完全可約的群及環之構造

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摘要：令 G 表一具有算子域 Ω 的羣而 Ω 假定至少包含所有 G 的內同構.

G 稱爲其子羣系 $\{I\}$ 的直積，若 G 的每一元素 a 可一意地表爲乘積 $a = a_1 \cdot a_2 \cdots a_n$ 而不等的 a_i 分別屬於 $\{I\}$ 中不同的 I_i.

G 的任一子羣 I 稱爲不可約的，若除其本身與單位羣外 I 不再含有 G 的子羣.

G 稱爲完全可約羣若對於 G 的任一子羣 N 恆有一子羣 N' 使 G 爲 N 及 N' 的直積.

首先我們證明了下面的主要定理：

一完全可約羣爲不可約子羣的直積.

其次，將上面的結果應用於環.

一環 R 可看作一以其自身爲左乘（或右乘）算子域的加羣. 若此加羣爲完全可約羣，則稱 R 爲左邊（或右邊）完全可約環. 此加羣的不可約子羣即爲 R 的最小左（或右）理想集合，遂有定理.

一左邊（或右邊）完全可約環爲其最小左（或右）理想集合的直和.

所謂環 R 的根基 $\bar{R}$ 即爲所有某次羃後爲零的左理想集合的和. 對於一左邊完全可約環的根基且有如下的定理.

一左邊完全可約環 R 的根基 $\bar{R}$ 有下列性質：

i) $\bar{R}^2 = 0$.

ii) $\bar{R}\,R = 0$.

iii) 設 l 爲 R 的任一非零最小左理想集合且含於 $\bar{R}$ 內者. 於是，若 $R\,l = 0$，則 l 由一元素 x 之倍數 $x, 2\,x, \cdots, p\,x\,(=0)$ 所組成，而 p 爲一質數；若 $R\,l \neq 0$，則 $l = l'\,x, x \varepsilon \bar{R}$ 而 l' 爲 R 的某一最小左理想集合.

至此即可論其根基爲零的左邊完全可約環. 此種環特稱爲半簡單環. 若任一左邊完全可約環除其本身及 0 外不再含有兩邊理想集合，則稱爲簡單環. 固然若一簡單環 R 有 $R^2 \neq 0$ 則爲一半簡單環. 反之我們可證.

任一半簡單環爲簡單環之直和且爲唯一的.

且有

任一半簡單環亦爲右邊完全可約環.

STRUCTURE OF COMPLETELY REDUCIBLE GROUPS AND RINGS

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Let C be a group with an operator domain. All subgroups in consideration will be admissible and normal. G is said to be a completely reducible group (we shall use the abbreviation C. R. group) if to every subgroup N of G, there is another subgroup N' of G, such that G is the direct product of N and N'. Assuming one of the chain conditions, Jacobson [1] has proved that a C. R. group is a direct product of a finite number of irreducible subgroups. In this paper, we shall investigate the structure of a C. R. group without using either of the chain conditions and we have established the fact that a C. R. group is always a direct product of a finite number or an infinite number of irreducible subgroups. By this, we have also obtained a generalization of Wedderburn-Artin's theory of rings without minimal condition.

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Part I. Structure of a C. R. Group

First, we shall give some definitions and some known results about C. R. groups. When homomorphisms or isomorphisms are spoken of we shall always mean $\Omega-$homomorphisms or $\Omega-$isomorphisms.

Def. 1. G is said to be a C. R. group, if to every subgroup N of G, there is another subgroup N′ of G, such that G is the direct product of N and N′.

Def 2. $\mathfrak{J}$ *is said to be an irreducible subgroup, if* $\mathfrak{J}$ *has no subgroups other than e, the identity subgroup, and itself.*

Def. 3. G is said to be a direct product of a system of (denumerably or non-denumerably many) irreducible subgroups $I_1, I_2, \cdots$, *if any element a of G can be uniquely expressed as* $a=a_1, a_2 \cdots a_n$ *where* a_i, *belongs to* I_i.

We denote by $N=(a_1, a_2, \cdots a_n)$ the smallest subgroup containing $a_1, a_2, \cdots a_n$, and N is said to have a finite basis.

Known results about C. R. groups:

1). A subgroup N of a C. R. group G is itself a C. R. group.

2). In a C. R. group G, the ascending chain condition holds if and only if the descending chain condition holds.

3). The ascending chain condition holds in a group G, if and only if every subgroup N has a finite basis.

THEOREM 1. *If G is a C. R. group, different from e, then G contains an irreducible subgroup* $\mathfrak{J}$, *different from e.*

Proof. Let a $(\neq e)$ be an element of G. We shall show that (a) satisfies the ascending chain condition.

Let M_1 be a subgroup of (a). By hypothesis $G=M_1 \times M_1'$ (direct product). Writing $a=a_1 \cdot a_1'$, where $a_1 \varepsilon M_1$, $a_1' \varepsilon M_1'$, we have $(a) \subset (a_1) \cdot (a_1')$, and this is a direct product, since $(a_1) \cap (a_1') \subset M_1 \cap M_1' = e$. From $a_1' = a_1^{-1} a$ follows that $(a_1') \subset M_1 \times (a) \subset (a)$. Therefore $(a) \subset (a_1) \times (a_1') \subset M_1 \times (a_1') \subset (a)$. Hence $(a)=(a_1) \times (a_1') = M_1 \times (a_1')$, and then $M_1 = (a_1)$.

This result, together with 1), 2), 3) shows that (a) contains irreducible subgroups. This proves the theorem.

THEOREM 2. *A C. R. group is a direct product of irreducible subgroups.*

Proof. Let $G (\neq e)$ be a C. R. group. By Theorem 1, there is an irreducible subgroup $\mathfrak{J}_1 (\neq e)$. As G is a C. R. group, $G = \mathfrak{J}_1 \times \mathfrak{J}_1'$. By 1), $\mathfrak{J}_1'$ is a C. R. group; if $\mathfrak{J}_1' \neq e$, we again have an irreducible subgroup $\mathfrak{J}_2$, then

$$G = \mathfrak{J}_1 \times (\mathfrak{J}_2 \times \mathfrak{J}_2') = (\mathfrak{J}_1 \times \mathfrak{J}_2) \times \mathfrak{J}_2'.$$

By transfinite induction, we arrive at the conclusion that G can be decomposed into a direct product of irreducible subgroups.

THEOREM 3. *If G is a direct product of irreducible subgroups, then G is a C. R. group.*

THEOREM 4. *If two decompositions in which corresponding factors are isomorphic are considered identical, then the decomposition of a C. R. group G into a direct product of irreducible subgroups is unique.*

THEOREM 5. *If G is a product of irreducible subgroups, then G is a C. R. group.*

Proofs of Theorems 3, 4, 5 were given by Krull [2] for Abelian gronps. The proofs for the general case are essentially the same.

Part II. Structure of a C. R. Ring

Def. 4. R is said to be a C. R. l. ring (completely reducible ring for left ideals), if to every left ideal l of R there is another left ideal l' such that R is a direct sum of l and l'.

Similarly we may define a C. R. r. ing (completely reducible ring for right ideals).

If R is a C. R. r. ring as well as a C. R. l. ring, we simply call it a C. R. ring.

Consider a C. R. l. ring R as an additive group with itself as operator domain, we obtain immediately:

THEOREM 6. *A C. R. l. ring is a direct sum of minimal left ideals.*

The sum of all nilpotent left ideals, i.e., the left ideals l for which there is a positive integer n such that $l^n=0$, form a maximal two-sided ideal $\bar{R}$ of R [3]. $\bar{R}$ is called the radical.

THEOREM 7. *If R is a C. R. l. ring, $\bar{R}$ its radical, then $\bar{R}^2=0$.*

Proof. We need only prove that if R is a C. R. l. ring, then for every nilpotent left ideal l, we have $l^2=0$. In fact, if $a, b, \varepsilon\ \bar{R}$, then $a \varepsilon l$, $b \varepsilon l'$ where l, l' are nilpotent, therefore $a, b, \varepsilon l+l'$. As the sum of nilpotent left ideals is again a nilpotent left ideal, $ab=0$ or $\bar{R}^2=0$.

Now let l be a nilpotent left ideal, and assume $l^2 \neq 0$, then, by 1) of part I we have $l=l^2+l'$ (direct sum) from $l\,l'\subset l^2, l\,l'\subset l'$ and $l^2=l$ (l^2+l') we have $l\,l'=0$ and $l^2=l^3$, this leads to the contradiction that l would not be nilpotent.

Def. 5. R is said to be a semi-simple ring if R is a C. R. l. ring and if the radecel $\bar{R}=0$.

Def. 6. R is said to be a simple ring if R is a C. R. l. ring and has no two sided ideal other than zero ideal and R itself.

If R is a simple ring with $R^2=0$ then R is an additive group of prime order with the multiplication ab$=0$ for any a, b of R. In the sequel, simple rings are understood to be non-trivial or $R^2\neq 0$.

THEOREM 8. *A simple ring is semisimple.*

Proof. As usual.

THEOREM 9. *A semi-simple ring is a direct sum of simple two-sided ideals.*

Proof. Let R be a semi-simple ring. Then by Theorem 4, R is a direct sum of minimal left ideals. The sum of all R-isomorphic minimal left ideals form a two sided simple ideal. [3]

Remarks

1. We may define a simple ring (in the most general sense) as a ring without two-sided ideals other than zero-ideal and R itself. Their structure can not be determined in general, unless certain other conditions are imposed. One of them, due to Jacobson, is that R contains minimal left (or right) ideals [4]. The simple ring in our sense is obviously a simple ring in Jacobson's sense. Converseįy, let R be a simple ring in Jacobson's sense and l be a minimal left ideal of R. Then the sum of all left ideals R-isomorphic to l, form a two sided ideal ($\neq 0$) of R, therefore must coincide with R. By Theorem 5, R is a C. R. l. ring.

2. A semi-simple ring R is a C. R. ring. This will be proved if it is proved for simpl rings. Let R be a simple ring. Then $\bar{R}=0$. Let l be a mininal left ideal. As $\bar{R}=0$, there is an idempotent element $e\neq 0$ wich belongs to l and $l\equiv Re$, then eR is minimal right ideal and R is sum of minimal right ideals, therefore R is a C. R. r. ring· This establishes

the fact. However, this is not true in general. For example, if we introduce a multiplication $(a, b)(c, d) = (ac, ad)$ into the vector space (a, b) over the rational field, we obtain a ring, which is a C. R. l. ring but not a C. R. r. ring.

5. In a C. R. l. ring it may be of some interest, to prove the following theorem: Every non-nilpotent left ideal contains an idempotent element $e \neq 0$.

Proof. Let R be a C. R. l. ring, and l a non-nilpotent left ideal. Then $l^2 \neq 0$. So, there is an element $\mu \varepsilon l$ such that $l\mu \neq 0$. Let l_1 be the left annihilator of μ in l. Then l_1 is a left ideal properly contained in l. By hypothesis, and l)

$$l = l_1 + l_1'$$

Therefore, $0 \neq l\mu = l_1'\mu \subset l$. Then $l = l_1'\mu + l'$. We have $\mu = e\mu + \lambda'$. If $e = 0$ we have $\mu = \lambda'$. As $l_1' \neq 0$, there exists $0 \neq \lambda \varepsilon l_1'$. From $\lambda\mu = \lambda\lambda'$, $\lambda\mu \varepsilon l_1'\mu$, $\lambda\lambda' \varepsilon l'$, we have $\lambda\mu = \lambda\lambda' = 0$. But then $\lambda \varepsilon l_1$ or $\lambda \varepsilon l_1 \cap l_1' = 0$, a contradiction. Therefore $e \neq 0$. Since $e\mu = e^2\mu + e\lambda'$, we have $(e^2 - e)\mu = -e\lambda' \varepsilon l'$. Similar arguments show that $e^2 - e = 0$ or $e^2 = e$.

THEOREM 10; *Let $\bar{R}$ be the radical of a C. R. l. ring R. Then*

i) $\bar{R}R = 0$

ii) The non-zero left ideals of $\bar{R}$ have only the form

(a) a cyclic group (x) of prime order and $Rx = 0$

(b) $l = l'x$ where l' is a minimal non-nilpotent left ideal and $x \varepsilon \bar{R}$.

Proof. By hypothesis $R = S + \bar{R}$ (direct sum). As $\bar{R}$ is two-sided, $\bar{R}S \subset \bar{R}$, $\bar{R}S \subset S$. Hence $\bar{R}S = 0$. So, $\bar{R}R = \bar{R}(S + \bar{R}) = 0$.

Now if l is a non-zero minimal left ideal of $\bar{R}$ and $l \neq 0$. Let x be a non-zero element of l.

Case a) $Rx = 0$. Obviously, l is a cyclic group (x) of prine order, and $Rx = 0$.

Case b) $Rx \neq 0$. As $\bar{R}^2 = 0$, $Rx = (S + \bar{R})x = Sx \neq 0$ As S is a direct sum of minimal left ideals of the form Re, where e is an idempotent, there exists a certain e for which $ex \neq 0$. So, $Rex = l'x$ is a minimal left ideal in $\bar{R}$.

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