III. Dynamic Programming: Method and Applications

Now we work with models where agents live forever (infinite planning horizon).

An example: infinitely-lived representative household

$$\max \quad \sum_{t=0}^{\infty} \beta^t u(c_t)$$

$$\text{s.t.} \quad c_t + k_{t+1} = y_t = k_t^{\alpha} \quad \forall t, \quad k_0 \text{ given}$$

Dynamic Programming (DP) is useful to solve this sort of "recursive problem".

Define a periodic return function: $r_t(x_t, u_t)$.

- $r_t(-)$ is concave
- x_t: vector of state variables
  - state variables characterize the economic system's current position
  - state variables are out of the control of an agent (cannot change in period t)
  - state variables can be exogenous variables or endogenous variables
  - state variables usually include information variables that help predict the future
- u_t: vector of control variables (choice variables)
  - control variables are under control of an agent (can change in period t)

Define a transition function: $g_t(x_t, u_t)$.

- a transition function maps the state of the model today into the state tomorrow
- assume $\{(x_{t+1}, x_t) : \ x_{t+1} \leq g_t(x_t, u_t), u_t \in R^k\}$ is convex and compact

General recursive problem is:

$$\max_{\{u_t\}_{t=0}^{\infty}} \quad \sum_{t=0}^{\infty} r_t(x_t, u_t)$$

$$\begin{array}{lrcl} \text{s.t.} & x_0 & \text{given} & \\ & x_1 & = & g_0(x_0, u_0) \\ & x_2 & = & g_1(x_1, u_1) \\ & & \vdots & \end{array}$$

Recursive: past x_t's and past u_t's have no *direct* effect on current and future returns (the only effect of past x_t's and past u_t's is through x_t).

Two further specializations:

- transition function is time invariant:

$$g_t(x_t, u_t) = g(x_t, u_t)$$

  for example,

$$c_t + k_{t+1} = k_t^{\alpha} \quad \Rightarrow \quad k_{t+1} = k_t^{\alpha} - c_t = g(k_t, c_t)$$

- periodic return function has the form:

$$r_t(x_t, u_t) = \beta^t r(x_t, u_t) \qquad 0 < \beta < 1$$

  for example,

$$U = \sum_{t=0}^{\infty} \beta^t \ln c_t$$

DP amounts to finding a *policy function* $h(x_t)$ mapping x_t into u_t, such that the sequence $\{u_t\}_{t=0}^{\infty}$ generated by iterating the two functions

$$u_t = h(x_t) \qquad \text{(policy function)}$$

$$x_{t+1} = g(x_t, u_t) \qquad \text{(transition function)}$$

starting from the initial condition x_0 solves the problem (**problem 1**)

$$\max_{\{u_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t r(x_t, u_t)$$

$$\begin{aligned} \text{s.t.} \quad & x_0 \quad \text{given} \\ & x_{t+1} = g(x_t, u_t) \quad \forall t \end{aligned}$$

A solution in the form of $u_t = h(x_t)$ and $x_{t+1} = g(x_t, u_t)$ is said to be *recursive.*

Example of a recursive problem:

$$\max_{\{c_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \ln c_t$$

$$\begin{aligned} \text{s.t.} \quad & k_0 \quad \text{given} \\ & k_{t+1} = k_t^{\alpha} - c_t \end{aligned}$$

The *Value Function* $V(x)$ gives the optimal value of the recursive problem:

$$V(x_0) = \max_{\{u_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t r(x_t, u_t)$$

where the maximization is subject to x_0 given and $x_{t+1} = g(x_t, u_t)$.

If we knew $V(-)$, then we could calculate the policy function $h(-)$ by solving the following problem (**problem 2**) for all possible values of the state variables (i.e. for each $x \in X$):

$$\max_{u} \{r(x, u) + \beta V(\tilde{x})\}$$

$$\text{s.t.} \quad x \text{ given}, \quad \tilde{x} = g(x, u)$$

We started with problem 1 which involves solving for an infinite sequence of control variables $\{u_t\}_{t=0}^{\infty}$. Now our task has become to solve problem 2. Problem 2 involves solving for the functions $V(x)$ and $h(x)$ that solve a continuum of maximization problems (one for each value of $x \in X$).

Our task has become jointly to solve for $V(x)$, $h(x)$. The (still) unknown value function $V(x)$ and policy function $h(x)$ are linked by the **Bellman Equation (BE)**

$$V(x) = \max_u \{r(x,u) + \beta V(\tilde{x})\} \qquad \text{(BE 1)}$$

$$\text{s.t.} \quad x \text{ given}, \quad \tilde{x} = g(x,u)$$

or

$$V(x) = \max_u \{r(x,u) + \beta V[g(x,u)]\} \qquad \text{(BE 2)}$$

$$\text{s.t.} \qquad x \text{ given}$$

The maximizer of the RHS of (BE 2) is a policy function $u = h(x)$ that satisfies

$$V(x) = r[x, h(x)] + \beta V\{g[x, h(x)]\}.$$

Solving the Bellman Equation:

Under the assumptions we make on the return function $r(-)$ and the set $\{(x_{t+1}, x_t) : x_{t+1} \leq g_t(x_t, u_t),\ u_t \in R^k\}$, it turns out that

1. The functional equation (BE 1) has a unique strictly concave solution.

2. This solution is approached in the limit as $j \to \infty$ by iterations on

$$V_{j+1} = \max_u \{r(x,u) + \beta V_j(\tilde{x})\}$$

$$\text{s.t.} \quad x \text{ given}, \quad \tilde{x} = g(x,u)$$

starting from any bounded continuous initial V_0.

3. There is a unique and time invariant optimal policy of the form $u_t = h(x_t)$, where $h(-)$ is chosen to maximize the RHS of BE 1.

4. Off corners, the limiting value function V is differentiable with

$$V'(x) = \frac{\partial r}{\partial x}[x, h(x)] + \beta \frac{\partial g}{\partial x}[x, h(x)] V'\{g[x, h(x)]\}.$$

This is a version of a *formula of Benveniste and Scheinkman* (1979) for the derivative of the optimal value function (or *envelope condition*). (See sections 4.1 and 4.2 of Stokey and Lucas (with Prescott) (1989) for the proof.)

We often encounter settings in which the transition law can be formulated so that the state x does not appear in it, so that $\frac{\partial g}{\partial x} = 0$, which makes the above equation become

$$V'(x) = \frac{\partial r}{\partial x}[x, h(x)].$$

For example, consider the following problem:

$$\max \quad \sum_t \ln(C_t)$$
$$\text{s.t.} \quad A_{t+1} = R(A_t - C_t)$$

In this original problem, A_t is state variable, C_t and A_{t+1} are control variables, and $A_{t+1} = R(A_t - C_t) = g(A_t, C_t)$ is the transition law.
If we define a new control variable $S_t = A_t - C_t$, then the transition law becomes $A_{t+1} = RS_t = g(S_t)$ and $\frac{\partial g_t}{\partial A_t} = 0$.

Four ways to solve a DP problem:

1. Value function iteration.
2. Guess and verify a solution for the policy function.
3. Guess and verify a solution for the value function.
4. Policy function iteration (Howard's improvement algorithm).

An Example: optimal growth model (Cass-Koopmans)

Planner chooses the sequence $\{c_t, k_{t+1}\}_{t=0}^{\infty}$ to maximize

$$\sum_{t=0}^{\infty} \beta^t \ln c_t$$
$$s.t. \qquad k_0 \text{ given}, \quad c_t + k_{t+1} = Ak_t^{\alpha}$$

- **Method 1: value function iteration**

  Bellman Equation (BE):

$$V(k) = \max_{c, \tilde{k}} \{\ln c + \beta\, V(\tilde{k})\}$$
$$s.t. \qquad k \text{ given}, \quad \tilde{k} = Ak^{\alpha} - c$$

- Start with $V_0 = 0$.
- First iteration: solve

$$V_1(k) = \max_{c,\tilde{k}} \{\ln c + \beta\ V_0(\tilde{k})\}$$
$$s.t. \quad c + \tilde{k} = Ak^\alpha$$

  Trivial solution is $\quad c = Ak^\alpha \quad$ and $\quad \tilde{k} = 0$.

  Accordingly: $\quad V_1(k) = \ln Ak^\alpha = \ln A + \alpha \ln k$.

- Second iteration: solve

$$\begin{aligned} V_2(k) &= \max_{c,\tilde{k}} \{\ln c + \beta V_1(\tilde{k})\} \\ &= \max_{\tilde{k}} \{\ln(Ak^\alpha - \tilde{k}) + \beta[\ln A + \alpha \ln \tilde{k}]\} \end{aligned}$$

  FOC for the problem on the right-hand-side (RHS) of BE:

$$\frac{-1}{Ak^\alpha - \tilde{k}} + \frac{\beta\alpha}{\tilde{k}} = 0 \quad \Rightarrow \quad \tilde{k} = \frac{\alpha\beta}{1+\alpha\beta}\ Ak^\alpha$$
$$\Rightarrow \quad c = Ak^\alpha - \tilde{k} = \frac{1}{1+\alpha\beta}\ Ak^\alpha$$

  Using the solutions for c and $\tilde{k}$ in the BE, we find

$$\begin{aligned} V_2(k) &= \ln\left(\frac{A}{1+\alpha\beta}\right) + \beta \ln A + \alpha\beta \ln\left(\frac{\alpha\beta A}{1+\alpha\beta}\right) + \alpha(1+\alpha\beta)\ln k \\ &= \text{constant} + \alpha(1+\alpha\beta)\ln k \end{aligned}$$

- Third iteration: solve

$$V_3(k) = \max_{\tilde{k}} \{\ln(Ak^\alpha - \tilde{k}) + \beta[\text{cst} + \alpha(1+\alpha\beta)\ln \tilde{k}]\}$$

$$\Rightarrow \quad \tilde{k} = \frac{\alpha\beta + (\alpha\beta)^2}{1+\alpha\beta+(\alpha\beta)^2}\ Ak^\alpha, \qquad c = \frac{1}{1+\alpha\beta+(\alpha\beta)^2}\ Ak^\alpha$$

  (1) $\quad c = Ak^\alpha \qquad \tilde{k} = 0$

  (2) $\quad c = \frac{1}{1+\alpha\beta}Ak^\alpha \qquad \tilde{k} = \frac{\alpha\beta}{1+\alpha\beta}\ Ak^\alpha$

  (3) $\quad c = \frac{1}{1+\alpha\beta+(\alpha\beta)^2}Ak^\alpha \qquad \tilde{k} = \frac{\alpha\beta+(\alpha\beta)^2}{1+\alpha\beta+(\alpha\beta)^2}\ Ak^\alpha$

  (∞) $\quad c = \frac{1}{(\frac{1}{1-\alpha\beta})}\ Ak^\alpha = (1-\alpha\beta)Ak^\alpha \qquad \tilde{k} = \alpha\beta\ Ak^\alpha$

Value function:

$$V(k)=\frac{1}{1-\beta}\left\{\ln[A(1-\alpha\beta)]+\frac{\alpha\beta}{1-\alpha\beta}\ln(A\alpha\beta)\right\}+\frac{\alpha}{1-\alpha\beta}\ln k.$$

- **Method 2: guess and verify a solution for the policy function**

∘ Bellman Equation:

$$V(k)=\max_{\tilde{k}}\ \left\{\ln(Ak^{\alpha}-\tilde{k})+\beta V(\tilde{k})\right\}$$

∘ **Guess** $\tilde{k}=\gamma Ak^{\alpha}$ where γ is an undetermined coefficient.

∘ **FONC**:

$$\frac{1}{Ak^{\alpha}-\tilde{k}}=\beta V'(\tilde{k})$$

∘ Find $V'(\tilde{k})$ using **Benveniste and Scheinkman/Envelope Condition**:

$$V'(k)=\frac{\alpha Ak^{\alpha-1}}{Ak^{\alpha}-\tilde{k}}=\frac{\alpha Ak^{\alpha-1}}{Ak^{\alpha}-\gamma Ak^{\alpha}}=\frac{\alpha Ak^{\alpha-1}}{(1-\gamma)Ak^{\alpha}}$$

$$V'(k)=\frac{\alpha}{(1-\gamma)k}\qquad\Rightarrow\qquad V'(\tilde{k})=\frac{\alpha}{(1-\gamma)\tilde{k}}$$

∘ FONC and Envelope Condition yield the **Euler Equation**:

$$\frac{1}{Ak^{\alpha}-\tilde{k}}=\beta\frac{\alpha}{(1-\gamma)\tilde{k}}$$

Using the guess again

$$\frac{1}{Ak^{\alpha}-\gamma Ak^{\alpha}}=\beta\frac{\alpha}{(1-\gamma)\gamma Ak^{\alpha}}$$

$$\frac{1}{(1-\gamma)Ak^{\alpha}}=\frac{\alpha\beta}{(1-\gamma)\gamma Ak^{\alpha}}\qquad\Rightarrow\qquad\gamma=\alpha\beta$$

∘ Policy functions:

$$\tilde{k}=\alpha\beta Ak^{\alpha},\qquad c=(1-\alpha\beta)Ak^{\alpha}.$$

- **Method 3: guess and verify a solution for the value function**

$$V(k) = \max_{\tilde{k}} \{\ln(Ak^\alpha - \tilde{k}) + \beta V(\tilde{k})\}$$

- ∘ **Guess** $V(k) = E + F \ln k$ where E and F are undetermined coefficients.
- ∘ **First Step**: Use the guess in BE and solve for a "preliminary" policy function.

  Bellman Equation is

$$E + F \ln k = \max_{\tilde{k}} \{\ln(Ak^\alpha - \tilde{k}) + \beta E + \beta F \ln(\tilde{k})\}$$

  Maximization problem on the RHS yields the preliminary policy function:

$$\tilde{k} = \frac{\beta F}{1 + \beta F} \, Ak^\alpha$$

- ∘ **Second Step**: Use solution for $\tilde{k}$ in BE and solve for the undetermined coefficients

$$E + F \ln k = \ln \left[\left(1 - \frac{\beta F}{1 + \beta F}\right) Ak^\alpha \right] + \beta E + \beta F \ln \left(\frac{\beta F}{1 + \beta F} Ak^\alpha \right)$$

$$E + F \ln k = \ln \left(\frac{A}{1 + \beta F} \right) + \alpha \ln k + \beta E + \beta F \ln \left(\frac{\beta F A}{1 + \beta F} \right) + \alpha \beta F \ln k$$

  Grouping terms:

$$E + F \ln k = \left[\ln \left(\frac{A}{1 + \beta F} \right) + \beta E + \beta F \ln \left(\frac{\beta F A}{1 + \beta F} \right) \right] + [\alpha + \alpha \beta F] \ln k$$

  For this equation to hold for any $\ln k$, it must be that

$$E \equiv \ln \left(\frac{A}{1 + \beta F} \right) + \beta E + \beta F \ln \left(\frac{\beta F A}{1 + \beta F} \right) \tag{1}$$

$$F \equiv \alpha + \alpha \beta F \tag{2}$$

  Restriction (2) implies

$$F = \frac{\alpha}{1 - \alpha \beta} \tag{3}$$

  Using result (3) in restriction (1) implies

$$E = \frac{1}{1 - \beta} \left[\ln \left(A(1 - \alpha \beta) \right) + \frac{\alpha \beta}{1 - \alpha \beta} \ln(\alpha \beta A) \right]$$

  Result (3) implies the policy functions:

$$\tilde{k} = \frac{\beta F}{1 + \beta F} Ak^\alpha \quad \Rightarrow \quad \tilde{k} = \alpha \beta Ak^\alpha, \quad c = (1 - \alpha \beta) Ak^\alpha.$$