aligned}
$$

which is the particular solution obtained from the second-order scalar representation.

Rather than obtaining the solution of the homogeneous system by factoring the characteristic equation, one can equivalently factor the polynomial in the time shift operators. Because the operator polynomials also provide a convenient way to obtain the nonhomogeneous solution (as was illustrated in Section 2.3), this approach essentially combines the homogeneous solution and the nonhomogeneous solution in a notationally and computationally convenient way.

Write (2.88) as

$$
\left(-L^{-1}+\frac{1}{\alpha_2}-\frac{\alpha_2}{\alpha_1}L\right)\gamma_i=\frac{\delta}{\alpha_1}\theta_i .
\tag{2.95}
$$

Let $H(L)=L^{-1}-1/\alpha_1+(\alpha_2/\alpha_1)L$ be the polynomial on the left-hand side of

(2.95) and let $P(z)=z^2-1/(\alpha_1)z+\alpha_2/\alpha_1$ be the characteristic polynomial in (2.89). The polynomial $H(L)$ can be factored into

$$\mu(1-\phi L^{-1})(1-\psi L), \tag{2.96}$$

where $\phi=-\mu^{-1}$, $\psi=-\mu^{-1}\alpha_2\alpha_1^{-1}$, and where μ is one of the solutions of $P(\mu)=0$; that is one of the roots of $P(\cdot)$. This can be seen by equating the coefficient of $H(L)$ and the polynomial in (2.96). Continuing to assume that only one of the roots of $P(\cdot)$ is greater than one in modulus (say λ_1) we set $\phi=\lambda_1^{-1}<1$. Since the product of the roots of $P(\cdot)$ equals $\alpha_2\alpha_1^{-1}$ we immediately have that $\psi=\lambda_2$. Thus, there is a unique factorization of the polynomial with ϕ and ψ both less than one in modulus.

Because $\psi=\lambda_2$, the stable solution (2.97) to the homogeneous difference equation can be written

$$(1-\psi L)\gamma_i^{(H)}=0. \tag{2.97}$$

The particular solution also can be written using the operator notation:

$$\gamma_i^{(P)}=\frac{\delta\alpha_1^{-1}\rho^i}{\mu(1-\phi L^{-1})(1-\psi L)}. \tag{2.98}$$

The complete solution is given by $\gamma_i=\gamma_i^{(H)}+\gamma_i^{(P)}$ which implies that

$$(1-\lambda_2 L)\gamma_i=(1-\lambda_2 L)\gamma_i^{(H)}+\left(1-\lambda_2 L^{-1}\right)\gamma_i^{(P)}. \tag{2.99}$$

The first term on the right-hand side of (2.99) equals zero. Therefore the complete solution is given by

$$\begin{aligned}\gamma_i&=\lambda_2\gamma_{i-1}+\frac{\delta\alpha_1^{-1}\rho^i}{\lambda_1\left(1-\lambda_1^{-1}L^{-1}\right)}\\&=\lambda_2\gamma_{i-1}+\frac{\delta\alpha_1^{-1}\rho^i}{\lambda_1\left(1-\rho\lambda_1^{-1}\right)}.\end{aligned} \tag{2.100}$$

This solution is equivalent to that derived by adding the particular solution in (2.95) to the solution of the homogeneous solution of (2.91).

Note that this procedure or solving (2.95) can be stated quite simply in two steps: (1) factor the lag polynomial into two stable polynomials, one involving positive powers of L (lags) and the other involving negative powers of L (leads), and (2) operate on both sides of (2.95) by the inverse of the polynomial involving negative powers of L.

It is clear from (2.94) that the γ_i weights are such that the solution for y_t can be represented as a first-order autoregressive process with a serially correlated error:

$$y_t = \lambda_2 y_{t-1} + \delta\alpha_1^{-1}(\lambda_1 - \rho)^{-1}u_t, \tag{2.101}$$

where

$$u_t = \rho u_{t-1} + \varepsilon_t.$$

In the papers by Sargent (1979), Taylor (1980a) and Hansen and Sargent (1980), the difference equation in (2.95) was written $\gamma_i = \mathrm{E}_t y_{t+i}$ and $\theta_i = \mathrm{E}_t u_{t+i}$, a form which can be obtained by taking conditional expectations in eq. (2.86). In other words rather than working with the moving average coefficients they worked directly with the conditional expectations. As discussed in Section 2.3 this requires the use of a non-standard lag operator.

2.5. *Rational expectations solutions as boundary value problems*

It is useful to note that the problem of solving rational expectations models can be thought of as a boundary value problem where final conditions as well as initial conditions are given. To see this consider the homogeneous equation

$$\gamma_{i+1} = \frac{1}{\alpha}\gamma_i \qquad i = 0,1,\dots. \tag{2.102}$$

The stationarity conditions place a restriction on the "final" value $\lim_{j\to\infty}\gamma_j = 0$ rather than on the "initial" value γ_0. As an approximation we want $\gamma_j = 0$ for large j. A traditional method to solve boundary value problems is "shooting": One guesses a value for γ_0 and then uses (2.102) to project (shoot) a value of γ_j for some large j. If the resulting $\gamma_j \neq 0$ (or if γ_j is further from 0 than some tolerance range) then a new value (chosen in some systematic fashion) of γ_0 is tried until one gets γ_j sufficiently close to zero. It is obvious in this case that $\gamma_0 = 0$ so it would be impractical to use such a method. But in nonlinear models the approach can be quite useful as we discuss in Section 6.

This approach obviously generalizes to higher order systems; for example the homogeneous part of (2.88) is

$$\gamma_{i+1} = \frac{1}{\alpha_1}\gamma_i - \frac{\alpha_2}{\alpha_1}\gamma_{i-1} \qquad i = 0,1,2,\dots. \tag{2.103}$$

with $\gamma_{-1} = 0$ as one initial condition and $\gamma_j = 0$ for some large j as the one "final" condition. This is a two point boundary problem which can be solved in the same way as (2.102).

3. Econometric evaluation of policy rules

Perhaps the main motivation behind the development of rational expectations models was the desire to improve policy evaluation procedures. Lucas (1976) argued that the parameters of the models conventionally used for policy evaluation – either through model simulation or formal optimal control – would shift when policy changed. The main reason for this shift is that expectations mechanisms are adaptive, or backward looking, in conventional models and thereby unresponsive to those changes in policy that would be expected to change expectations of future events. Hence, the policy evaluation results using conventional models would be misleading.

The Lucas criticism of conventional policy evaluation has typically been taken as destructive. Yet, implicit in the Lucas' criticism is a constructive way to improve on conventional evaluation techniques by modeling economic phenomena in terms of "structural" parameters; by "structural" one simply means invariant with respect to policy intervention. Whether a parameter is invariant or not is partly a matter of researcher's judgment, of course, so that any attempt to take the Lucas critique seriously by building structural models is subject to a similar critique that the researcher's assumption about which parameters are structural is wrong. If taken to this extreme that no feasible structural modeling is possible, the Lucas critique does indeed become purely destructive and perhaps even stifling.

Hansen and Sargent (1980), Kydland and Prescott (1982), Taylor (1982), and Christiano (1983) have examined policy problems where only the parameters of utility functions or production functions can be considered invariant or structural. Taylor (1979, 1980b) has considered models where the parameters of the wage and price setting functions are invariant or structural.

The thought experiments described in Section 2 whereby multiplier responses are examined should be part of any policy evaluation technique. But it is unrealistic to think of policy as consisting of such one-shot changes in the policy instrument settings. They never occur. Rather, one wants to consider changes in the way the policymakers respond to events – that is, changes in their policy rules. For this we can make use of stochastic equilibrium solutions examined in Section 2. We illustrate this below.

3.1. Policy evaluation for a univariate model

Consider the following policy problem which is based on model (2.1). Suppose that an econometric policy advisor knows that the demand for money is given by

$$m_t - p_t = -\beta(\mathrm{E}_t p_{t+1} - p_t) + u_t. \tag{3.1}$$

Here there are two shocks to the system, the supply of money m_t and the demand for money u_t. Suppose that $u_t = \rho u_{t-1} + \varepsilon_t$, and that in the past the money supply was fixed: $m_t = 0$; suppose that under this fixed money policy, prices were thought to be too volatile. The policy advisor is asked by the Central Bank for advice on how m_t can be used in the future to reduce the fluctuations in the price level. Note that the policy advisor is not asked just what to do today or tomorrow, but what to do for the indefinite future. Advice thus should be given as a contingency rule rather than as a fixed path for the money supply.

Using the solution technique of Section 2, the behavior of p_t during the past is

$$p_t = \rho p_{t-1} - \frac{\varepsilon_t}{1+\beta(1-\rho)}. \tag{3.2}$$

Conventional policy evaluation might proceed as follows: first, the econometrician would have estimated ρ in the reduced form relation (3.2) over the sample period. The estimated equation would then serve as a model of expectations to be substituted into (3.1); that is, $\mathrm{E}_t p_{t+1} = \rho p_t$ would be substituted into

$$m_t - p_t = -\beta(\rho p_t - p_t) + u_t. \tag{3.3}$$

The conventional econometricians model of the price level would then be

$$p_t = \frac{m_t - u_t}{1+\beta(1-\rho)}. \tag{3.4}$$

Considering a feedback policy rule of the form $m_t = g u_{t-1}$ eq. (3.4) implies

$$\operatorname{var} p_t = \frac{1}{[1+\beta(1-\rho)]^2(1-\rho^2)}\sigma_\varepsilon^2\left[g^2+1-2g\rho\right]. \tag{3.5}$$

If there were no cost to varying the money supply, then eq. (3.5) indicates that the best choice for g to minimize fluctuation in p_t is $g = \rho$.

But we know that (3.5) is incorrect if $g \neq 0$. The error was to assume that $\mathrm{E}_t p_{t+1} = \rho p_t$ regardless of the choice of policy. This is the expectations error that rational expectations was designed to avoid. The correct approach would have been to substitute $m_t = g u_{t-1}$ directly into (3.1) and calculate the stochastic equilibrium for p_t. This results in

$$p_t = \frac{-1-\beta(1-g)}{(1+\beta)(1+\beta(1-\rho))} u_t + \frac{g}{1+\beta} u_{t-1}. \tag{3.6}$$

Note how the parameters of (3.6) depend on the parameters of the policy rule. The variance of p_t is

$$\operatorname{Var} p_t = \frac{1}{(1+\beta)^2(1-\rho^2)} \left[\frac{(1+\beta(1-g))^2}{(1+\beta(1-\rho))^2} - \frac{2g(1+\beta(1-g))\rho}{1+\beta(1-\rho)} + g^2 \right] \sigma_\varepsilon^2. \tag{3.7}$$

The optimal policy is found by minimizing Var p_t with respect to g.

This simple policy problem suggests the following approach to macro policy evaluation: (1) Derive a stochastic equilibrium solution which shows how the endogeneous variables behave as a function of the parameters of the policy rule; (2) Specify a welfare function in terms of the moments of the stochastic equilibrium, and (3) Maximize the welfare function across the parameters of the policy rule. In this example the welfare function is simply Var p. In more general models there will be several target variables. For example, in Taylor (1979) an optimal policy rule to minimize a weighted average of the variance of real output and the variance of inflation was calculated.

Although eq. (3.1) was not derived explicitly from an individual optimization problem, the same procedure could be used when the model is directly linked to parameters of a utility function. For instance, the model of Example (5) in Section 2.2 in which the parameters depend on a firm's utility function could be handled in the same way as the model in (3.1).

3.2. *The Lucas critique and the Cowles Commission critique*

The Lucas critique can be usefully thought of as a dynamic extension of the critique developed by the Cowles Commission researchers in the late 1940s and early 1950s and which gave rise to the enormous literature on simultaneous equations. At that time it was recognized that reduced forms could not be used for many policy evaluation questions. Rather one should model structural relationships. The parameters of the reduced form are, of course, functions of the structural parameters in the standard Cowles Commission setup. The discussion by Marschak (1953), for example, is remarkably similar to the more recent rational expectations critiques; Marschak did not consider expectations variables, and in this sense the rational expectations critique is a new extension. But earlier analyses like Marschak's are an effort to explain why structural modeling is necessary, and thus has much in common with more recent research.

3.3. Game-theoretic approaches

In the policy evaluation procedure discussed above, the government acts like a dominant player with respect to the private sector. The government sets g and the private sector takes g as given. The government then maximizes its social welfare function across different values of g. One can imagine alternatively a game theoretic setup in which the government and the private sector each are maximizing utility. Chow (1983), Kydland (1975), Lucas and Sargent (1981), and Epple, Hansen, and Roberds (1983) have considered this alternative approach. It is possible to specify the game theoretic model as a choice of parameters of decision rules in the steady state or as a formal non-steady state dynamic optimization problem with initial conditions partly determining the outcome. Alternative solution concepts including Nash equilibria have been examined.

The game-theoretic approach naturally leads to the important time inconsistency problem raised by Kydland and Prescott (1977) and Calvo (1979). Once the government announces its policy, it will be optimal to change it in the future. The consistent solution in which everyone expects the government to change is generally suboptimal. Focussing on rules as in Section 3.1 effectively eliminates the time inconsistency issue. But even then, there can be temptation to change the rule.

4. Statistical inference

The statistical inference issues that arise in rational expectations models can be illustrated in a model like that of Section 2.

4.1. Full information estimation

Consider the problem of estimating the parameters of the structural model

$$y_t = \alpha \underset{t}{\mathrm{E}} y_{t+1} + \delta x_t + v_t, \tag{4.1}$$

where v_t is a serially uncorrelated random variable. Assume (for example) that x_t has a finite moving average representation:

$$x_t = \varepsilon_t + \theta_1\varepsilon_{t-1} + \cdots + \theta_q\varepsilon_{t-q}, \tag{4.2}$$

where ε_t is serially uncorrelated and assume that $\operatorname{Cov}(v_t, \varepsilon_s) = 0$ for all t and s.

To obtain the full information maximum likelihood estimate of the structural system (4.1) and (4.2) we need to reduce (4.1) to a form which does not involve expectations variables. This can be done by solving the model using one of the techniques described in Section 2. Using the method of undetermined coefficients, for example, the solution for y_t is

$$y_t = \gamma_0\varepsilon_t + \cdots + \gamma_q\varepsilon_{t-q} + v_t, \tag{4.3}$$

where the γ parameters are given by

$$\begin{pmatrix} \gamma_0 \\ \gamma_1 \\ \vdots \\ \gamma_q \end{pmatrix} = \delta \begin{pmatrix} 1 & \alpha & \alpha^2 & \cdots & \alpha^{q-1} & \alpha^q \\ 0 & 1 & \alpha & \cdots & \alpha^{q-2} & \alpha^{q-1} \\ \vdots & & & & \vdots & \vdots \\ 0 & & & & 1 & \alpha \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ \theta_1 \\ \vdots \\ \theta_q \end{pmatrix}. \tag{4.4}$$

Eqs. (4.2) and (4.3) together form a two dimensional vector model.

$$\begin{pmatrix} y_t \\ x_t \end{pmatrix} = \begin{pmatrix} \gamma_0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \varepsilon_t \\ v_t \end{pmatrix} + \begin{pmatrix} \gamma_1 & 0 \\ \theta_1 & 0 \end{pmatrix}\begin{pmatrix} \varepsilon_{t-1} \\ v_{t-1} \end{pmatrix} + \cdots + \begin{pmatrix} \gamma_q & 0 \\ \theta_q & 0 \end{pmatrix}\begin{pmatrix} \varepsilon_{t-q} \\ v_{t-q} \end{pmatrix}. \tag{4.5}$$

Eq. (4.5) is an estimatable reduced form system corresponding to the structural form in (4.1) and (4.2).

If we assume that (v_t, ε_t) is distributed normally and independently, then the full-information maximum likelihood estimate of $(\theta_1, \dots, \theta_q, \alpha, \delta)$ can be obtained using existing methods to estimate multivariate ARMA models. See Chow (1983, Section 6.7 and 11.6). Note that the coefficients of the ARMA model (4.5) are constrained. There are *cross-equation* restrictions in that the θ and γ parameters are related to each other by (4.4). In addition, relative to a fully unconstrained ARMA model, the off-diagonal elements of the autoregression are equal to zero.

Full information estimation maximum likelihood methods for linear rational expectations models have been examined by Chow (1983), Muth (1981), Wallis (1980), Hansen and Sargent (1980, 1981), Dagli and Taylor (1985), Mishkin

(1983), Taylor (1979, 1980a), and Wickens (1982). As in this example, the basic approach is to find a constrained reduced form and maximize the likelihood function subject to the constraints. Hansen and Sargent (1980, 1981) have emphasized these cross-equation constraints in their expositions of rational expectations estimation methods. In Muth (1981), Wickens (1982) and Taylor (1979) multivariate models were examined in which expectations are dated at $t-1$ rather than t and $E_{t-1}y_t$ appears in (4.1) rather than $E_t y_{t+1}$. More general multivariate models with leads and lags are examined in the other papers.

For full information estimation, it is also important that the relationship between the structural parameters and the reduced form parameters can be easily evaluated. In this example the mapping from the structural parameters to the reduced form parameters is easy to evaluate. In more complex models the mapping does not have a closed form; usually because the roots of high-order polynomials must be evaluated.

4.2. *Identification*

There has been relatively little formal work on identification in rational expectations models. As in conventional econometric models, identification involves the properties of the mapping from the structural parameters to the reduced form parameters. The model is identified if the structural parameters can be uniquely obtained from the reduced form parameters. Over-identification and under-identification are similarly defined as in conventional econometric models. In rational expectations models the mapping from reduced form to structural parameters is much more complicated than in conventional models and hence it has been difficult to derive a simple set of conditions which have much generality. The conditions can usually be derived in particular applications as we can illustrate using the previous example.

When $q=0$, there is one reduced form parameter γ_0, which can be estimated from (4.2) and (4.3), recalling that Cov $(v_t, \varepsilon_t)=0$, and two structural parameters δ and α in eq. (4.4). Hence, the model is not identified. In this case, $\delta=\gamma_0$ is identified from the regression of y_t on the exogenous x_t, but α is not identified. When $q=1$, there are three reduced form parameters γ_0, γ_1 and θ_1 which can be estimated from (4.2) and (4.3), and three structural parameters δ, α, and θ_1. (θ_1 is both a structural and reduced form parameter since x_t is exogenous). Hence, the model is exactly identified according to a simple order condition. More generally, there are $q+2$ structural parameters $(\delta, \alpha, \theta_1, \dots, \theta_q)$ and $2q+1$ reduced form parameters $(\gamma_0, \gamma_1, \dots, \gamma_q, \theta_1, \dots, \theta_q)$ in this model. According to the order conditions, therefore, the model is overidentified if $q>1$.

Treatments of identification in more general models focus on the properties of the cross-equation restrictions in more complex versions of eq. (4.4). Wallis (1980) gives conditions for identification for a class of rational expectations models; the conditions may be checked in particular applications. Blanchard (1982) has derived a simple set of identification restrictions for the case where x_t in (4.2) is autoregressive and has generalized this to higher order multivariate versions of (4.1) and (4.2).

4.3. Hypothesis testing

Tests of the rational expectations assumption have generally been constructed as a test of the cross-equation constraints. These constraints arise because of the rational expectations assumption. In the previous example, the null hypothesis that the cross-equation constraints in (4.5) hold can be tested against the alternative that (4.5) is a fully unconstrained moving average model by using a likelihood ratio test. Note, however, that this is a joint test of rational expectations and the specification of the model. Testing rational expectations against a specific alternative like adaptive expectations usually leads to non-nested hypotheses.

In more general linear models, the same types of cross-equation restrictions arise, and tests of the model can be performed analogously. However, for large systems the fully unconstrained ARMA model may be difficult to estimate because of the large number of parameters.

4.4. Limited information estimation methods

Three different types of "limited information" estimates have been used for rational expectations models. These can be described using the model in (4.1) and (4.2). One method investigated by Wallis estimates (4.2) separately in order to obtain the parameters $\theta_1, \dots, \theta_q$. These estimates then are taken as given (as known parameters) in estimating (4.3). Clearly this estimator is less efficient than the full information estimator, but in more complex problems the procedure saves considerable time and effort. This method has been suggested by Wallis (1980) and has been used by Papell (1984) and others in applied work.

A second method proposed by Chow (1983) and investigated by Chow and Reny (1983) was mentioned earlier in our discussion of nonuniqueness. This method does not impose the saddle point stability constraints on the model. It leads to an easier computation problem than does imposing the saddle point constraints. If the investigator does not have any reason to impose this constraint, then this could prove quite practical.

A third procedure is to estimate eq. (4.1) as a single equation using instrumental variables. Much work has been done in this area in recent years, and because of computational costs of full information methods it has been used frequently in applied research. Consider again the problem of estimating eq. (4.1). Let $e_t = \mathrm{E}_t y_{t+1} - y_{t+1}$ be the forecast error for the prediction of y_t. Substitute $\mathrm{E}_t y_{t+1}$ into

(4.1) to get

$$y_t = \alpha y_{t+1} + \delta x_t + v_t - \alpha e_{t+1}. \tag{4.6}$$

By finding instruments of variables for y_{t+1} that are uncorrelated with v_t and e_{t+1} one can estimate (4.6) using the method of instrumental variables. In fact this estimate would simply be the two stage least squares estimate with y_{t+1} treated as if it were a right-hand side endogenous variable in a conventional simultaneous equation model. Lagged values of x_t could serve as instruments here. This estimate was first proposed by McCallum (1976).

Several extensions of McCallum's method have been proposed to deal with serial correlation problems including Cumby, Huizinga and Obstfeld (1983), McCallum (1979), Hayashi and Sims (1983), Hansen (1982), and Hansen and Singleton (1982). A useful comparison of the efficiency of these estimators is found in Cumby, Huizinga and Obstfeld (1983).

5. General linear models

A general linear rational expectations model can be written as

$$B_0 y_t + B_1 y_{t-1} + \cdots + B_p y_{t-p} + A_1 \underset{t}{\mathrm{E}} y_{t+1} + \cdots + A_q \underset{t}{\mathrm{E}} y_{t+q} = C u_t, \tag{5.1}$$

where y_t is a vector of endogenous variables, u_t is a vector of exogenous variables or shocks, and A_i, B_i and C are matrices containing parameters.

Two alternative approaches have been taken to solve this type of model. Once it is solved, the policy evaluation and estimation methods discussed above can be applied. One approach is to write the model as a large first-order vector system directly analogous to the 2-dimensional vector model in eq. (2.50). The other approach is to solve (5.1) directly by generalizing the approach taken to the second-order scalar model in eq. (2.86). The first approach is the most straightforward. The disadvantage is that it can easily lead to very large (although sparse) matrices with high-order polynomials to solve to obtain the characteristic roots. This type of generalization is used by Blanchard and Kahn (1980) and Anderson and Moore (1984) to solve deterministic rational expectations models.

5.1. A general first-order vector model

Equation (5.1) can be written as

$$\underset{t}{\mathrm{E}} z_{t+1} = A z_t + D u_t, \tag{5.2}$$

by stacking $y_t, y_{t-1}, \dots, y_{t-p}$ into the vector z_t much as in eq. (2.50). (It is necessary that A_q be nonsingular to write (5.1) as (5.2)). Anderson and Moore (1984) have developed an algorithm that reduces equations with a singular A_q into an equivalent form with a nonsingular matrix coefficient of y_{t+q} and have applied it to an econometric model of the U.S. money market. (Alternatively, Preston and Pagan (1982, pp. 297–304) have suggested that a "shuffle" algorithm described by Luenberger (1977) be used for this purpose). In eq. (5.2) let z_t be an n-dimensional vector and let u_t be an m dimensional vector of stochastic disturbances. The matrix A is $n \times n$ and the matrix D is $n \times m$.

We describe the solution for the case of unanticipated temporary shocks: $u_t = \varepsilon_t$ where ε_t is a serially uncorrelated vector with a zero mean. Alternative assumptions about u_t can be handled by the methods discussed in Section 2.2. The solution for z_t can be written in the general form:

$$z_t = \sum_{i=0}^{\infty} \Gamma_i \varepsilon_{t-i}, \tag{5.3}$$

where the Γ_i are $n \times m$ matrices of unknown coefficients. Substituting (5.3) into (5.2) we get

$$\begin{aligned} \Gamma_1 &= A\Gamma_0 + D, \\ \Gamma_{i+1} &= A\Gamma_i \qquad i = 1, 2, \dots . \end{aligned} \tag{5.4}$$

Note that these matrix difference equations hold for each column of Γ_i separately; that is

$$\begin{aligned} \gamma_1 &= A\gamma_0 + d, \\ \gamma_{i+1} &= A\gamma_i \qquad i = 1, 2, \dots, \end{aligned} \tag{5.5}$$

where γ_i is any one of the $n \times 1$ column vectors in Γ_i and where d is the corresponding column of D. Eq. (5.5) is a deterministic first-order vector difference equation analogous to the stochastic difference equation in (5.2). The solution for the Γ_i is obtained by solving for each of the columns of Γ_i separately using (5.5).

The analogy from the 2-dimensional case is now clear. There are n equations in (5.5). In a given application we will know some of the elements of γ_0, but not all of them. Hence, there will generally be more than n unknowns in (5.5). The number of unknowns is $2n - k$ where k is the number of values of γ_0 which we know. For example, in the simple bivariate case of Section 2 where $n = 2$, we know that the second element of γ_0 equals 0. Thus, $k = 1$ and there are 3 unknowns and 2 equations.

To get a unique solution in the general case, we therefore need $(2n-k)-n=n-k$ additional equations. These additional equations can be obtained by requiring that the solution for y_t be stationary or equivalently in this context that the γ_i do not explode. If there are exactly $n-k$ distinct roots of A which are greater than one in modulus, then the saddle point manifold will give exactly the number of additional equations necessary for a solution. The solution will be unique. If there are less than $n-k$ roots then we have the same nonuniqueness problem discussed in Section 2.

Suppose this root condition for uniqueness is satisfied. Let the $n-k$ roots of A that are greater than one in modulus be $\lambda_1,\ldots,\lambda_{n-k}$. Diagonalize A as $H^{-1}\Lambda H=A$. Then

$$H\gamma_{i+1}=\Lambda H\gamma_i \qquad i=1,2,\ldots. \tag{5.6}$$

$$\begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix}\begin{pmatrix} \gamma_{i+1}^{(1)} \\ \gamma_{i+1}^{(2)} \end{pmatrix}=\begin{pmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{pmatrix}\begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix}\begin{pmatrix} \gamma_i^{(1)} \\ \gamma_i^{(2)} \end{pmatrix} \qquad i=1,2,\ldots, \tag{5.7}$$

where Λ_1 is a diagonal matrix with all the unstable roots on the diagonal. The γ vectors are partitioned accordingly and the rows (H_{11}, H_{12}) of H are the characteristic vectors associated with the unstable roots. Thus, for stability we require

$$H_{11}\gamma_1^{(1)}+H_{12}\gamma_1^{(2)}=0. \tag{5.8}$$

These $n-k$ equations define the saddle point manifold and are the additional $n-k$ equations needed for a solution. Having solved for γ_1 and the unknown elements of γ_0 we then obtain the remaining γ_i coefficients from

$$\gamma_i^{(1)}=-H_{11}^{-1}H_{12}\gamma_i^{(2)} \qquad i=2,\ldots, \tag{5.9}$$

$$\gamma_{i+1}^{(2)}=\Lambda_2\gamma_i^{(2)} \qquad i=1,2,\ldots. \tag{5.10}$$

5.2. *Higher order vector models*

Alternatively the solution of (5.1) can be obtained directly without forming a large first order system. This method is essentially a generalization of the scalar method used in Section 2.4. Very briefly, by substituting the general solution of y_t into (5.1) and examining the equation in the Γ_i coefficients the solution can be obtained by factoring the characteristic polynomial associated with these equations.

This approach has been used by Hansen and Sargent (1981) in an optimal control example where $p=q$ and $B_i=hA_i'$. In that case, the factorization can be shown to be unique by an appeal to the factorization theorems for spectral density matrices. A similar result was used in Taylor (1980a) in the case of a factoring spectral density functions.

In general econometric applications, these special properties on the A_i and B_i matrices do not hold. Whiteman (1983) has a proof that a unique factorization exists under conditions analogous to those placed on the roots of the model in Section 5.1. Dagli and Taylor (1983) have investigated an iterative method to factor the polynomials in the lag operator in order to obtain a solution. This factorization method was used by Rehm (1982) to estimate a 7-equation rational expectations model of the U.S. using full information maximum likelihood.

6. Techniques for nonlinear models

As yet there has been relatively little research with nonlinear rational expectations models. The research that does exist has been concerned more with solution and policy evaluation rather than with estimation. Fair and Taylor (1983) have investigated a full-information estimation method for a non-linear model based on a solution procedure described below. However, this method is extremely expensive to use given current computer technology. Hansen and Singleton (1982) have developed and applied a limited-information estimator for nonlinear models.

There are a number of alternative solution procedures for nonlinear models that have been investigated in the literature. They generally focus on deterministic models, but can be used for stochastic analysis by stochastic simulation techniques.

Three methods are reviewed here: (1) a "multiple shooting" method, adopted for rational expectations models from two-point boundary problems in the differential equation literature by Lipton, Poterba, Sachs, and Summers (1982), (2) an "extended path" method based on an iterative Gauss–Seidel algorithm examined by Fair and Taylor (1983), and (3) a nonlinear stable manifold method examined by Bona and Grossman (1983). This is an area where there is likely to be much research in the future.

A general nonlinear rational expectation model can be written

$$f_i\left(y_t, y_{t-1}, \dots, y_{t-p}, \underset{t}{\mathrm{E}} y_{t+1}, \dots, \underset{t}{\mathrm{E}} y_{t+q}, \alpha_i, x_t\right) = u_{it}, \tag{6.1}$$

for $i = 1, \dots, n$, where y_t is an n dimensional vector of endogenous variables at time t, x_t is a vector of exogenous variables, α_i is a vector of parameters, and u_{it} is a vector of disturbances. In some write-ups, (e.g. Fair–Taylor) the viewpoint date on the expectations in (6.1) is based on information through period $t-1$

rather than through period t. For continuity with the rest of this paper, we continue to assume that the information is through period t, but the methods can easily be adjusted for different viewpoint dates. We also distinguish between exogenous variables and disturbances, because some of the nonlinear algorithms can be based on known future values of x_t rather than on forecasts of these from a model like (2.2).

6.1. *Multiple shooting method*

We described the shooting method to solve linear rational expectations models in Section 2.5. This approach is quite useful in nonlinear models. The initial conditions are the values for the lagged dependent variables and the final conditions are given by the long-run equilibrium of the system. In this case, a system of nonlinear equations must be solved using an iterative scheme such as Newton's method. One difficulty with this technique is that (6.1) is explosive when solved forward so that very small deviations of the endogenous variables from the solution can lead to very large final values. If this is a problem then the shooting method can be broken up in the series of shootings (multiple shooting) over intervals smaller than $(0, j)$. For example three intervals would be $(0, j_1)$, (j_1, j_2) and (j_2, j) for $0 < j_1 < j_2 < j$. In effect the relationship between the final values and the initial values is broken up into a relationship between intermediate values of these variables. The intervals can be made arbitrarily small. This approach has been used by Summers (1981) and others to solve rational expectations models of investment and in a number of other applications. It seems to work very well.

6.2. *Extended path method*

This approach has been examined by Fair and Taylor (1983) and used to solve large-scale nonlinear models. Briefly it works as follows. Guess values for the $\mathrm{E}_t y_{t+j}$ in eq. (6.1) for $j = 1, \ldots, J$. Use these values to solve the model to obtain a new path for y_{t+j}. Replace the initial guess with the new solution and repeat the process until the path y_{t+j}, $j = 1, \ldots, J$ converges, or changes by less than some tolerance range. Finally, extend the path from J to $J+1$ and repeat the previous sequence of iterations. If the values of y_{t+j} on this extended path are within the tolerance range for the values of $J+1$, then stop; otherwise extend the path one more period to $J+2$ and so on. Since the model is nonlinear, the Gauss–Seidel method is used to solve (6.1) for each iteration given a guess for y_{t+j}. There are no general proofs available to show that this method works for an arbitrary nonlinear model. When applied to the linear model in Section (2.1) with $|\alpha| < 1$ the method is shown to converge in Fair and Taylor (1983). When $|\alpha| > 1$, the iterations diverge. A convergence proof for the general linear model is not yet available, but many experiments have indicated that convergence is achieved under the usual saddle path assumptions. This method is expensive but is fairly easy to use. An empirical application of the method to a modified version of the Fair model is found in Fair and Taylor (1983) and to a system with time varying parameters in Taylor (1983). Carlozzi and Taylor (1984) have used the method to calculate stochastic equilibria. This method also appears to work well.

6.3. *Nonlinear saddle path manifold method*

In Section (2.4) we noted that the solution of the second-order linear difference eq. (2.88) is achieved by placing the solution on the stable path associated with the saddle point line. For nonlinear models one can use the same approach after linearizing the system. The saddle point manifold is then linear. Such a linearization, however, can only yield a local approximation.

Bona and Grossman (1983) have experimented with a method that computes a nonlinear saddle-point path. Consider a deterministic univariate second-order version of (6.1):

$$f(y_{t+1}, y_t, y_{t-1}) = 0, \qquad i = 1, 2, \dots . \tag{6.2}$$

A solution will be of the form

$$y_t = g(y_{t-1}), \tag{6.3}$$

where we have one initial condition y_0. Note that eq. (6.2) is a nonlinear version of the homogeneous part of eq. (2.88) and eq. (6.3) is a nonlinear version of the saddle path dynamics (2.91).

Bona and Grossman (1983) compute $g(\cdot)$ by a series of successive approximations. If eq. (6.3) is to hold for all values of the argument of g then

$$f(g(g(x)), g(x), x) = 0, \tag{6.4}$$

must hold for every value of x (at least within the range of interest). In the application considered by Bona and Grossman (1983) there is a natural way to write (6.4) as

$$g(x) = h(g(g(x)), g(x), x), \tag{6.5}$$

for some function $h(\cdot)$. For a given x eq. (6.5) may be solved using successive approximations:

$$g_{n+1}(x) = h(g_n(g_n(x)), g_n(x), x), \qquad n = 0,1,2,\dots. \tag{6.6}$$

The initial function $g_0(x)$ can be chosen to equal the linear stable manifold associated with the linear approximation of $f(\cdot)$ at x.

Since this sequence of successive approximations must be made at every x, there are two alternative ways to proceed. One can make the calculations recursively for each point y_t of interest; that is, obtain a function g for $x = y_0$, a new function for $x = y_1$ and so on. Alternatively, one could evaluate g over a grid of the entire range of possible values of x, and form a "meta function" g which is piecewise linear and formed by linear interpolation for the value of x between the grid points. Bona and Grossman (1983) use the first procedure to numerically solve a macroeconomic model of the form (6.2).

It is helpful to note that when applied to linear models the method reduces to a type of undetermined coefficients method used by Lucas (1975) and McCallum (1983) to solve rational expectations models (a different method of undetermined coefficients than that applied to linear process (2.4) in Section 2 above). To see this, substitute a linear function $y_t = gy_{t-1}$ into

$$y_{t+1} = \frac{\alpha_2}{\alpha_1} y_t - \frac{1}{\alpha_1} y_{t-1}, \tag{6.7}$$

the deterministic difference equation already considered in eq. (2.88). The resulting equation is

$$\left(g^2 - \frac{2}{\alpha_1} g + \frac{1}{\alpha_1}\right) y_{t-1} = 0. \tag{6.8}$$

Setting the term in parenthesis equal to zero, yields the characteristic polynomial of (6.7) which appears in eq. (2.89). Under the usual assumption that one root is inside and one root is outside the unit circle a unique stable value of g is found and is equal to stable root λ_2 of (2.89).

7. Concluding remarks

As its title suggests, the aim of this chapter has been to review and tie together in an expository way the extensive volume of recent research on econometric techniques for macroeconomic policy evaluation. The table of contents gives a good summary of the subjects that I have chosen to review. In conclusion it is perhaps useful to point out in what ways the title is either overly inclusive or not inclusive enough relative to the subjects actually reviewed.

All of the methods reviewed – estimation, solution, testing, optimization – involve the rational expectations assumption. In fact the title would somewhat more accurately identify the methods reviewed if the work "new" were replaced by "rational expectations". Some other new econometric techniques not reviewed here that have macroeconomic policy applications include the multivariate time series methods (vector auto-regressions, causality, exogeneity) reviewed by Geweke (1983) in Volume 1 of the *Handbook of Econometrics*, the control theory methods reviewed by Kendrick (1981) in Volume 1 of the *Handbook of Mathematical Economics*, and the prediction methods reviewed by Fair (1986) in this volume. On the other hand some of the estimation and testing techniques reviewed here were designed for other applications even though they have proven useful for policy.

Some of the topics included were touched on only briefly. In particular the short treatment of limited information estimation techniques, time inconsistency, and stochastic general equilibrium models with optimizing agents does not give justice to the large volume of research in these areas.

Most of the research reviewed here is currently very active and the techniques are still being developed. (About $\frac{2}{3}$ of the papers in the bibliography were published between the time I agreed to write the review in 1979 and the period in 1984 when I wrote it.) The development of computationally tractable ways to deal with large and in particular non-linear models is an important area that needs more work. But in my view the most useful direction for future research in this area will be in the applications of the techniques that have already been developed to practical policy problems.

References

Anderson, Gary and George Moore (1984) "An Efficient Procedure for Solving Linear Perfect Foresight Models". Board of Governors of the Federal Reserve Board, unpublished manuscript.

Anderson, T. W. (1971) *The Statistical Analysis of Time Series*. New York: Wiley.

Baumol, W. J. (1970) *Economic Dynamics: An Introduction, 3d ed.* New York: Macmillan.

Birkhoff, Garret and G. C. Rota (1962) *Ordinary Differential Equations*. Waltham: Blaisdell, 2nd Edition.

Blanchard, Olivier J. (1979) "Backward and Forward Solutions for Economies with Rational Expectations", *American Economic Review*, 69, 114–118.

Blanchard, Olivier J. (1982) "Identification in Dynamic Linear Models with Rational Expectations". Technical Paper No. 24, National Bureau of Economic Research.

Blanchard, Oliver and Charles Kahn (1980) "The Solution of Linear Difference Models under Rational Expectations", *Econometrica*, 48, 1305–1311.

Blanchard, Olivier and Mark Watson (1982) "Rational Expectations, Bubbles and Financial Markets", in: P. Wachtel, ed., *Crises in The Economic and Financial Structure*. Lexington: Lexington Books.

Bona, Jerry and Sanford Grossman (1983) "Price and Interest Rate Dynamics in a Transactions Based Model of Money Demand". University of Chicago, unpublished paper.

Buiter, Willem H. and Marcus Miller (1983) "Real Exchange Rate Overshooting and the Output Cost of Bringing Down Inflation: Some Further Results", in: J. A. Frenkel, ed., *Exchange Rates and International Macroeconomics*. Chicago: University of Chicago Press for National Bureau of Economic Research.

Cagan, Phillip (1956) "The Monetary Dynamics of Hyperinflation", in: M. Friedman, ed., *Studies in the Quantity Theory of Money*. Chicago: University of Chicago Press.
Calvo, Guillermo (1978) "On The Time Consistency of Optimal Policy in a Monetary Economy", *Econometrica*, 46, 1411–1428.
Calvo, Guillermo (1980) "Tax-Financed Government Spending in a Neo-Classical Model with Sticky Wages and Rational Expectations", *Journal of Economic Dynamics and Control*, 2, 61–78.
Carlozzi, Nicholas and John B. Taylor (1984) "International Capital Mobility and the Coordination of Monetary Rules", in: J. Bandhari, ed., *Exchange Rate Management under Uncertainty*. MIT Press, forthcoming.
Chow, G. C. (1983) *Econometrics*. New York: McGraw Hill.
Chow, Gregory and Philip J. Reny (1984) "On Two Methods for Solving and Estimating Linear Simultaneous Equations with Rational Expectations". Princeton University, unpublished paper.
Christiano, Lawrence J. (1984) "Can Automatic Stabilizers be Destablizing: An Old Question Revisited", *Carnegie–Rochester Conference Series on Public Policy*, 20, 147–206.
Cumby, Robert E., John Huizinga and Maurice Obstfeld (1983) "Two-Step Two-Stage Least Squares Estimation in Models with Rational Expectations", *Journal of Econometrics*, 21, 333–355.
Dagli, C. Ates and John B. Taylor (1985) "Estimation and Solution of Linear Rational Expectations Models Using a Polynomial Matrix Factorization", *Journal of Economic Dynamics and Control*, forthcoming.
Dhrymes, Pheobus J. (1971) *Distributed Lags: Problems of Estimation and Formulation*. San Francisco: Holden–Day.
Dixit, Avinash (1980) "A Solution Technique for Rational Expectations Models with Applications to Exchange Rate and Interest Rate Determination". Princeton University, unpublished paper.
Dornbusch, Rudiger (1976) "Expectations and Exchange Rate Dynamics", *Journal of Political Economy*, 84, 1161–1176.
Epple, Dennis, Lars P. Hansen and William Roberds (1983) "Linear Quadratic Games of Resource Depletion", in: Thomas J. Sargent, ed., *Energy, Foresight, and Strategy*. Washington: Resources for the Future.
Evans, George and Seppo Honkapohja (1984) "A Complete Characterization of ARMA Solutions to Linear Rational Expectations Models". Technical Report No. 439, Institute for Mathematical Studies in the Social Sciences, Stanford University.
Fair, Ray (1986) "Evaluating the Predictive Accuracy of Models", in: Z. Griliches and M. Intriligator, eds., *Handbook of Econometrics*. Amsterdam: North-Holland, Vol. III.
Fair, Ray and John B. Taylor (1983) "Solution and Maximum Likelihood Estimation of Dynamic Nonlinear Rational Expectations Models", *Econometrica*, 51, 1169–1185.
Fischer, Stanley (1979) "Anticipations and the Nonneutrality of Money", *Journal of Political Economy*, 87, 225–252.
Flood, R. P. and P. M. Garber (1980) "Market Fundamentals versus Price-Level Bubbles: The First Tests", *Journal of Political Economy*, 88, 745–770.
Futia, Carl A. (1981) "Rational Expectations in Stationary Linear Models", *Econometrica*, 49, 171–192.
Geweke, John (1984) "Inference and Causality in Economic Time Series Models", in: Z. Griliches and M. Intriligator, eds., *Handbook of Econometrics*. Amsterdam: North-Holland, Vol. II.
Gourieroux, C., J. J. Laffont and A. Monfort (1982) "Rational Expectations in Linear Models: Analysis of Solutions", *Econometrica*, 50, 409–425.
Hansen, Lars P. (1982) "Large Sample Properties of Generalized Method of Moments Estimators", *Econometrica*, 50, 1029–1054.
Hansen, Lars P. and Thomas J. Sargent (1980) "Formulating and Estimating Dynamic Linear Rational Expectations Models", *Journal of Economic Dynamics and Control*, 2, 7–46.
Hansen, Lars P. and Thomas J. Sargent (1981) "Linear Rational Expectations Models for Dynamically Interrelated Variables", in: R. E. Lucas and T. J. Sargent, eds., *Rational Expectations and Econometric Practice*. Minneapolis: University of Minnesota Press.
Hansen, L. P. and K. Singleton (1982) "Generalized Instrumental Variables Estimation of Nonlinear Rational Expectations Models", *Econometrica*, 50, 1269–1286.
Harvey, Andrew C. (1981) *Time Series Models*. New York: Halsted Press.

Hayashi, Fumio and Christopher Sims (1983) "Nearly Efficient Estimation of Time Series Models with Predetermined, but not Exogenous, Instruments", *Econometrica*, 51, 783–798.
Kendrick, David (1981) "Control Theory with Applications to Economics", in: K. Arrow and M. Intriligator, eds., Amsterdam: North-Holland, Vol. I.
Kouri, Pentti J. K. (1976) "The Exchange Rate and the Balance of Payments in the Short Run and in the Long Run: A Monetary Approach", *Scandinavian Journal of Economics*, 78, 280–304.
Kydland, Finn E. (1975) "Noncooperative and Dominant Player Solutions in Discrete Dynamic Games", *International Economic Review*, 16, 321–335.
Kydland, Finn and Edward C. Prescott (1977) "Rules Rather Than Discretion: The Inconsistency of Optimal Plans", *Journal of Political Economy*, 85, 473–491.
Kydland, Finn and Edward C. Prescott (1982) "Time to Build and Aggregate Fluctuations", *Econometrica*, 50, 1345–1370.
Lipton, David, James Poterba, Jeffrey Sachs and Lawrence Summers (1982) "Multiple Shooting in Rational Expectations Models", *Econometrica*, 50, 1329–1333.
Lucas, Robert E. Jr. (1975) "An Equilibrium Model of the Business Cycle", *Journal of Political Economy*, 83, 1113–1144.
Lucas, Robert E. Jr. (1976) "Econometric Policy Evaluation: A Critique", in: K. Brunner and A. H. Meltzer, eds., *Carnegie Rochester Conference Series on Public Policy*. Amsterdam: North-Holland, 19–46.
Lucas, Robert E. Jr. and Thomas J. Sargent (1981) "Introduction", to their *Rational Expectations and Econometric Practice*. University of Minneapolis.
Luenberger, David G. (1977) "Dynamic Equations in Descriptor Form", *IEEE Transactions on Automatic Control*, AC-22, 312–321.
Marschak, Jacob (1953) "Economic Measurements for Policy and Prediction", in: W. C. Hood and T. C. Koopmans, eds., *Studies in Econometric Method*. Cowles Foundation Memograph 14, New Haven: Yale University Press.
McCallum, Bennett T. (1976) "Rational Expectations and the Natural Rate Hypothesis: Some Consistent Estimates", *Econometrica*, 44, 43–52.
McCallum. Bennett T. (1979) "Topics Concerning the Formulation, Estimation, and Use of Macroeconometric Models with Rational Expectation", *American Statistical Association, Proceedings of the Business and Economics Section*, 65–72.
McCallum, Bennett T. (1983) "On Non-Uniqueness in Rational Expectations: An Attempt at Perspective", *Journal of Monetary Economics*, 11, 139–168.
Mishkin, Frederic S. (1983) *A Rational Expectations Approach to Macroeconometrics: Testing Policy Ineffectiveness and Efficient-Markets Models*. Chicago: University of Chicago Press.
Muth, John F. (1961) "Rational Expectations and The Theory of Price Movements", *Econometrica*, 29, 315–335.
Muth, John F. (1981) "Estimation of Economic Relationships Containing Latent Expectations Variables", reprinted in: R. E. Lucas and T. J. Sargent, eds., *Rational Expectations and Econometric Practice*. Minneapolis: University of Minnesota Press.
Papell, David (1984) "Anticipated and Unanticipated Disturbances: The Dynamics of The Exchange Rate and The Current Account", *Journal of International Money and Finance*, forthcoming.
Preston, A. J. and A. R. Pagan (1982) *The Theory of Economic Policy*. Cambridge: Cambridge University Press.
Rehm, Dawn (1982) *Staggered Contracts, Capital Flows, and Macroeconomic Stability in The Open Economy*. Ph.D. Dissertation, Columbia University.
Rodriquez, Carlos A. (1980) "The Role of Trade Flows in Exchange Rate Determination: A Rational Expectations Approach", *Journal of Political Economy*, 88, 1148–1158.
Sargent, Thomas J. (1979) *Macroeconomic Theory*. New York: Academic Press.
Sargent, Thomas J. and Neil Wallace (1973) "Rational Expectations and The Dynamics of Hyperinflation", *International Economic Review*, 14, 328–350.
Sargent, Thomas J. and Neil Wallace (1975) "'Rational' Expectations, The Optimal Monetary Instrument, and The Optimal Money Supply Rule", *Journal of Political Economy*, 83, 241–254.
Summers, Lawrence H. (1981) "Taxation and Corporate Investment: A q-Theory Approach", *Brookings Papers on Economic Activity*, 1, 67–127.
Taylor, John B. (1977) "Conditions for Unique Solutions in Stochastic Macroeconomic Models with Rational Expectations", *Econometrica*, 45, 1377–1385.

Taylor, John B. (1979) "Estimation and Control of a Macroeconomic Model with Rational Expectations", *Econometrica*, 47, 1267–1286.
Taylor, John B. (1980a) "Aggregate Dynamics and Staggered Contracts", *Journal of Political Economy*, 88, 1–23.
Taylor, John B. (1980b) "Output and Price Stability: An International Comparison", *Journal of Economic Dynamics and Control*, 2, 109–132.
Taylor, John B. (1982) "The Swedish Investment Fund System as a Stabilization Policy Rule", *Brookings Papers on Economic Activity*, 1, 57–99.
Taylor, John B. (1983) "Union Wage Settlements During a Disinflation", *American Economic Review*, 73, 981–993.
Wallis, Kenneth F. (1980) "Econometric Implications of The Rational Expectations Hypothesis", *Econometrica*, 48, 49–73.
Whiteman, Charles H. (1983) *Linear Rational Expectations Models: A User's Guide*. Minneapolis: University of Minnesota.
Wickens, M. (1982) "The Efficient Estimation of Econometric Models with Rational Expectations", *Review of Economic Studies*, 49, 55–68.
Wilson, Charles (1979) "Anticipated Shocks and Exchange Rate Dynamics", *Journal of Political Economy*, 87, 639–647.